Game Theory
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Lecture 04
Combinatorial Games: The Game of Hex

In the previous lecture, we discussed Zermelo's theorem which says that one of the players has a winning strategy or the game ends in a draw. In this lecture, we will look at some examples and determine which player has a winning strategy. In particular, we look at the games where either a win or a loss occurs and there is no possibility of a draw.

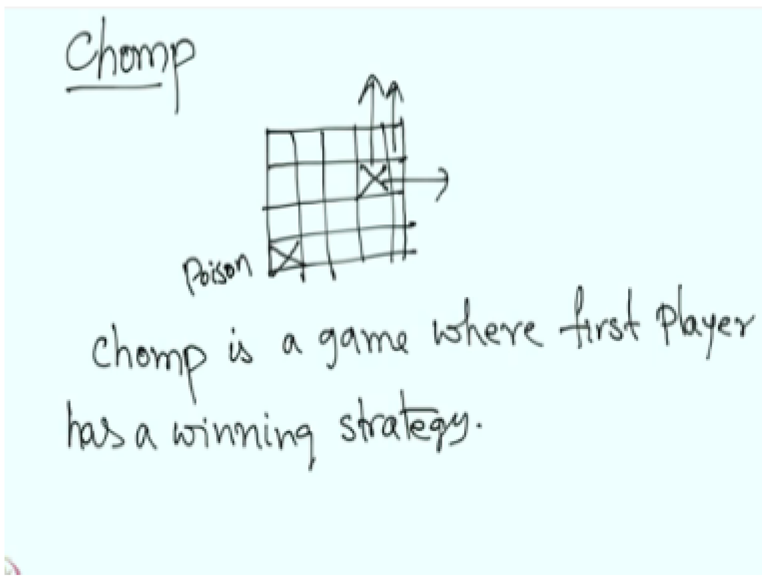


Figure 1: Refer Slide Time: 00:51

We start with the game of Chomp. As we have seen earlier, in Chomp, there is a finite rectangular collection of cells, as can be seen in the slide above. Each cell is considered as a piece of chocolate with the bottom left most cell being poisonous. Now, the players take turns alternatively choosing a cell and taking all the cells to the top and right to their chosen cell including the cell chosen by them. Hence, as the game terminates when a player takes(or is forced to take) the poisonous cell, we can never have a draw. It will always end in a either a win or loss.

So, which player wins this game? In fact, chomp is a game where the first player has a winning strategy. Let us look at how Player 1 should play this game to force a win. To start, let us consider a square, instead of a rectangle.

In the first move, Player 1 will mark the cell one to the top and right of the poisonous cell(as seen in the slide above) Referring to the instance above, once Player 1 makes this move, 3 cells to the top and 3 cells to the right of the poisonous cell remain in the game. Now, interestingly, whatever

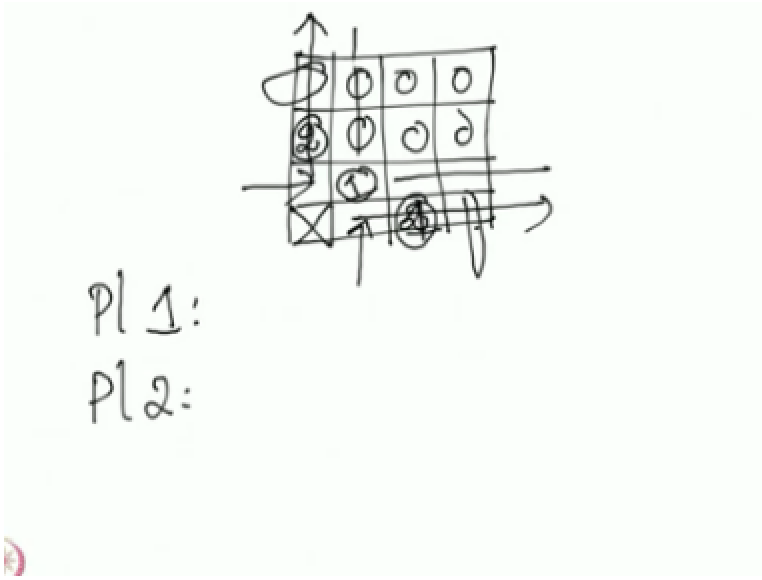


Figure 2: Refer Slide Time: 02:32

Player 2 plays, player 1 mimics her and plays it in the other side. For example in the above, if Player 2 plays a cell to the top of the poisonous cell, then Player 1 will respond by playing the corresponding cell to the right of the poisonous cell.

Hence, from this point in the game, whatever Player 2 does, Player 1 has a corresponding(symmetric) move for it. In the end, Player 2 will be left to play with only the poisonous cell remaining. This would lead to her taking the poisonous cell and hence, losing. Therefore, we conclude that Player 1 is going to win this game. Thus, the game of Chomp has a winning strategy for Player 1 when the board is given by a square.

Now, consider a rectangular array of cells. The remainder of this proof is quite intuitive. Suppose, Player 2 has a winning strategy. If player 2 has a winning strategy, then we do the following: We consider 2 games, the first game being a dummy game and the second game being the real game. In the first game, Player 1 plays the top right most cell. Now, given that Player 2 plays the move according to her winning strategy, Player 1 observes what Player 2 does and plays the same move in the second game as his first move in that game. Note that, now the two games are in the same position(refer to the slide given below).

The only difference in the two games is that in the dummy game, it is Player 1's turn next, whereas in the real game, it is Player 2's turn next. Now, it is simple to show that Player 1 wins in the real game. Whatever Player 2 moves in the second game, Player 1 observes and repeats in the first game. Moreover, whatever Player 2 moves in the first game, Player 1 observes and repeats in the second game. Also, because Player 2 always plays according to her winning strategy, if Player 2 can force a win in the first game, Player 1 can force a win in the second(real) game. This is a contradiction as we had assumed that Player 2 had a winning strategy. Hence, as a draw is not possible in this game, by Zermelo's Theorem, Player 1 must have a winning strategy. This concludes the idea behind the proof.

Here, we are using a strategy stealing argument. Whatever Player 2 does in the dummy game,

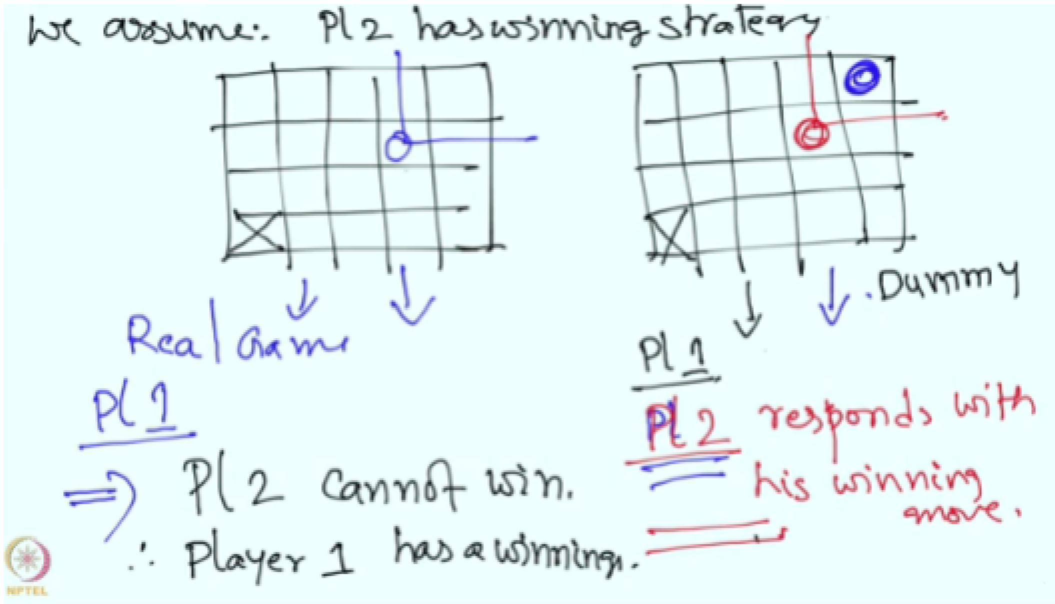


Figure 3: Refer Slide Time: 06:31

Player 1 follows it in the real game. We explain this with the following analogy: Suppose you are playing chess with both Kasparov and Karpov at the same time and suppose you play with one of them as white and with the other, say Kasparov, as black. You observe what Kasparov moves with white and replicate that strategy in the game with Karpov. Then, you are sure to win one of those 2 games or at the least, you will end in a draw.

Now, let us move on to the game of Hex.

As we have seen earlier, the game of Hex is played between two players, the blue player(Player 1) and the red player(Player 2). The goal of this game is to make a path from one side to the other, as can be seen in the slide above. The blue player wishes to mark the hexagonal cells such that he makes a path(horizontally) from his side to the other side. Similarly, the red player would like to make a path(vertically) from her side to the other side. They make their moves alternatively, where in each move, the players pick one of the hexagon cells and mark them with their respective color. This game was invented by Piet Hein, a Danish scientist, mathematician, writer and poet, in 1942. It was rediscovered by John Nash at Princeton in 1948, and became popular at Princeton. In this game, we want to address the following three questions:

- How do we solve this game?
- Does the game have a winning strategy for either player?
- Can there be a draw?

We proceed with the following theorem.

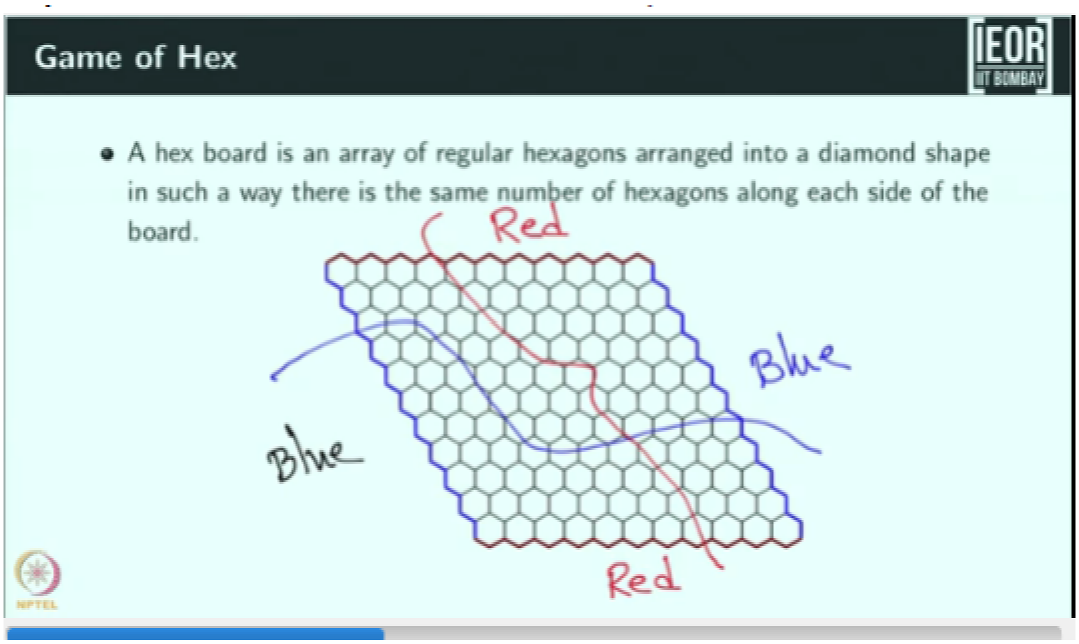


Figure 4: Refer Slide Time: 09:56

Theorem. *There can be no draw in the Game of Hex.*

Proof. We give a very intuitive proof to the above theorem. Let the board with hexagonal cells, be made out of paper. Let the two players, blue and red move in such a way that the blue player moves by colouring his cell blue and the red player moves by cutting out her cell from the paper board. In the end, when each cell has been chosen either by the red player or the blue player, the sheet is either connected or disconnected with at least two pieces. Then, taking all the cases, we see that either the red player wins or the blue player wins this game.

In fact, this is one of the underlying ideas in this proof of the Jordan Curve theorem which is quite an important result in mathematics. And we can also use the game of Hex to prove the Jordan curve theorem. Now, let us go to the next result which says that the first player has a winning strategy in this game.

Theorem. *The first player has a winning strategy in the game of Hex.*

Proof. Suppose, the second player has a winning strategy. Because moves by the players are symmetric, it is possible for the first player to adopt the second player's winning strategy as follows:

The first player, on his first move, colors an arbitrarily chosen hexagonal cell. Subsequently, for each move by the second player, the first player responds with the appropriate move dictated by the second player's winning strategy. This is called "stealing the strategy" and is used by Nash in his proof.

If the strategy requires that the first player move in the spot that he chose in his first turn, and there are empty hexagons left, he just picks another arbitrary spot and moves there instead. This becomes the new extra hexagon for him. Having an extra hexagon on the board can never hurt the first player - it can only help him. In this way, the first player, too, is guaranteed to win, implying that both players have winning strategies, a contradiction. This completes the proof.

This game of hex is interesting for several reasons, one of them being that it can be used to prove the Brouwer's Fixed Point theorem, a very important theorem which we are going to use extensively, throughout this course. Before proceeding, let us state the *Brouwer's Fixed Point theorem.*

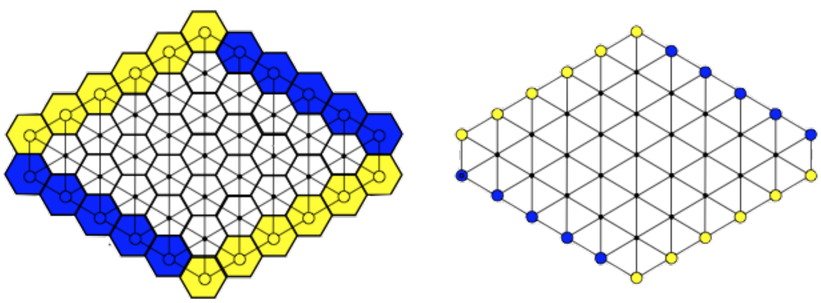
Brouwer's Fixed point Theorem. *Let $f : [0,1] \to [0,1]$ be a continuous function. Then, $\exists\, x \in [0,1]$ such that,*

$$f(x) = x$$

where such an x is called a 'fixed point' of f.

Note that, we can take an n dimensional function as well. Also, instead of taking $[0,1]$ we can take any compact and convex set K and the fixed point of f will then lie in K. Let us now see the proof of this using the game of Hex.

Consider the dual lattice of the Hex board. For each of the hexagonal cells, we connect the central points as shown in the figure below.

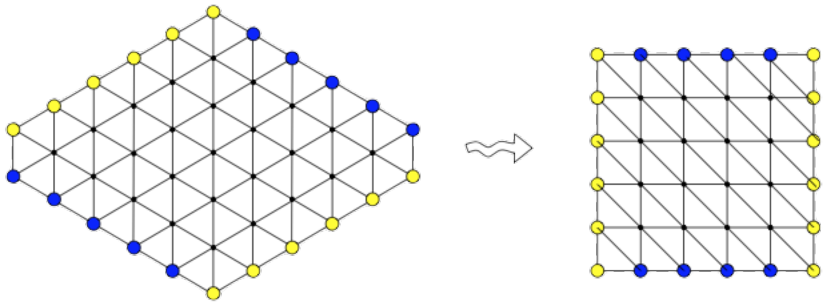


Now, coloring the hexagons is equivalent to coloring the nodes and a path here, from one side to the other, is a path in the game of Hex which the players want to achieve. The set of nodes can be described as $\{au + bv : a, b \in \mathbb{Z}\}$ with $u = (0,1), v = (\frac{\sqrt{3}}{2}, \frac{1}{2})$. Two nodes x, y are neighbours if $\|x - y\| = 1$.

By applying the linear transformation G defined by

$$G(u) = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \qquad \textit{and} \qquad G(v) = (0,1)$$

we obtain a more convenient representation of this lattice.



(Please note that the above figure is only for illustration purposes)

Now the game of Hex can be thought of as a game on the square of this lattice. Let f be a continuous function from the unit square to itself. We need to show that f has a fixed point. For this we show that for any $\varepsilon > 0$, there is a point x such that $|f(x) - x| \leq \sqrt{2}\varepsilon$. And, from compactness, as $\varepsilon \to 0$, this would converge to a fixed point. This would complete the proof.

Fix $\varepsilon > 0$ and choose $0 < \delta \leq \varepsilon$ such that,

$$|x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon.$$

Also, note that f being a continuous function on a compact set, is uniformly continuous(Heine-Cantor theorem).

Let f_1 be the first component of f, being a continuous function from a square to a square, and let f_2 be its second component. Consider the regions

$$\begin{aligned} H^+ &= \{(x,y) : f_1(x,y) - x \geq \varepsilon\} \\ H^- &= \{(x,y) : x - f_1(x,y) \geq \varepsilon\} \\ V^+ &= \{(x,y) : f_2(x,y) - y \geq \varepsilon\} \\ V^- &= \{(x,y) : y - f_2(x,y) \geq \varepsilon\} \end{aligned}$$

Observe that no vertex of H^+ is adjacent to a vertex of H^-. This follows from the definitions of H^+ and H^-. In fact, no vertex of H^+ will have a first coordinate equal to 1 and no vertex of H^- will have a first coordinate equal to 0. Therefore, if one of the players marks their cells only from H^+ and H^-, it will not form a winning set.

We know from the above theorems, that there is a winning strategy. Thus, $H^+ \cup H^-$ cannot form a winning set for a hex player wishing to join the sides $x_1 = 0$ and $x_1 = 1$. Similar statements apply for V^+ and V^- and hence $V^+ \cup V^-$ cannot form a winning set for a player wishing to join the sides $x_2 = 0$ and $x_2 = 1$.

Thus $H^+ \cup H^- \cup V^+ \cup V^-$ can not cover all the vertices. This means that there must be a hexagon with its central point not in any of the four sets. This middle point x will satisfy

$$|f(x) - x| \leq \sqrt{2}\varepsilon.$$

The compactness arguments now, provide a fixed point for the function f and thus, this game of Hex provides a proof for the Brouwer Fixed Point theorem. Although we have proved the above only in two dimensions, this argument can be extended to higher dimensions as well.