Game Theory
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Lecture - 40
Matching Problems

In this lecture, we will study about matching problems.

Matching Problems

Class of problems which involve matching the members of one group of agents with one or more members of a second disjoint group of agents, all of whom have preferences over the possible resulting matches.

So, this matching problem; these are a , so what I mean to say that there is a group of people here this side, there is another group of people here and then each person has preference over the other side.
This problem is basically introduced by David gale and Lloyd Shapley.

Examples:

- One to one matching:
  1. Medical Internships in US Hospitals.
  2. Matching of kidneys(or other human organs) to patients in need of a transplant.

- One to many matching:
  1. College admissions, students have preferences over the set of colleges and colleges take students on the basis of their ranks. Here, one college is allocated to many students.

Stable Matching:

A matching is a mapping from the elements of one set to the elements of the other set. Let us consider two matching (M1,W1) and (M2,W2). A matching (M1,W1) is not stable if:

1. There is an element M1 of the first matched set which prefers some given element W2 of the second matched set over the element to which M1 is already matched (i.e., W1), and
2. W2 also prefers M1 over the element to which W2 is already matched(i.e., M2).

Here, (M1,W2) is called blocking pair.
A matching is said to be stable when there does not exist any match (M1,W2) in which both M1,W2 prefer each other to their current partner under the matching. In other words, A matching is said to be stable if there does not exits any blocking pair.

Stable matching problem is the problem of finding a stable matching between two sets of elements (having same cardinality) given an ordering of preferences for each element.

Stable matching algorithm:

The stable marriage problem has been stated as follows:

Given n men and n women, where each person has ranked all members of the opposite sex in order of preference, match the men and women together such that there are no two people of opposite sex who would both rather have each other than their current partners. When there are no such pairs of people, the set of marriages is said to be stable.

Algorithm:
Gale and Shapley developed an algorithm which shows that a marriage market always admits a stable matching.

1. Each man proposes to his most preferred woman.
2. Each woman evaluates here proposers, including the man she is tentatively matched to, if there is one, and rejects all but the most preferred one. She becomes tentatively matched to this latter man.
3. Each rejected man proposes to his next preferred woman.
4. Repeat step (2) and (3) until each women has a tentative match.

The idea in this algorithm is that you are not accepting until the algorithm terminates, so you are deferring the acceptance, so therefore it is also known as a deferred acceptance (DAA).

Question: Is this algorithm terminate in a finite time? If yes why?

Answer: Yes. In this algorithm, every man will propose atmost n women and there are total of n mens. Thus, algorithm can run at the most n^2 rounds.

Question: Why DAA provides stable match?

Answer: So, let us say the Bharat prefers women Anita than the one, he is currently matched.This means Bharat must have proposed to Anita earlier because she was on top of his list than the current person. But got rejected by Anita. Thus it concludes Bharat must have been rejected by everyone else who he preferred more than the current person. So this implies no men can get better than he is currently with in the stable match.
Now we will look at women side, if there is some other men whom she prefers, then the two

things can happen either that person never proposed to her or she must have rejected that person, if she has not rejected then she surely wouldn't have received a proposal from that person. So this implies no women can get better than she is currently with in the stable match. Thus, DAA provides stable match.

Example of stable matching

For a given set of men $= \{M_1, M_2, M_3\}$ and set of women $= \{W_1, W_2, W_3\}$, preferences are as follows

Men	Preference		Women	Preference
M_1	$\mathrm{W}_2 W_1 W_3$		W_1	$\mathrm{M}_1 M_3 M_2$
M_2	$\mathrm{W}_1 W_2 W_3$		W_2	$\mathrm{M}_3 M_1 M_2$
M_3	$\mathrm{W}_1 W_2 W_3$		W_3	$\mathrm{M}_1 M_2 M_3$

Stable matching:

1. When men proposes: $\{(W_1, M_3), (W_2, M_1), (W_3, M_2)\}$.
2. When women proposes: $\{(W_1, M_1), (W_2, M_3), (W_3, M_2)\}$.

Remark 1: Among all possible different stable matchings, it always yields the one that is best for all men (one who proposes) among all stable matchings, and worst for all women (who gets the proposal)
Remark 2: If you take any other stable matches, that matches are always lies between these two stable mathings(i.e., men optimal and women optimal).
Remark 3: It is a truthful mechanism from the point of view of the one who is proposing (say men). That is, no man can get a better matching for himself by misrepresenting his preferences.

Decenteralized algorithm

1. Start with any match.
2. Find blocking pair and then interchange.
3. Go on and repeat.

Question: Will this algorithm reach stable match?
Answer: No. Donald Knuth provided the counter example for the same. Also suggested some changes in the algorithm. He further asked what changes in algorithm will lead to reach stable match? Answer to this question is positively given by Alvin Roth and Vand Vate. They suggested following algorithm:

1. Start with any match.
2. Find a blocking pair. Instead of interchanging them. Pair the blocking pair. Let their current partner to be single. It will lead to another matching where some are single and some are matched.

3. Repeat.

Further they proved that with probability 1, this algorithm converges to stable match. (details is not discussed here).

Other aspects in stable matching

Two-sided: Until now, we have discussed two sided matching and also we have assumed strict preferences. Now, if we relax this assumption (i.e., a person can prefer two person equally). We also haven't assumed the possibility where a person wants to be single, instead of matching with someone. Now, if we relax this assumption also. The Gale-Shapley algorithm can be modified to entertain these changes.

One-sided: In the one sided market, there are only single set of people. For example, allocation of students in hostel rooms. Two out of all student need to be matched for a room, this has to be done for every room. Here, each student have preference over the remaining students. Next question is the existence of stable matching. In one sided market, stable match need not exist.