Game Theory
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Lecture 28
Fictitious Play

In the previous lectures, we introduced evolutionary game theory. The major idea was to introduce evolutionary stable strategies and connect them with replicator dynamics. One interesting fact about replicator equations is that the strategies which are performing better are adapted in the population. Then, as the time progresses the people adapt to a fitter strategy and eventually, it leads to a evolutionary stable strategy. In fact, the one of the very interesting ideas is that this sub-rationality is connected with rational behaviour which is given by a Nash Equilibrium.

The evolutionary stable strategy is, in that sense, a very interesting subject on its own. The replicator dynamics provide a nice way of learning the equilibrium. Under what equilibria will the stable points under this dynamics be? This question is very important and people have tried variety of methods. One of the methods that we have seen earlier is fictitious play.

Now we will look at *fictitious play* in a more detailed manner. Let us go back to the Hawk-Dove game. Recall the model of the same:

- There is a large population of some species.
- There are two types of behaviours: Hawk(aggressive) and Dove(passive)
- The game matrix is given by:

P1

P2		H	D
	H	$\frac{V-C}{2}, \frac{V-C}{2}$	$0, V$
	D	$V, 0$	$\frac{V}{2}, \frac{V}{2}$

- With $V > C$, Hawk is a strictly dominant strategy. This game becomes Prisoners' Dilemma. If $C > V$, the game turns into what is called the game of chicken.
- In this game, there are two asymmetric Nash Equilibria where one plays Hawk and the other is Dove. There is a symmetric Nash Equilibrium.

Lemma. The Hawk-Dove game with $C > V$, has one ESS, given by $x = \left(\frac{V}{C}, 1 - \frac{V}{C}\right)$.

This is left for the reader to verify. Let us now shift our focus to Fictitious Play. Consider the underlying game (A, B). The game proceeds in several rounds.

	P1	P2
Round 1	a_0	b_0
Round 2	$a_1 \in PBR(b_0)$	$b_1 \in PBR(a_0)$
Round 3	$a_2 \in PBR\left(\frac{b_0+b_1}{2}\right)$	$b_2 \in PBR\left(\frac{a_0+a_1}{2}\right)$

and so on.

Fictitious play says that

$$\frac{a_1 + a_2 + .. + a_n}{n+1} \to x^*$$
$$\frac{b_1 + b_2 + .. + b_n}{n+1} \to y^*$$

and (x^*, y^*) is Nash Equilibrium. Julia Robinson proved this conjecture for zero-sum games. We will not go into the proof of this. Note that, the above average need not converge to x^* and y^*. Before looking at this, let us introduce Fictitious Play more formally.

- There are two players.
- For $t = 0, 1, 2..$, let $\eta_i : S_{-i} \to \mathbb{N}$ be the number of times i has observed s_{-i} in the past. $\eta_i^0(s_i)$ represents the starting point.
- Beliefs are formed according to

$$\mu_i^t(s_{-i}) = \frac{\eta_i^t(s_{-i})}{\sum\limits_{\bar{s}_{-i} \in S_{-i}} \eta_i^t(\bar{s}_{-i})}$$

- Once beliefs are formed, a player chooses his action at time t to maximize his payoff. That is,

$$s_i^t \in \arg\max_{s_i \in S_i} u_i(s_i, \mu_i^t)$$

  Note that, this s_i need not be unique because there may be multiple best responses.

Let us look at an example. Consider the matrix game given by, This can be solved by Domination.

P1

P2		L	R
	U	$3,3$	$0,0$
	D	$4,0$	$1,1$

In fact, it can be verified that the unique Nash Equilibrium is (D, R). Assume that $\eta_1^0 = (3,0)$ and $\eta_2^0 = (1, 2.5)$. Then, we have the following:

- In period 1, we have $\eta_1^0 = (3,0)$ and $\eta_2^0 = (1, 2.5)$. Therefore, $\mu_1^0 = (1,0)$, $\mu_2^0 = \left(\frac{1}{3.5}, \frac{2.5}{3.5}\right)$. So, play follows $s_1^0 = D$ and $s_2^0 = L$.
- In period 2, $\eta_1^1 = (4,0)$ and $\eta_2^1 = (1, 3.5)$. Hence, considering the values of μ_1^1 and μ_2^1, we have $s_1^1 = D$ and $s_2^1 = R$.
- In period 3, $\eta_1^1 = (4,1)$ and $\eta_2^1 = (1, 4.5)$. Hence, $s_1^2 = D$ and $s_2^2 = R$.

So, (D,R) has been reached which they continue playing throughout afterwards. Moreover, this is a Nash equilibrium.

Let $\{s^t\}$ be the sequence of strategy profiles generated by Fictitious Play. The sequence $\{s^t\}$ converges to s if there exists T such that $s^t = s$ for all $t \geq T$.
Theorem. Firstly, if s^t converges to $\bar{s}$, then $\bar{s}$ is Nash Equilibrium. Secondly, suppose for some t, $s^t = s^*$ where s^* is a strict Nash Equilibrium, then $s^\tau = s^*$ for all $t \geq \tau$.

The proof is not that hard and is left for the reader as an exercise.
Result: Let s^t converges to a mixed strategy profile σ in a time-average sense if for all i and for all $s_i \in S_i$, we have

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{1}_{\{s_i^t = s_i\}} = \sigma(s_i)$$

where $\mathbb{1}$ is the indicator function. In other words, $\mu_{-i}^t \to \sigma_i(s_i)$.
Theorem. If $s^t \to \sigma$ in time-average sense, then σ is a Nash Equilibrium.

The proof is left for the leader as an exercise. In fact, another interesting exercise would be to check the above for the game of matching pennies.

Non-Convergence

This is due to Shapley. He considers a modified version of the Rock-paper-scissors game and shows that Fictitious Play does not converge. Consider the following game matrix for the same:

P1

P2	R	S	P
R	0,0	1,0	0,1
S	0,1	0,0	1,0
P	1,0	0,1	0,0

Check that the unique mixed Nash Equilibrium is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Let us start with the initial beliefs $\eta_1^0 = (1,0,0)$ and $\eta_2^0 = (0,1,0)$. Fictitious Play follows the following pattern:

- In period 0, we have $\eta_1^0 = (1,0,0)$ and $\eta_2^0 = (0,1,0)$. The play will be (P,R).
- In period 1, the play will be (P,R). This continues until $P2$ switches to S.
- Once, $P2$ switches to S, play continues with (P,S) until $P1$ switches to R.
- Then, it continues with (R,S) until $P2$ switches to P.

This keeps cycling and every time a particular strategy is played after a cycle, the amount of time increases. Therefore, this would never lead to convergence. For example, (P,S) is played for some k periods, then (R,S) is played for some βk periods, then (R,S) is played for some $\beta^2 k$ periods, and so on.