Game Theory
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Lecture - 32
Cooperative Games: The Nash Bargaining Problem - I

Introduction

Consider an example of Prisoner's dilemma: In this game, two criminals (say C1 and C2) are arrested and each is held in solitary confinement. The prosecutors does not have the evidence for their crime, so they offer each prisoner the opportunity to either betray the other by testifying that the other committed the crime or cooperate by remaining silent.

If both prisoners betray each other, each serves 5 years in prison. If C1 betrays C2 but C2 remains silent, prisoner C1 is set free and prisoner C2 serves 10 years in prison or vice versa. If each remains silent, then each serves just one year in prison. The Nash equilibrium in this example is for both players to betray each other. Even though mutual cooperation leads to a better outcome but if one prisoner chooses mutual cooperation and the other does not then that prisoner's outcome is worse. So, in some sense can we really come up with certain contracts which will ensure that they will get a better outcome than the Nash equilibrium outcome (i.e., betraying each other).

Nash Bargaining problem:

This is a situation where

- Individuals have the possibility of concluding mutually beneficial agreements. So that they know by cooperating they can get better utilities.
- There is conflict of interest on which agreement to conclude.
- No agreement until every player approves.

If there is no agreement, player will get disagreement payoff.

There are several places where such situations arise. For example, the management labour attrition where the management negotiates with the labour union.

So, basically labour union have certain utility over what agreement they should go with the management and this is one very nice example where this subject has been applied and another example in this direction is, duopoly market, when the two firms are competing in a market, they can actually go for this bargaining, so that both of them can benefit by staying in the market.

Definition

A two-person bargaining problem consists of:

1. A feasible set of allocations $\mathbb{F}$, a closed and convex subset of $\mathbb{R}^2\mathbb{R}^2$, the elements of which are interpreted as agreements.

2. A disagreement point $v = (v_1, v_2) \in \mathbb{R}^2$, where v_1 and v_2 are the payoffs to player 1 and player 2 respectively. We have assumed that $v \in F$.

The problem is nontrivial if $\mathbb{F} \cap \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq v_1, x_2 \geq v_2\}$ is non-empty. F is assumed to be convex because for any two feasible allocations, a convex combination of them is typically also feasible.

Now, consider a bi-matrix game $G = \{S_1, S_2, u_1, u_2\}$, where S_1 and S_2 are strategy spaces; u_1 and u_2 are respective payoffs of player 1 and player 2. Let $\mathbb{F}$ to be the set of allocations under correlated strategies.

$$\mathbb{F} = \{(u_1(\mu), u_2(\mu)) \ : \ \mu \in \mathbb{P}(S_1 \times S_2)$$

It is straightforward to prove that F is convex and compact.
Next, we have to address the selection of disagreement vector. There are several possibilities for disagreement vector $v = (v_1, v_2)$

1. We can choose v_1 and v_2 as follows:

$$v_1 = \min_{\sigma_2 \in \Delta(S_2)} \max_{\sigma_1 \in \Delta(S_1)} u_1(\sigma_1, \sigma_2)$$

$$v_2 = \min_{\sigma_1 \in \Delta(S_1)} \max_{\sigma_2 \in \Delta(S_2)} u_2(\sigma_1, \sigma_2)$$

2. We can choose v as Nash payoff vector.

Nash actually solves bargaining problem using certain axioms.

Axioms of Nash:

1. Strong efficiency

2. Individual rationality

3. Scale covariance

4. Independence and irrelevant alternatives

5. Symmetry

Strong efiiciency

Given F, an allocation $x = (x_1, x_2) \in F$ is said to be a strongly pareto efficient, if there exists no $y = (y_1, y_2) \in F$, such that, $y_1 \geq x_1$ and $y_2 \geq x_2$, with strict inequality satisfied for at least one inequality.

an allocation $x = (x_1, x_2) \in F$ is said to be a weakly pareto efficient, if there exists no $y = (y_1, y_2) \in F$, such that, $y_1 > x_1$ and $y_2 > x_2$.

The axiom asserts that any solution of bargaining problem should be feasible and strongly efficient. If it is not strongly efficient then that means both the players have other option where both of them is getting higher payoffs. So, therefore they would like to go for that solution instead of this one. Thus, assuming that the solution should be strongly efficient is a quite a natural axiom.

Individual rationality

Let $f(F, v) = (f_1(F, v), f_2(F, v))$ be Nash bargaining solution. This axiom says any player should get atleast disagreement payoff, i.e.,

$$f(F, v) \geq v; i.e., f_1(F, v) \geq v_1 \text{ and } f_1(F, v) \geq v_2$$

This axiom is again natural to consider because if any player is getting less than the disagreement payoff for a solution, then they would not like to consider this solution.

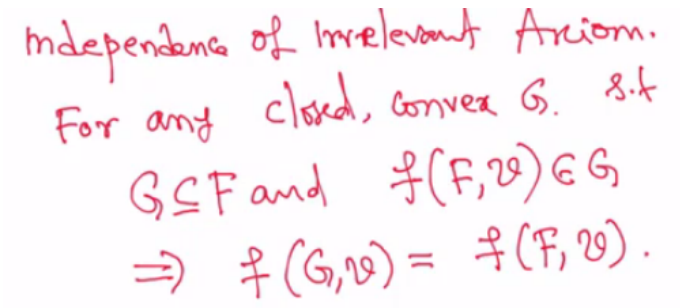
Scale covariance

This axiom says that if all the vectors are scaled and translated by something, then the solution should also get have the same effect.

Consider $\lambda_1, \lambda_2, \mu_1$ and μ_2, where $\lambda_1, \lambda_2 > 0$. Define
$G = \{(\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2) : (x_1, x_2) \in F\}$
$w = (\lambda_1 v_1 + \mu_1, \lambda_2 v_2 + \mu_2)$

According to this axiom, solution of this new bargaining problem (G,w) is given by
$f(G, w) = (\lambda_1 f_1(F, v_1) + \mu_1, \lambda_2 f_2(F, v_2) + \mu_2)$

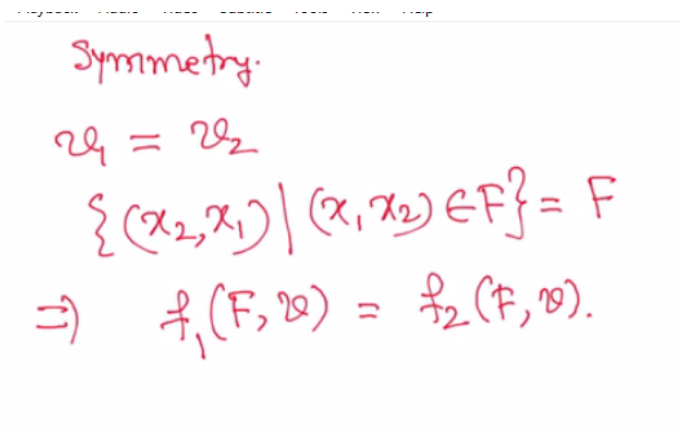
Independence and irrelevant alternatives



This axiom is implying that if certain allocations from F are removed which are not solutions, in such a way that the G still happens to be convex, then the solution will also continue to be same.

Symmetry

For (F, v), when v_1, v_2 the disagreement payoffs are same and the set F is symmetric, then the solution should also have symmetry.



Under these five axioms Nash proposed a following theorem.

Theorem 1. *Given two person bargaining problem (F,v), ∃ a unique solution function f that satisfies above five axioms. The solution satisfies*

$$f(F, v) \in arg\,max_{(x_1,x_2)\in F; x_1\geq v_1, x_2\geq v_2}(x_1 - v_1)(x_2 - v_2)$$