Game Theory
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Lecture - 34
Cooperative Games: The Nash Bargaining Problem - III

In the previous lecture, we have introduced Nash bargaining problem and we derived the Nash bargaining rule satisfying certain axioms. Now, we will look at one example to illustrate certain aspects.

Example

Version 1: There are two players and they have 30 units of money. The question is how to share this 30 units among them. Consider disagreement vector $v = (0,0)$ and take $F = \{(y_1, y_2) : 0 \leq y_1 \leq 30, 0 \leq y_2 \leq 30 - y_1\}$. Let assume players value the money linearly, i.e., players are risk neutral. Using all axioms, the solution $f(F,v)$ came out to be (15,15).

Version 2: Now assume player 1 is risk neutral and player 2 is risk averse (i.e., utility is proportional to square root of money). Therefore, $F = \{(y_1, y_2) : 0 \leq y_1 \leq 30, 0 \leq y_2 \leq \sqrt{30 - y_1}\}$ and $v = (0,0)$. The solution $f(F,v)$ is $(20, \sqrt{10})$.

Other Solution Concepts

1. Principle of equal gains: It says that *you should do this for me, since I am doing this for you.* This principle gives **egalitarian solution**.
2. Principle of greatest good: *you should do this for me, because it helps me more than it hurts you.* This principle gives **utilitarian solution**.

Let us take bargaining problem (f, v). The egalitarian solution f(F,v) is a unique point $(x_1, x_2) \in F$ that is weakly efficient and satisfies $x_1 - v_1 = x_2 - v_2$.

The utilitarian solution f(F,v) is a unique point $(x_1, x_2) \in F$ that satisfies

$$x_1 + x_2 = \max_{(y_1,y_2)\in F} y_1 + y_2$$

Both solution violates the axiom of scale covariance.

λ-Egalitarian solution

Given numbers $\lambda_1, \lambda_2 > 0, \mu_1, \mu_2$ and

$$L(y) = \{(\lambda_1 y_1 + \mu_1, \lambda_2 y_2 + \mu_2) | y \in \mathcal{R}^2\}$$

Given (F, v),

$$L(F) = \{L(y) | y \in F\}$$

The egalitarian solution of (L(F),L(v)) is L(x), where x is unique weakly efficient point in F such that the following satisfies, $\lambda(x_1 - v_1) = \lambda(x_2 - v_2)$. This is called λ-egalitarian solution.

λ-Utilitarian solution

The egalitarian solution of (L(F),L(v)) is L(z), where z is unique weakly efficient point in F such that the following satisfies,

$$\lambda_1 z_1 + \lambda_2 z_2 = \max_{(y_1,y_2)\in F} \lambda_1 y_1 + \lambda_2 y_2$$

This is called λ-utilitarian solution.

Relationship with Nash Bargaining

Theorem 1. *Let (F,v) be essential two person bargaining problem. Suppose $x^* \in F, x^* \geq v$. Then x^* is Nash bargaining solution for (F,v) iff $\exists$ strictly positive numbers λ_1, λ_2 s.t.*

$$\lambda_1 x_1^* - \lambda_1 v_1 \lambda_2 x_2^* - \lambda_2 v_2$$
$$\lambda_1 x_1^* - \lambda_2 x_2^* = \max_{y\in F}(\lambda_1 y_1 + \lambda_2 y_2)$$

This theorem interprets that the Nash Bargaining solution is same as λ-egalitarian solution as well as λ-utilitarian solution.

Multi-person bargaining game

Let set of players $N = \{1, 2, \ldots, n\}$ $n > 2$. Also, let $F \subseteq \mathcal{R}^n, v \in \mathcal{R}^n$. Thus, (F,v) is multi-person bargaining problem. Nash product for this game is given by $\Pi_{i\in N}(x_i - v_i)$. Now, we will see some example of multi-person bargaining game and their solution.

Example

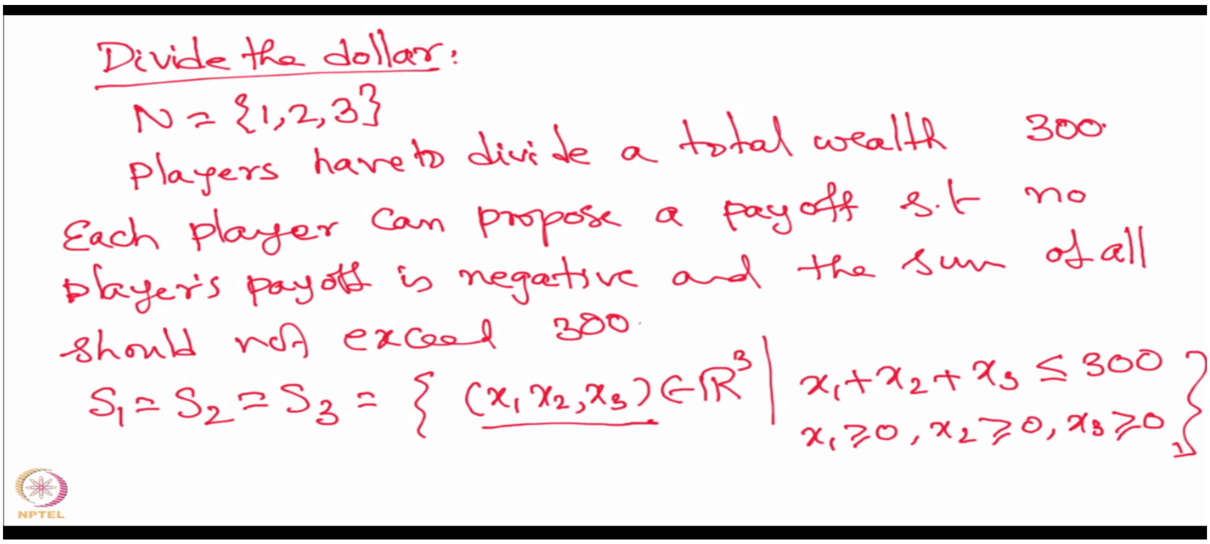

Figure 1: Divide the dollar game

There are multiple version of this game. We will see solution for some versions.

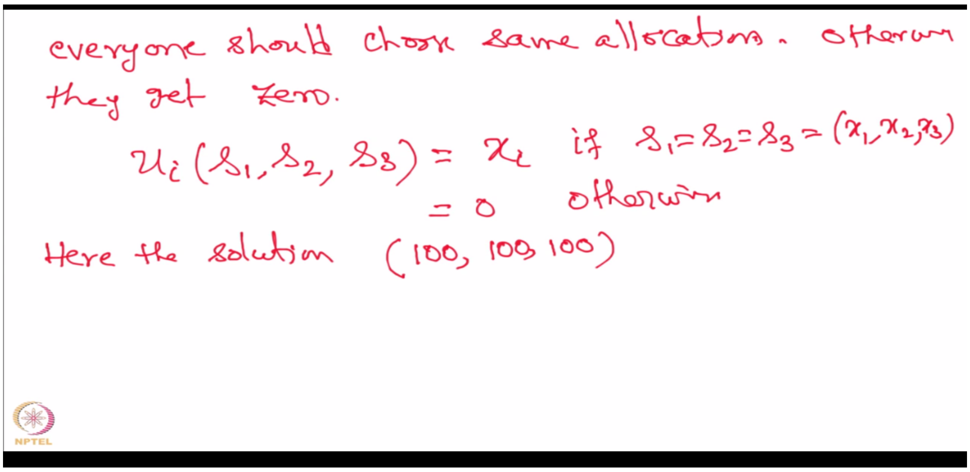

Figure 2: Version 1 of divide the dollar game

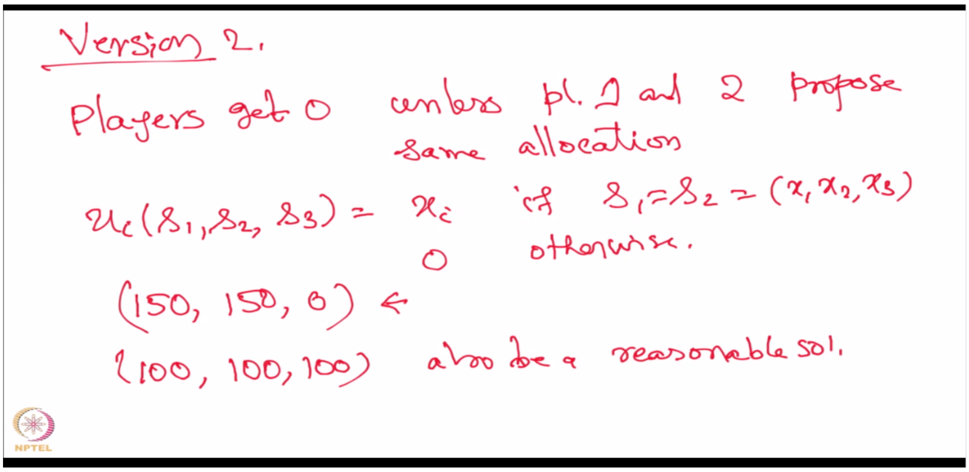

Figure 3: Version 1 of divide the dollar game

For version 1, the solution is (100,100,100). For version 2, there are two reasonable solution, (150,150,0) and (100,100,100). For solution (150,150,0) is a reasonable solution because player 1 and 2 are the ones whose decision is affecting their payoff. So, they can ignore the third player and think it as a two-person bargaining problem and hence the solution is (150,150,0). Justification of second solution is homework.

Games in characterstic form

Transferable utility(TU) games

Let N is the set of all players and $v : 2^N \longrightarrow \mathcal{R}$. If $S \subseteq N$ is a coalition then $v(S)$ denotes worth of that coalition and $v(\phi) = 0$. Game (N, v) with above properties is known as TU games.

Non-transferable utility(NTU) games

Game on the set of players N, where

1. If $C \subseteq N$ is a coalition then $v(C) \subseteq \mathcal{R}^{|C|}$ and it is non-empty,closed and convex subset.
2. $\{x|x \in V(C), x_i \geq v_i \, \forall i \in C\}$ is bounded subset of $\mathcal{R}^{|C|}$, where

$$v_i = \max\{y_i | y \in V(\{i\})\} < \infty \, \forall \, i \in N$$