Game Theory
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Lecture - 37
Cooperative Games: Characterization of Games with Non-Empty Core

In the previous lecture, we introduced core and discussed some examples. Now, we will see when the core is non-empty.

Characterization of Games with Non-Empty Core

Shapley and Bondereva independently gave this characterization. Given TU game (N, v), where N is the set of players and v is the worth of coalitions. let us consider following LP

$$\begin{aligned} &\min x_1 + x_2 + \cdots + x_n \\ &\text{s.t.} \sum_{i \in C} x_i \geq v(C)\ \forall C \subseteq N \\ &(x_1, x_2, \ldots, x_n) \in \mathcal{R}^n \end{aligned}$$

From above LP, we know $x_1^* + x_2^* + \cdots + x_n^* \geq v(N)$. Now, if we are able to show $\exists\, (x_1^*, x_2^*, \ldots, x_n^*)$ optimal solution of above LP, such that $x_1^* + x_2^* + \cdots + x_n^* = v(N)$ then this implies core of (N, v) is non-empty. Otherwise, i.e., when $x_1^* + x_2^* + \cdots + x_n^* > v(N)$, we can say that core is empty.

Since above optimization problem is linear and objective value is lower bounded and we are looking for minimum. Thus, we can say this minimization problem will have a solution. This gives an existence of optimal solution $(x_1^*, x_2^*, \ldots, x_n^*)$ of above LP. Now only thing remained to prove is $x_1^* + x_2^* + \cdots + x_n^* = v(N)$.

Primal Dual pair of this LP can be written as follows:

Primal: $\min \sum_{i \in N} x_i$

s.t $\sum_{i \in C} x_i \geq v(C)$, $C \subseteq N$

$x \in \mathbb{R}^n$

Dual $\max \sum_{C \subseteq N} \alpha(C) v(C)$

s.t $\sum_{C \ni i} \alpha(C) = 1 \quad \forall i \in N$

$\alpha(C) \geq 0 \quad \forall C \subseteq N$

From strong duality theorem, we know that ”*if primal has an optimal solution, then dual also have an optimal solution and moreover optimal value for both are same.*” Thus, $x^* \in \mathcal{R}^n$ and $\alpha^*(C)$ such that,

$$\sum_{i \in C} x_i^* \geq v(C),\ \alpha^*(C) \geq 0\ \forall C \subseteq N$$
$$\sum_{C\{i\}} \alpha^*(C) = 1,\ \forall i \in N$$

Now due to feasibility of the dual problem, we have

$$\sum_{C\{i\}} \alpha^*(C) = 1,\ \forall i \in N \implies \sum_{C \subseteq N} \alpha^*(C) v(C) \leq v(N)$$

This implies, optimal value is going to be $v(N)$. This condition is basically characterizes the non-emptiness of the core. This condition is known as balancedness condition.

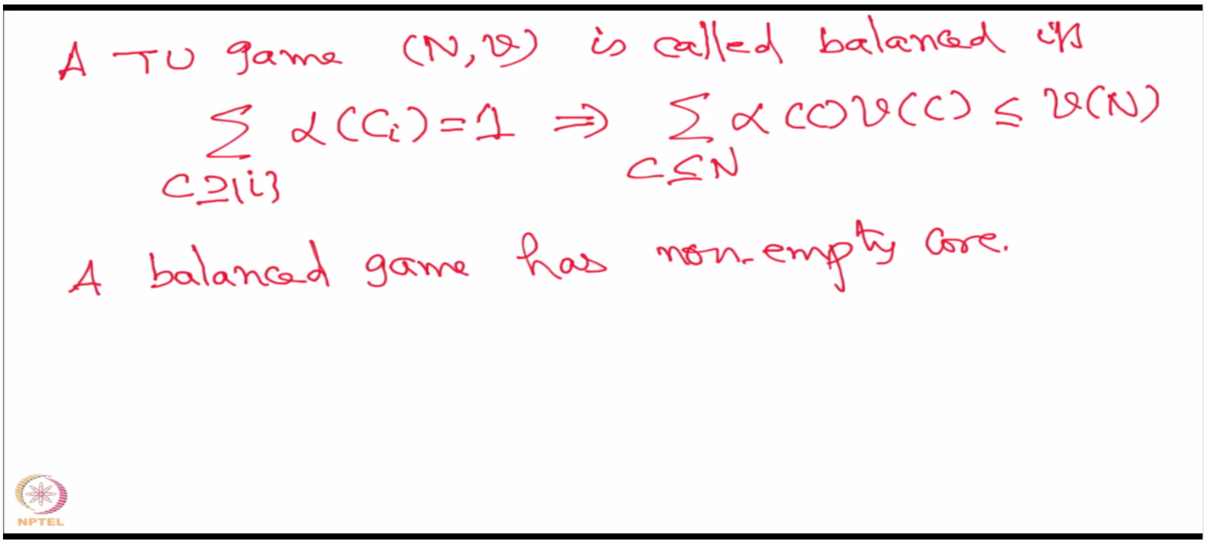


Figure 1: Balanced Game

Now, we will study another important solution concept for cooperative games, which is called Shapley value. Which we can define irrespective of the fact that game is balanced or not.

Shapley Value

For a cooperative game (N, v), Shapley value is a solution concept $\phi(v)$ which is basically an allocation for each player, satisfies certain axioms.

Shapley's Axioms

1. Symmetry

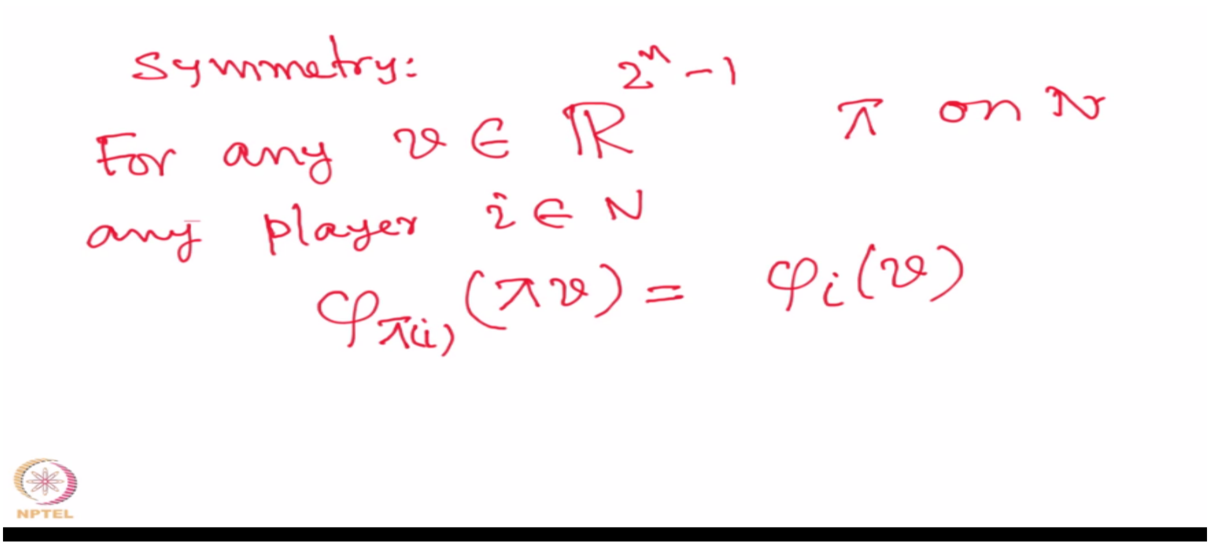
2. Linearity

3. Carrier

Permuted players: For a given game (N, v), take a permutation on N. Then, consider $(N, \pi v)$ is a game such that, $\pi v(\{\pi i \ i \in C\}) = v(C)$. Basically, what we are doing is that we are permuting the players. Instead of giving them index from 1 to n, we are giving them index $\pi i, \pi 2, \ldots, \pi n$.

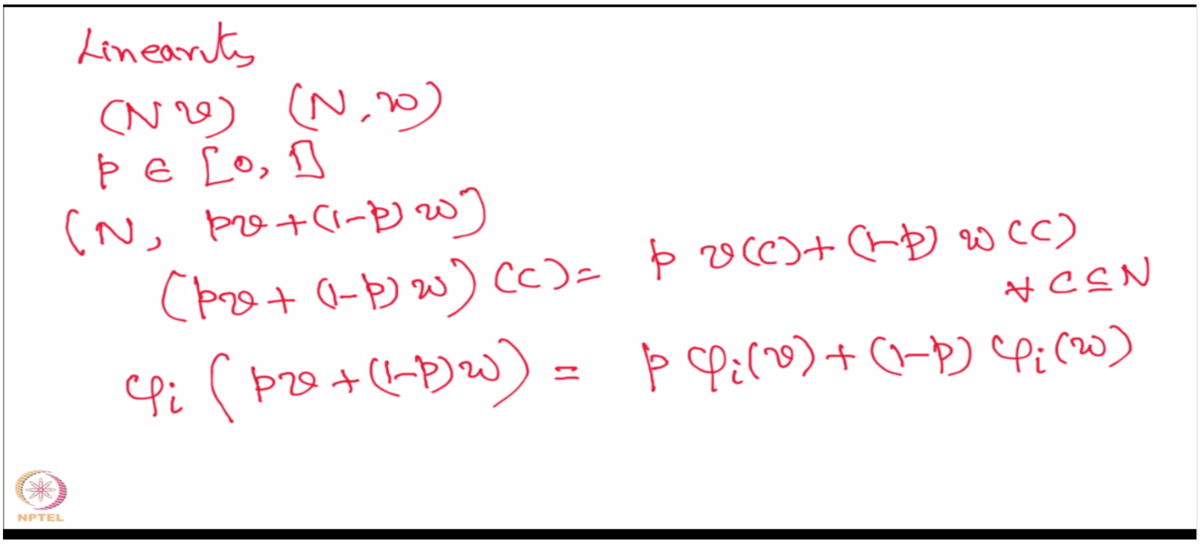
For example, Consider a game (N, v), where $N = \{1, 2, 3\}$ and permutation over players is as follows: $\pi(1) = 3, \pi(2) = 1, \pi(3) = 2$.. Therefore, in a game $(N, \pi v)$, πv can be represented as, $\pi v(1) = v(2); \pi v(2) = v(3); \pi v(3) = v(1); \pi v(1, 2) = v(2, 3); \pi v(1, 3) = v(2, 1); \pi v(12, 3) = v(3, 2); \pi v(1, 2, 3) = v(2, 1, 3)$.

Symmetry



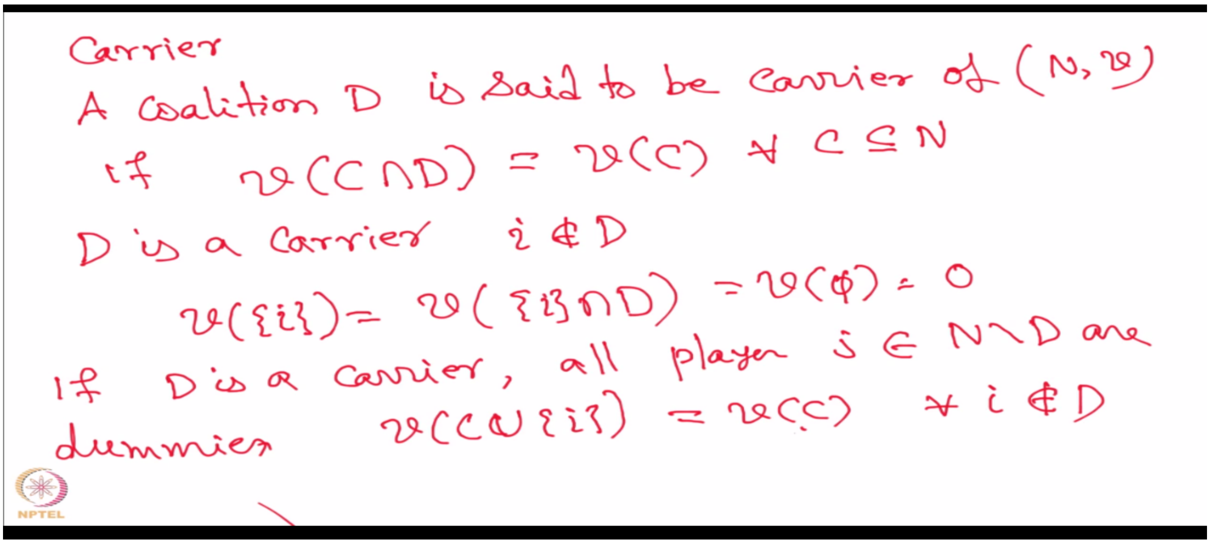
This axiom says that the shapley value of the player does not depend on the identity (i.e., shapley value of player i will be equal to shapley value of its permuted player).

Linearity



Linearity axiom says that if a player i is in the linear sum of any two games, then the shapley value is the linear sum of shapley value in those two games.

Carrier



Carrier axiom says that if a player is dummy then he should get zero value. (i.e., $\phi_i(v) = 0\,\forall i \notin D$.

So, these are the axioms that Shapley introduced and using these axioms he characterizes Shapley value as a unique solution concept. We will continue the proof of this Shapley result in the next lecture.