Game Theory
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Lecture - 38
Cooperative Games: Shapley Value

In the previous lecture, we have introduced several axioms which are necessary for understanding the Shapley value. In this lecture, we will state and prove Shapley theorem.

Theorem 1 (Shapley)**.** *There is exactly one mapping*

$$\phi : \mathcal{R}^{2^n-1} \longrightarrow \mathcal{R}^n$$

that satisfies symmetry,linearity and carrier axiom. The map is given by

$$\phi_i(v) = \sum_{C \subseteq N \setminus \{i\}} \frac{|C|!(n-|C|-1)!}{n!} \left[v(C \cup \{i\}) - v(C)\right] \tag{1}$$

Proof. We will prove this theorem in two parts,

1. We will show that the mapping satisfies three Shapley's axioms.
2. Uniqueness.

Part 1:

1. Linearity: We need to show

   $$\phi_i(pv + (1-p)w) = p\phi_i(v) + (1-p)\phi_i(w)$$

   By definition, for any coalition C,

   $$(pv + (1-p)w)(C) = pv(C) + (1-p)w(C)$$

   Now, from(1)

   $$\phi_i(pv + (1-p)w) = \sum_{C \subseteq N \setminus \{i\}} \frac{|C|!(n-|C|-1)!}{n!} \left[(pv + (1-p)w)(C \cup \{i\}) - (pv + (1-p)w)(C)\right]$$
   $$\phi_i(pv + (1-p)w) = p \sum_{C \subseteq N \setminus \{i\}} \frac{|C|!(n-|C|-1)!}{n!} \left[v(C \cup \{i\}) - v(C)\right]$$
   $$+ (1-p) \sum_{C \subseteq N \setminus \{i\}} \frac{|C|!(n-|C|-1)!}{n!} \left[w(C \cup \{i\}) - w(C)\right]$$
   $$\phi_i(pv + (1-p)w) = p\phi_i(v) + (1-p)\phi_i(w)$$

2. Symmetry: In any coalition, $\phi_i(v)$ depends only on $|C|$ and whether C contains i or not. Therefore, relabelling has no effect on the value. Therefore, symmetry holds.

3. Carrier: Suppose D is a carrier of the game (N,v). Therefore,

$$v(C) = v(C \cap D)\ \forall C \subseteq N$$
$$v(\{i\}) = 0\ \forall i \in N \backslash D, v(D) = v(N)$$

It is straightforward to show to verify that $\phi_i(v) = 0,\ \forall i \in N \backslash D$. Now, we need to show for carrier D,

$$\phi_i(v) = 0,\ \forall i \in N \backslash D$$

We know, $\sum_{i \in N} \phi_i(v) = v(N) = v(D)$ and we also know $\phi_i(v) = 0,\ \forall i \in N \backslash D$
This implies, $\phi_i(v) = 0,\ \forall i \in N \backslash D$

Part 2:

1. Let (N, z) be a cooperative game that assigns zero worth to every possible coalition C. Thus, $\phi_i(z) = 0$ (by carrier axiom).

2. $(pv + (1-p)w)(C) = pv(C) + (1-p)w(C)$ (by linearity axiom).

Let $L(N)$ denotes all subsets of N and $|L(N)| = 2^n - 1$. Therefore, $\mathcal{R}^{|L(N)|}$ space contain all the cooperative games. Let $D \subseteq N$, define (N, w_D) such that $w_D(C) = 1$ if $D \subseteq C$, otherwise 0. For every subset, we can define such game. Thus there are total of 2^n -1 such games. We will now show that this forms a basis. For this, we need to show that set of all these 2^n -1 games is linearly independent. Let D_1 and D_2 are two coalitions such that $D_1 \neq D_2$. Construct two games (N, w_{D_1}) and (N, w_{D_2}). Then we need to show that if $\alpha w_{D_1} + \beta w_{D_2} = z(zero)$, then $\alpha = \beta = 0$.

For any C, we know $w_{D_1}(C) = 1$ if $D_1 \subseteq C$, otherwise 0. And $w_{D_2}(C) = 1$ if $D_2 \subseteq C$, otherwise 0. Now, choose C such that $D_1 \subseteq C$ and $D_1 \nsubseteq C$. (Such C can be achievable since $D_1 \neq D_2$) Thus

$$\alpha w_{D_1}(C) + \beta w_{D_2}(C) = \alpha$$

now if $\alpha w_{D_1}(C) + \beta w_{D_2}(C)$ is zero, then α has to be zero. Similarly, we can show $\beta = 0$. Therefore, $\alpha w_{D_1} + \beta w_{D_2} = z$, implies $\alpha = \beta = 0$. Thus set of two games is linearly independent, and we can generalize it for set of all these 2^n -1 games. Therefore, this is a basis. From linear algebra, we know any linear function on a vector space, if it fixes the values on its basis, such a linear transformation is unique. So to prove uniqueness, it is enough to show that the mapping fixes its value on these simple games. Now, we will prove this. Choose a carrier $D \subseteq N$, we have game (N, w_D) and $\phi_i(w_D) = \frac{1}{|D|}$. We can verify from the formula in theorem that value coincides with this (which is easy to show) and therefore we have showed that the mapping fixes its value on these simple games. This complete the proof of uniqueness. □

Shapley value, which is represented as,

$$\phi_i(v) = \sum_{C \subseteq N \backslash \{i\}} \frac{|C|!(n - |C| - 1)!}{n!} \left[v(C \cup \{i\}) - v(C)\right]$$

is the weighted sum expectation of a marginal contribution of a player that he/she is contributing to any coalition C.

For certain class of games, Shapley value belongs to core and such games are known as convex games.

Convex Games

If we consider two coalition C and D such that $C \subseteq D \subseteq N$ and player $i \notin D$, then games where marginal contribution of player i for any coalition D is always bigger than marginal contribution for C are called convex games. i.e.,

$$v(D \cup \{i\}) - v(D) \geq v(C \cup \{i\}) - v(C)$$

Theorem 2. *For convex games, shapley value belongs to core.*