Game Theory

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Lecture - 33
Cooperative Games: The Nash Bargaining Problem - II

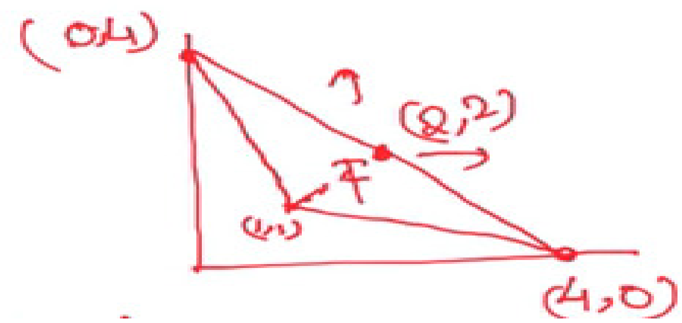
In this lecture, we will prove Nash bargaining solution theorem. Before that we will look at small example of Nash bargaining problem.

Example

Let us assume F to be convex hull of $\{(4,0),(0,4),(1,1)\}$ and let disagreement vector is (1,1). Consider a maximization problem,

$$\begin{aligned} &\max\ (x_1-1)(x_2-1) \\ &\text{s.t. } x_1 \geq 1, x_2 \geq 2, \\ &(x_1,x_2) \in F \end{aligned}$$

It is easy to see that $(2,2)$ is the solution of above maximization problem. This implies $f(F,(1,1)) = (2,2)$. One can see this solution satisfies all five axioms.



Nash Bargaining Solution

Theorem 1. *Given two person bargaining problem (F,v), $\exists$ a unique solution function f that satisfies above five axioms (strong efficiency, individual rationality, scale covariance, independence and irrelevant alternatives, symmetry). The solution satisfies*

$$f(F,v) \in arg\,max_{(x_1,x_2)\in F; x_1\geq v_1, x_2\geq v_2}(x_1-v_1)(x_2-v_2)$$

Proof. First, we will prove this result for class of problems called essential bargaining problem and then we generalize for all classes.
(F,v) is called essential bargaining problem if there exists at least one allocation $y \in F$ that is strictly better for both the players than the disagreement allocation, i.e., $y_1 > v_1$ and $y_2 > v_2$.

Let (F, v) is essential, therefore $\exists$ some $y \in F$ such that $y_1 > v_1$ and $y_2 > v_2$.. Consider optimization problem,

$$\begin{aligned} &\max\ (x_1 - v_1)(x_2 - v_2) \\ &\text{s.t. } x_1 \geq 1, x_2 \geq 2, \\ &(x_1, x_2) \in F \end{aligned} \tag{1}$$

where the term $(x_1 - v_1)(x_2 - v_2)$ is called Nash product (say $N(x_1, x_2)$). Now we look at function F from (x_1, x_2) to $(x_1 - v_1)(x_2 - v_2)$. This function F is strictly quasi concave [1].

It is easy to show that a strict quasi concave function will always have unique optimal solution (here it is maxima). Thus from this we can conclude that the Nash product has unique maximizer, say (x_1*, x_2^*). Let $F \subseteq \mathcal{R}$ be convex and closed, then $F \cap \{(x_1, x_2) : x_1 \geq v_1, x_2 \geq v_2\}$ is non-empty and bounded.
Now we need to prove following two parts

Part 1: Define $f(F, v) = (x_1^*, x_2^*)$, f satisfies all five axioms.
Part 2: Suppose $f(F, v)$ satisfies five axioms, then $f(F, v) = (x_1^*, x_2^*)$

Proof of Part 1: We need to prove (x_1^*, x_2^*), satisfies all five axioms.

1. Strong Efficiency: We can easily see that, if $(x_1, x_2) \leq (y_1, y_2)$, then $N(x_1, x_2) \leq N(y_1, y_2)$. Therefore, if (x_1^*, x_2^*) is maximizing the Nash product then there cannot be any vector which is higher than (x_1^*, x_2^*). Thus, (x_1^*, x_2^*) is strongly efficient.

2. Individual Rationality (IR): Since (x_1^*, x_2^*) is the optimal solution of (1), thus we have $x_1^* \geq v_1$ and $x_2^* \geq v_2$. Therefore, IR holds.

3. Scale covariance: Let us take $\lambda_1, \lambda_2 > 0, \mu_1, \mu_2$ and

$$G = \{(\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2) | (x_1, x_2) \in F\}$$

corresponding optimization problem is,

$$\begin{aligned} &\max_{(y_1, y_2) \in G} (y_1 - (\lambda_1 v_1 + \mu_1))(y_2 - (\lambda_2 v_2 + \mu_2)) \\ or &\max_{(x_1, x_2) \in F} \lambda_1 (x_1 - v_1) \lambda_2 (x_2 - v_2) \end{aligned}$$

This implies, $f(G, (\lambda_1 v_1 + \mu_1, \lambda_2 v_2 + \mu_2)) = (\lambda_1 f_1(F, v_1) + \mu_1, \lambda_2 f_2(F, v_2) + \mu_2)$

4. Independence and irrelevant alternatives: Let $G \subseteq F$, is closed and convex. Let (x_1^*, x_2^*) is optimal to (F,v). Also, (y_1^*, y_2^*) is optimal to (G,v). Therefore, $N(x_1^*, x_2^*) \geq N(y_1^*, y_2^*)$.

[1] A function $f : S \longrightarrow \mathcal{R}$ is said to be strict quasi concave if

$$f(\lambda x + (1 - \lambda)y) > \min\{f(x), f(y)\} \forall x, y \in S, \lambda \in (0, 1)$$

Since, (y_1^*, y_2^*) is optimal to (G,v), therefore Nash product corresponding to G is maximum. Thus, $N(y_1^*, y_2^*)$ must be maximum over all Nash products inside G.But (x_1^*, x_2^*) is in G by the independence and irrelevant alternatives axiom, therefore $N(x_1^*, x_2^*) \leq N(y_1^*, y_2^*)$. Thus $N(x_1^*, x_2^*) = N(y_1^*, y_2^*)$. This implies, (x_1^*, x_2^*) and (x_1^*, x_2^*) are also equal. Because F is essential and Nash product is strictly quasi concave and therefore it has a unique optimal solution. Thus, the axiom holds.

5. Symmetry: We have, $(x_1^*, x_2^*) \in F \implies$ $(\mathrm{x}_2^*, x_1^*) \in F$, and $v_1 = v_2$. Thus, (x_1^*, x_2^*) maximizes $(x_1 - v_1)(x_2 - v_1)$ and (x_2^*, x_1^*) also maximizes $(x_1 - v_1)(x_2 - v_1)$. Since Nash product is strictly quasi concave, it attains a unique optimal solution. This implies $x_1^* = x_2^*$. Thus symmetry holds.

Proof of Part 2: We need to show that, if $f(F, v)$ satisfies five axioms, then $f(F, v) = (x_1^*, x_2^*)$.

Note that $x_1^* > v_1$ and $x_2^* > v_2$, since F is essential. Consider

$$L(x_1, x_2) = (\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2)$$

Where,

$$\lambda_1 = \frac{1}{x_1^* - v_1}, \lambda_2 = \frac{1}{x_2^* - v_2}, \mu_1 = \frac{-v_1}{x_1^* - v_1}, \mu_2 = \frac{-v_2}{x_2^* - v_2}$$

Thus,

$$L(x_1, x_2) = \left(\frac{x_1 - v_1}{x_1^* - v_1}, \frac{x_2 - v_2}{x_2^* - v_2} \right)$$

. Notice $L(v_1, v_2) = (0, 0)$ and $L(x_1^*, x_2^*) = (1, 1)$. Define G = $\{L(x_1, x_2) | (x_1, x_2) \in F\}$. Therefore, the problem (F,v) is now transformed to (G,(0,0)). it is easy to verify that (1,1) is strong pareto efficient of G. therefore, the solution of (G,(0,0)) is $f(G, (0, 0)) = (1, 1)$. The Nash product in $(G, (0, 0)$ is $x_1 x_2$. Also, $x_1 + x_2 \leq 2$ (we can prove this by contradiction). Now , we know G is bounded. So, we can always find rectangle H which is symmetric about line $x_1 = x_2$ and HG and it is convex and bounded. further, choose H such that (1,1)$\in G$ is on the boundary of H. now, strong efficiency implies $f(H, (0, 0)) = (1, 1)$. Using independennce and irrelevant alternatives and scale covariance $f(H, (0, 0)) = f(G, (0, 0)) = L(f, f(v))$. Now this implies $L(f, f(v)) = (1, 1)$. Therefore, $f(F, v) = (x_1^*, x_2^*)$. This completes the proof for essential part.

Now we will consider non-essential part. Consider (F,v) which is inessential. F is convex implies that $\exists$ atleast one player i such that $y_1 \geq v_1$ and $y_2 \geq v_2$ and this implies $y_i = v_i \; \forall (y_1, y_2) \in F$. Without loss of generality, we can consider $y_1 \geq v_1$, $y_2 \geq v_2 \implies y_1 = v_1 \; \forall \; (y_1, y_2) \in F$. Suppose x^* is an allocation in F, i.e., best for player 2 subject to constraint $x_1 = v_1$. Note that under inessential, Nash product is always zero. This implies x^* is strongly pareto efficient. And all the other can also be easily satisfied. Thus we can say that $f(F, v) = (x_1^*, x_2^*)$. This completes the proof of the theorem. □