Game Theory

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Lecture 30
Potential Games

In this session, we introduce an important class of games called potential games. These potential games are very important because of their applicability in variety of scenarios. We start with the motivation.

Consider a finite game with strategy sets X_i. All players have the same payoff, given by $P : X \to \mathbb{R}$ such that,

$$u_i(x_1, x_2, .., x_n) = p(x_1, x_2, .., x_n)$$

Take $\bar{x} = (\bar{x}_1, \bar{x}_2, .., \bar{x}_n) \in \arg\max_{x \in X} p(x_1, x_2, ..., x_n)$. Then, $\bar{x}$ is a Nash Equilibrium in pure strategies. Note that, there may be other Nash Equilibria as well.

Let us now look at the *Best Response Dynamics*.

- Start from an arbitrary strategy profile, $(x_1, x_2, ..., x_n) \in X$.
- If a player i ha a strategy x_i' which gives better payoff(strictly increases his payoff) such that $u_i(x_i', x_{-i}) > u_i(x_i, x_{-i})$.
- If yes, then replace x_i with x_i'. Otherwise, we are at Nash Equilibrium.

The Best Response Dynamics for general games however need not converge. Now, let's look at *Payoff Equivalence*. Let us consider a general game with payoffs $u_i : X \to \mathbb{R}$. Suppose we add a constant to the payoff such that

$$\overline{u}_i(x_1, x_2, ..., x_n) = u_i(x_1, x_2, ..., x_n) + c_i$$

What if $c_i = c_i(x_{-i})$? We can simply see that in this case the best response is function is the same as in the original game with payoff function u_i. Whatever maximizes u_i would maximize $\overline{u}_i$ as well. Hence, the Nash Equilibria will also be the same.

Definition. The payoffs u_i and $\overline{u}_i$ are said to be *difference-equivalent* for player i if the difference $\overline{u}_i(x_1, x_2, ..., x_n) - u_i(x_1, x_2, ..., x_n) = c_i$ does not depend on her decision x_i but only on the strategies of the other players.

Theorem. Finite games with difference-equivalent payoffs have the same pure Nash equilibria.

Let us now introduce some notation that we will require to define potential games.

$$\Delta f(x_i', x_i, x_{-i}) = f(x_i', x_{-i}) - f(x_i, x_{-i})$$
$$\Delta \overline{u}(x_i', x_i, x_{-i}) = \Delta u_i(x_i', x_i, x_{-i})$$

Definition. A game is called a *potential game* if it is difference-equivalent to a game with common payoffs. That is, if there exists a function $p : X \to \mathbb{R}$ such that for all $i, x_{-i} \in X_{-i}$, $x_i, x_i' \in X_i$, we have

$$\Delta u_i(x_i', x_i, x_{-i}) = \Delta p(x_i', x_i, x_{-i})$$
$$u_i(x_i', x_{-i}) - u_i(x_i, x_{-i}) = p(x_i', x_{-i}) - p(x_i, x_{-i})$$

then we say that this game is a potential game and p is a potential function.

Moreover, from the arguments that we have seen already, we can say that if a game is a potential game then it admits at least one pure Nash Equilibrium and the Best Response Dynamics converges. Let us now look at an example:

Consider the bimatrix game given by,

P1

		L	R
P2	U	10, 10	0, 11
	D	11, 0	1, 1

An exercise for the reader would be to verify that

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

acts as a potential function for player 1. For player 2, the difference is when the first row is fixed, which is $11 - 10 = 1 - 0 = 1$. When the second row is fixed, we have $1 - 0 = 2 - 1 = 1$. Similarly for player 1, this property holds. Hence, it is a potential for this game.

There is another important class of games called *Routing Games*.

- Consider n drivers travelling between different origins and destinations in a city.
- The transport network is given by a network graph (N, A).
- Travel time of an arc $a \in A$ is non-negative increasing function $t_a = t_a(n_a)$ of the load, where n_a is the number of drivers using arc a. Due to congestion, travel time increases as more people use that arc.
- One pure strategy for i is a route $a_1, a_2, ..., a_l$ that is a sequence of arcs connecting his origin and the destination.
- The travel time is given by

$$u_i(r_1, r_2, .., r_n) = \sum_{a \in r_i} t_a(n_a)$$

  where $n_a = |\{j : a \in r_j\}|$.

Theorem. (*Rosenthal*) The Routing game has a potential given by

$$p(r_1, r_2, .., r_n) = \sum_{a \in A} \sum_{k=0}^{n_a} t_a(k)$$

where $n_a = |\{j : a \in r_j\}|$.

We introduce another class of games called *Congestion Games*.

- Each player $i = 1, 2, ..., n$ has to perform a certain task which requires some resources taken from a set R.
- The strategy set X_i for player i contains all subsets $x_i \subseteq R$.
- Each resource $r \in R$ has a cost $C_r(n_r)$, where n_r depends on the number of players using the resource.
- Players pay only for the resources that they use. For player i,

$$u_i(x_1, x_2, .., x_n) = \sum_{r \in x_i} C_r(n_r)$$

An exercise for the reader is to verify that

$$p(x_1, x_2, ..., x_n) = \sum_{r \in R} \sum_{k=1}^{n_r} C_r(k)$$

is the potential function for this game. An interesting fact is that for a potential game, Fictitious Play converges. This ends the module for non-cooperative games.