Game Theory
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Lecture - 35
Cooperative Games: Transferable Utility Games

Transferable Utility Games

Utility is transferable if one player can losslessly transfer part of its utility to another player. Cooperative game with transferable utility known as transferable utility (TU) game.

Examples

Divide the dollar game

If there are 3 players and all of them together worth 300 (i.e., $v(\{1, 2, 3\})$), they have to share this 300 rupees. This sharing can be done in many ways. We will see following two version.

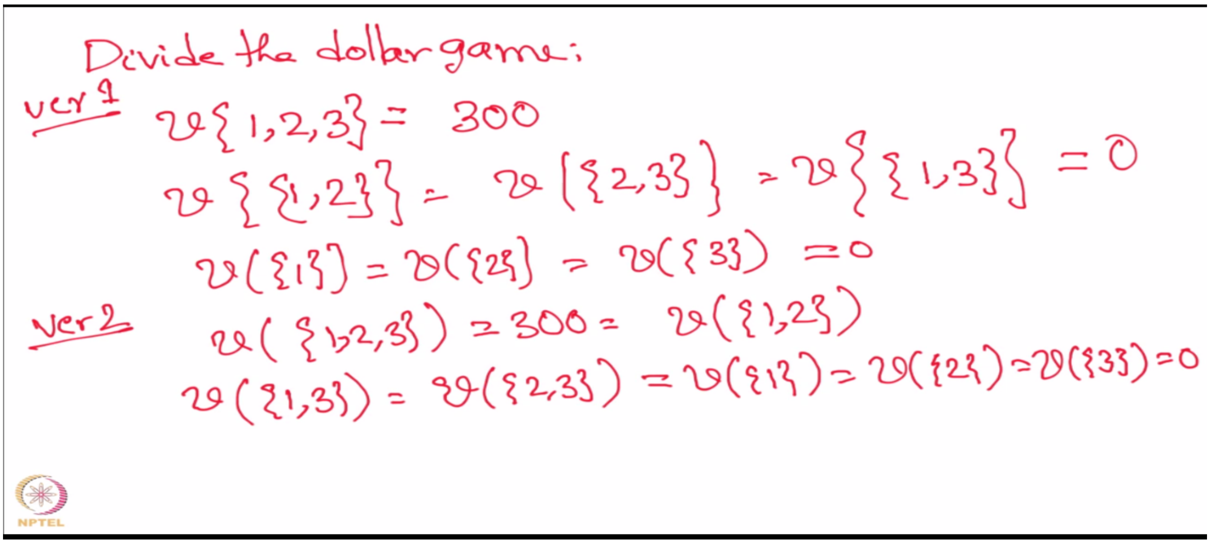

Figure 1: Example 1

These versions give us the characteristic form representation for the game. For each form, what is the corresponding solution, we will try to build that solution concepts in few upcoming lectures.

Another example is voting game. So, there is some 4 parties 1 2 3 4. Each party some number of members shown in figure 2. So, let us say to pass any bill, 51 votes are required. So, this situation can be modeled as a TU game with 4 players, which we will further see in this lecture.

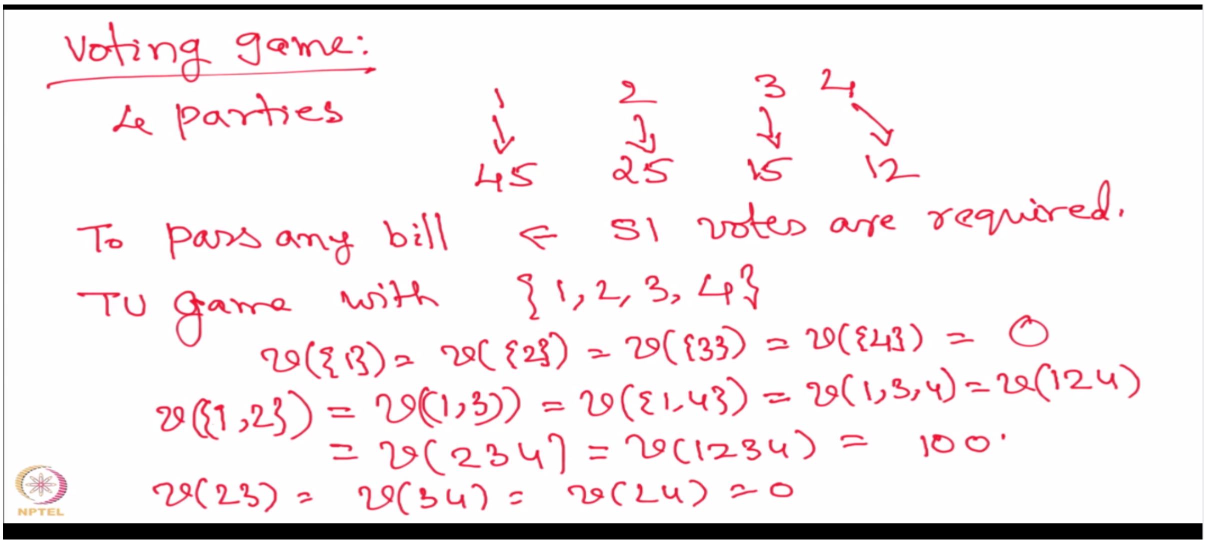

Figure 2: Example 2

Representation of TU Games

Minimax Representation

Let $G = (N, \{S_i\}, \{u_i\})$ is strategic form game with transferable utility. Where,

- N= set of players.
- S_i = strategy set of player i.
- u_i= transferable utility of player i.

Let $C \subseteq N$ is the coalition and $N \backslash C$ are the number of players which are not in C. Let

$$S_{N \backslash C} = \times_{j \in N \backslash C} S_j$$

$$S_C = \times_{j \in C} S_j$$

Let Δ_{S_C} and $\Delta_{N \backslash C}$ denotes set of correlated strategies of players in C and $N \backslash C$ respectively. Utility of player i in coalition C is given by

$$u_i(\sigma_C, \sigma_{N \backslash C}) = \sum_{s_C \in S_C} \sum_{s_{N \backslash C} \in S_{N \backslash C}} \sigma_C(s_C) \sigma_{N \backslash C} S_C(N \backslash C) u_i(\sigma_C, \sigma_{N \backslash C})$$

Worth of coalition C is given by

$$v(C) = \min_{\sigma_{N \backslash C}} \max_{\sigma_C} \sum_{i \in C} u_i(\sigma_C, \sigma_{N \backslash C})$$

This is the representation of TU games. One thing to note is that this is not the only way to represent TU games. There are other ways of representing but we will not discuss that here.

Few properties of this TU games

1. Super additive
2. Imputation
3. Strategic equivalence

Super additive

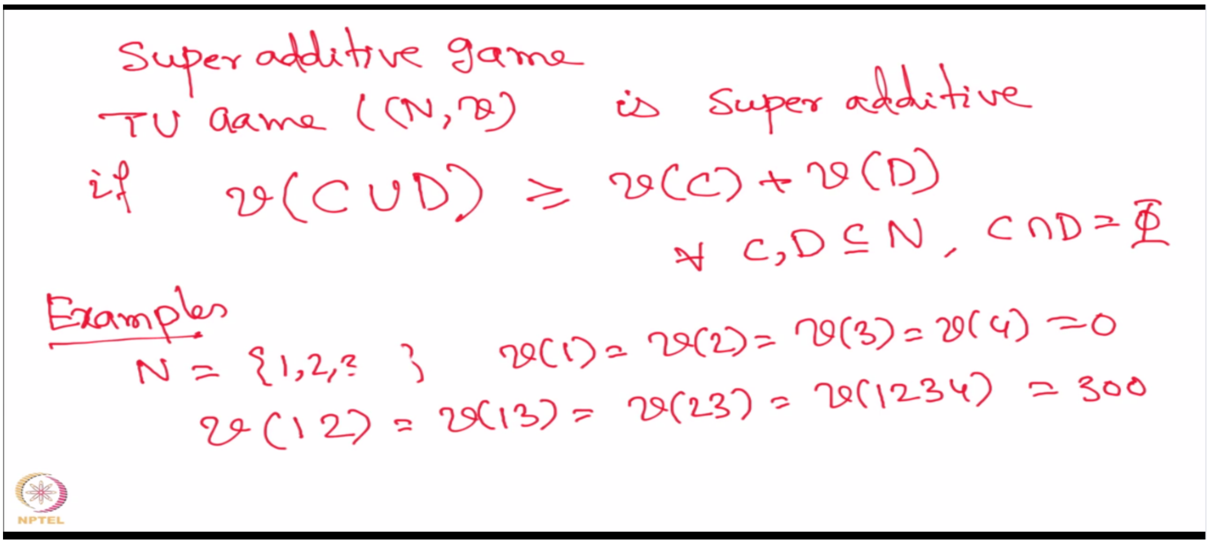

Figure 3: TU Game is Super additive

Where C and D are any two coalition and in example mentioned above, we have considered four players; $N = \{1, 2, 3, 4\}$. Worth of each coalition is mentioned above. One can easily see that this TU game is super additive.

Super additive Cover

Super additive cover of Game (N, v) is the super-additive game (N, w) such that

$$v(C) \leq w(C) \; \forall \; C \subseteq N$$

Characterization of super additive cover

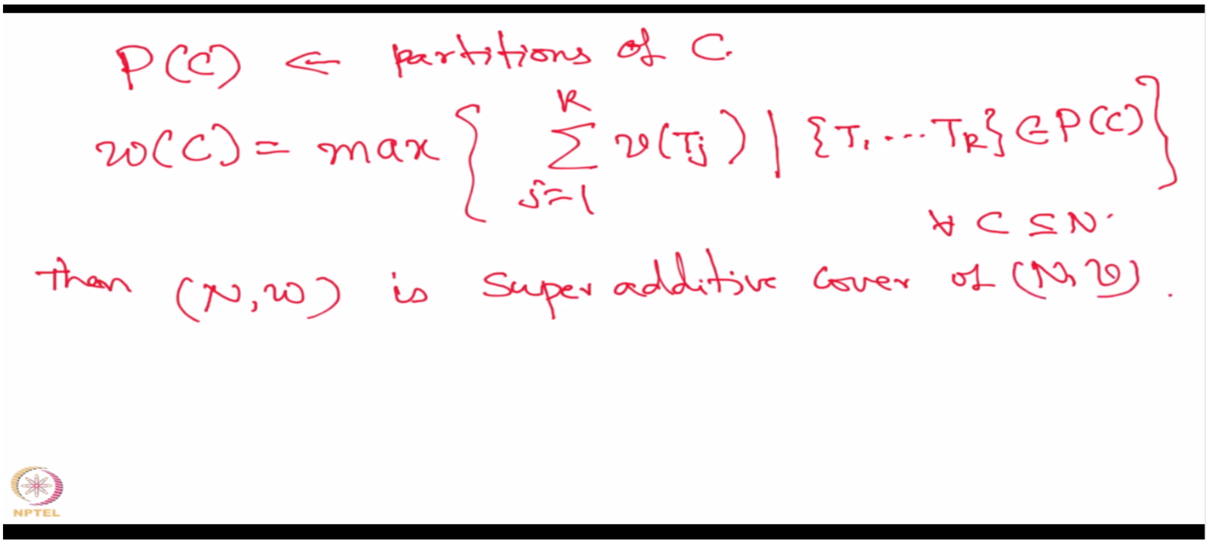

Figure 4: Characterization of super additive cover

Imputation

In TU game (N, v), imputation is an allocation $(x_1, x_2, \ldots, x_n) \in \mathcal{R}^n$ which satisfies following two properties:

1. **Individual rationality:** Any player i will join any coalition only if he gets atleast $v(i)$.

$$x_i \geq v(i)$$

2. **Collective rationality:** Each player is said to be collective rational if

$$\sum_{i \in N} x_i = v(N)$$

Essential and Inessential game:

Essential & Inessential games:

$(N, v) \leftarrow$ super additive

is said to be essential if

$$\sum_{i \in C} v(i) < v(N)$$

In essential. if

$$\sum_{i \in N} v(i) = v(N)$$

If (N, v) is inessential then

$$\sum_{i \in C} v(i) = v(C) \quad \forall \; C \subseteq N.$$

$\therefore$ Inessential game

only imputation is $(v(1), v(2), \ldots v(n))$

For essential games

there are infinitely many imputations.

Strategic Equivalence of TU games:

Strategic equivalence of TU games:

(N, v) (N, w) are strategically equivalent

if $\exists\ c_1, c_2, \ldots c_n \in \mathbb{R}$, $b > 0$ s.t

$$w(C) = b\left(v(C) + \sum_{i \in C} c_i\right)$$

Important result:

Any superadditive, essential n-person ch. form game G is strategically equivalent to a unique game with

$$N = \{1, 2, \ldots n\}$$

$$v(1) = v(2) = \cdots = v(n) = 0$$

$$v(N) = 1$$

$$0 \leq v(C) \leq 1, \quad C \subseteq N$$

This game is called the 0-1 normalization of the original game.

It is very easy to prove above result. So, this introduces some important concepts regarding these TU games and next we need to discuss the solution concepts, which we will do it in the next lecture.