Game Theory

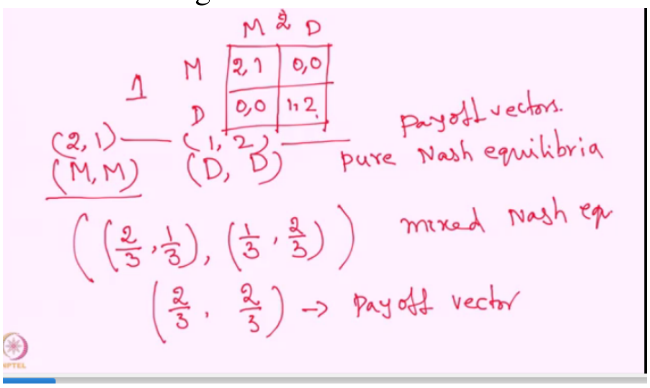
Prof. K. S. Mallikarjuna Rao
Department of Industrial Engineering & Operations Research
Indian Institute of Technology - Bombay

Lecture - 31
Cooperative Games: Correlated Equlibria

Correlated Equilibrium

Example 1: Battle of sexes

Figure 1: Battle of sexes



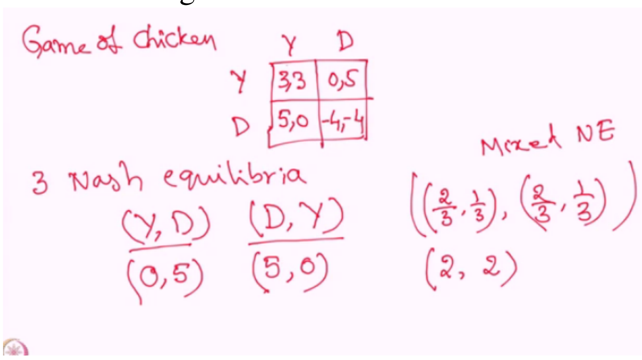
- There are 2 players. Say P1 and P2.
- Each player have two options: Movie (M) and Dance program (D).
- P1 wants to go to M, while P2 wants to go to D.
- If both of them go together then they have a positive utility and if they go to different places, there is no utility, so this is a coordination game.
- Payoff matrix for this game is shown in Fig .
- As we have seen earlier, there are two pure NE for this game, i.e., MM and DD with payoff vector $(2, 1)$ and $(1, 2)$ respectively.
- $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$ is mixed NE for this game with payoff vector $\left(\frac{2}{3}, \frac{2}{3}\right)$, which is fair but lower than the worst outcomes in the pure NE.
- Now, consider a situation where a "trusted" authority flips a fair coin and based on the outcome of the coin toss, tells the players what they should do. So, for example, if the coin shows heads, P1 is told to choose M and P2 is told to choose M. Similarly, both players are told to choose D when the outcome is tails. It is important to note that no individual party has an incentive to deviate from what they are told to do. In this case,

when P1 is told to choose M, he knows that P2 is told to choose M as well. So, P1 has no incentive to deviate and switch to D as the payoff would be lower (0 compared to 2). The advantage of following such a procedure is that the expected rewards are now higher $\left(\frac{3}{2}, \frac{3}{2}\right)$ compared to that of $\left(\frac{2}{3}, \frac{2}{3}\right)$ from the mixed NE.

- The idea is that each player chooses their action according to their observation of the value of the third party signal. A strategy assigns an action to every possible observation a player can make. If no player would want to deviate from the recommended strategy (assuming the others don't deviate), the distribution is called a correlated equilibrium.

Example 2: Game of chicken

Figure 2: Game of chicken



- A game in which two drivers (say P1 and P2) drive towards each other on a collision course: one must deviate(or yield), or both may die in the crash, but if one driver deviates and the other does not, the one who deviated will be called a "chicken".

- Payoff of this game are as follows:

  1. If both crashes (i.e., if both drives), (-4, -4).
  2. If P1 drives and P2 yields, (5,0).
  3. If P2 drives and P1 yields, (0,5).
  4. If both yields, (3,3)

- There are 3 NE of this game : 2 pure NE - (Y,D) and (D,Y) with payoffs (0,5) and (5,0) respectively ; one mixed NE - $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right)$ with payoff (2,2).

- We note once again that the trusted party only tells each player what he/she is supposed to do. The trusted party does not reveal what the other player is supposed to do. It is a correlated equilibrium if no player wants to deviate from the trusted party's instruction. So, in the Chicken example, if the trusted party tells player 2 to choose D, then P2 has no incentive to Y. This is because P2 knows that the outcome must have been (Y,D) and that P1 will obey the instruction to Y. Next, let us consider the case when P2 is told to choose Y. Then P2 knows that the outcome must have been either (D,Y) or (Y,Y), each happening with equal probability. P2's expected payoff on playing Y conditioned on the fact that P2 is told to choose Y is 1/2 3 + 1/2 0 = 1.5. In the above expression, 3 is the payoff from P1 also playing Y, i.e., the outcome was (Y,Y) and 0 is the payoff P2 gets when P1 plays D, i.e., outcome was (D,Y). If P2 decides to deviate, i.e., play D when told to play Y, then the expected payoff is 1/2 * 5 + 1/2 * (-4) = $0.5 < 1.5$. So, the expected payoff on deviating is lower than the payoff on obeying the instruction of the trusted party. Therefore P2 doesn't deviate. Since the game is symmetric, P1 also has no incentive to deviate from the instruction of the trusted party. Note that in the case of the correlated equilibrium, the expected reward for each player is = 1/2*5 + 1/2*0 = 2.5 (Since we are assuming that trusted party will tell to choose Y or D with equal probability). This is higher than the expected reward of 2 in the randomized Nash. Therefore, rewards can be made better by correlation.

- Now, if trusted party chooses between $\{(Y, D), (D, Y)\text{and}(Y, Y)\}$ with probability $\frac{1-p}{2}$, $\frac{1-p}{2}$ and p respectively.
- Conditional probability that a player will yield, if it is given that other player also yields is given by $p_y = \frac{p}{\frac{1-p}{2}+p}$ = $\frac{2p}{1+p}$.
- Conditional probability that a player will drive, if it is given that other player also yields is given by $p_D = \frac{\frac{1-p}{2}}{\frac{1-p}{2}+p}$ = $\frac{1-p}{1+p}$.
- Players utility is going to be $3p_Y$ for yielding and $5p_Y - 4p_D$ for driving.
- player will not deviate from the instruction as long as $3p_Y \geq 5p_Y - 4p_D$ i.e., $p \leq \frac{1}{2}$.
- Each player's utility under this correlated strategy is going to be $p*3+\frac{1-p}{2}*5+\frac{1-p}{2}*0 = \frac{p+5}{2}$, which is 2.75 for $p = 2$.
- Under this correlated strategy, payoff vector is (2.75,2.75). Total payoff is 5.5 which is greater than any NE.

Correlated Strategy

Consider G = (S_1, S_2, u_1, u_2), where S_1, S_2 are finite strategy spaces.
A correlated strategy is a probability distribution $\mu \in P(S_1 \times S_2)$ and it is said to be correlated equilibrium if $\forall i,\ \forall s_i, t_i \in S_i$,

$$\sum_{s_{-i} \in S_i} \mu(s_{-i}; s_i) u_i(s_{-i}; s_i) \geq \sum_{s_{-i} \in S_i} \mu(s_{-i}; s_i) u_i(s_{-i}; t_i)$$

Few interesting properties

Proposition 1. *If μ and ν are two correlated equilibrium then $\epsilon\mu + (1-\epsilon)\nu$ is also a correlated equilibrium $\forall \epsilon \in (0, 1)$. i.e., Set of correlated equilibria is convex*

Proposition 2. *If μ_n's are correlated equilibria and $\mu_n \longrightarrow \mu$,then μ is also a correlated equilibria.*

Proposition 3. *Every Nash equilibrium is a correlated equilibrium.*

- Even though correlated equilibrium concept is defined for the non-cooperative games, this correlation is a cooperative aspect of the game and under certain assumptions we can show that if the Nash equilibrium is unique, the correlated equilibrium will also be unique.