Game Theory

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Lecture 29
Brown-Von Neumann-Nash Dynamics

In the previous lecture, we saw fictitious play and some important results. In this session, we will see another method to solve zero-sum games. In this method, we use differential equations and this method is called Brown-Von Neumann Nash Dynamics or BNN Dynamics. We start with a zero-sum game with the corresponding game matrix A. We assume a symmetric game, and hence A is a skew-symmetric matrix.
Exercise: Let A be a matrix game. Consider another matrix game

$$B = \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix}$$

Then B is a symmetric game.

The exercise is to see that the Saddle Point Equilibria(SPE) of A and the SPE of B are related. We assume a symmetric game $A = -A^T$. An advantage of the above is that the value $v(A) = 0$. Let $P2$ choose y at time 0. If Player 1's value corresponding to y is not 0, then y will be perturbed.

There are m pure strategies. Take any (e_1, y) where e_i is a pure strategy with $i = 1, 2, ..m$ and y is any strategy. Note that,

$$u_i(y) = e_i^T Ay \quad \text{value player 1 gets}$$

y is the minmax strategy if $u_i(y) \leq 0$ for all $i = 1, 2, .., m$. The minmax strategy for player 2 is the one which gives

$$\min_{y \in \Delta_2} \max_{x \in \Delta_1} x^T Ay$$

We know that because it is a symmetric game, the above is always greater than or equal to 0. Because the value is 0, the above has to be equal to 0 for some strategy.

Define

$$\phi_i(y) = \max\{0, u_i(y)\}$$
$$\phi(y) = \phi_1(y) + \phi_2(y) + .. + \phi_m(y)$$

where $\phi_i(y)$ is the return of player 1 if $u_i(y) \geq 0$ as $u_i(y)$ is the payoff that player 1 is receiving. BNN dynamics are given by the following equations:

$$\frac{dy_i(t)}{dt} = \phi_i(y(t)) - \phi(y(t))y_i(t) \quad \text{for } t > 0 \text{ and } i = 1, ..., m.$$
$$y(0) \in \Delta_1 = \Delta_2$$

We extend ϕ to the whole domain by,

$$\phi(y) = \phi\left(\frac{y}{|y|}\right) \quad \text{if } y \notin \Delta$$

If $y = 0$, then $\phi(0) = 0$. Moreover, these ϕ's are continuous functions. Also, as $u_i(y)$ are linear in y, $\phi_i(y)$ is a Lipschitz continuous function. Also, as $\phi(y)$ is the sum of $\phi_i(y)$'s, $\phi(y)$ is also a Lipschitz continuous function. Hence, we can immediately say that by applying the Cauchy-Picard's theorem, this system of differential equations has a solution and it is unique.

This however, does not guarantee that $y(t) \in \Delta$. We will now try to prove that $y(t)$ indeed lies in Δ. Consider the following auxiliary ODE :

$$\frac{dx_i(t)}{dt} = \phi_i(x(t))$$
$$x(0) = y(0) \in \Delta$$

Without loss of generality, we can assume that $\phi_i(x(0)) > 0$. Therefore, this auxiliary equation has unique solution. Also, all the points lie in $\mathbb{R}^m \setminus 0$. As $x_i(t)$ is non-decreasing, we have $x_i(t) \geq x_i(0)$ and we have without loss of generality, $x_i(0) > 0$ and $x_i(t) > 0$ for all t.

Now consider,

$$\frac{d\alpha(t)}{dt} = \sum_{i=1}^{m} x_i(\alpha(t)) \quad t > 0$$
$$\alpha(0) = 0$$

x_i's are lipschitz continuous function. This is so because x_i's are solutions of the system of differential equations given above, and their derivatives are ϕ_i which are bounded(remember $\phi(y) = \phi\left(\frac{y}{|y|}\right)$). Therefore, applying the Caucy-Picard theorem again, there exists a unique solution $\alpha(t)$.

$$\frac{d\alpha(t)}{dt} = \sum_{i=1}^{m} x_i(\alpha(t)) \geq \sum_{i=1}^{m} x_i(0) = 1$$
$$\Rightarrow \alpha(t) = t \;\; \forall t$$

Define

$$y_i(t) = \frac{x_i(\alpha(t))}{\sum_{j=1}^{m} x_j(\alpha(t))}$$
$$\Rightarrow y(t) \in \Delta \;\; \forall t \geq 0$$

Claim: $y(t)$ is the solution of BNN Dynamics equations.
We have

$$y_i(t) \sum_{j=1}^{m} x_j(\alpha(t)) = x_i(\alpha(t))$$

Differentiating both sides with respect to t, we have

$$y_i'(t)\sum_{j=1}^{m} x_j(\alpha(t)) + y_i(t)\sum_{j=1}^{m} x_j'(\alpha(t)).\alpha'(t) = x_i'(\alpha(t)).\alpha'(t)$$

Applying a bit of algebra here gives us

$$y_i'(t) + y_i(t)\sum \phi_j(y(t)) = \phi_i(y(t))$$

This implies that $y(t)$ is the solution of BNN.

With an abuse of notation, we use $\phi_i(t) = \phi_i(y(t))$. Define

$$\psi(t) = \sum \left(\phi_i(t)\right)^2$$

Suppose $\phi_i(t) > 0$. Then,

$$\begin{aligned}\frac{d\phi_i(t)}{dt} &= e_i A \frac{dy(t)}{dt}\\ &= \sum a_{ij}\phi_j(t) - \phi(t)\sum a_{ij} y_j(t)\end{aligned}$$

We therefore have,

$$\begin{aligned}2\phi_i(t).\frac{d\phi_i(t)}{dt} &= \frac{d\left(\phi_i(t)\right)^2}{dt}\\ &= 2\sum_j a_{ij}\phi_i(t)\phi_j(t) - 2\phi(t)\sum_j a_{ij}\phi_i(t) y_j(t)\end{aligned}$$

If $\phi_i(t) = 0$, this equality is true. Summing over all i, we have

$$\sum \frac{d\left(\phi_i(t)\right)^2}{dt} = 2\sum_{i,j} a_{ij}\phi_i(t)\phi_j(t) - 2\phi(t)\sum_{i,j} a_{ij}\phi_i(t) y_j(t)$$

As $A = -A^T$, $a_{ij} = -a_{ji}$. Hence, the second term is

$$\sum \phi_i(t)\sum_{i,j} a_{ij} y_j(t) = \psi(t)$$

as defined earlier. Hence, we proved the following lemma:

Lemma. $\psi(t)$ satisfies

$$\frac{d\psi(t)}{dt} = -2\phi(t)\psi(t)$$

From ideas in differential equations, we can prove that

$$\sqrt{\psi(t)} \le \phi(t) \le \sqrt{m\psi(t)}$$

This comes from the above differential equation on $\psi(t)$ using integration by parts(left for the reader as an exercise).
Hence, $\psi(t)$ is decreasing. Moreover, $\phi(t) > 0$ as long as $\psi(t) > 0$. Now, we have

$$\frac{d\psi(t)}{dt} \leq -2\Big(\phi(t)\Big)^{3/2}$$
$$\Rightarrow \psi(t) \leq \frac{\psi(0)}{\left(1+\sqrt{|y(0)|t}\right)^2}$$

where the latter comes from the Gronwall's Lemma. Now, if $\psi(t) = 0$, the above is also true. Hence, we can prove that

$$\psi(t) = \psi(0).\exp\{-2\int_0^t \phi(s)ds\}$$

From here we can see that

$$\psi(t) \to 0 \text{ as } t \to \infty$$
$$\Rightarrow \phi_i(y(t)) \to 0 \text{ as } t \to \infty$$

This implies that $y(t)$ converges to a minmax strategy as $t \to \infty$. Thus, this proves the following theorem:
Theorem. BNN dynamics are asymptotically stable and any limit of the trajectory is a Saddle Point Equilibrium.