Game Theory
Prof. K. S. Mallikarjuna Rao
Department of Industrial Engineering & Operations Research
Indian Institute of Technology – Bombay

Lecture - 05
Combinatorial Games: Nim Games

In this session. We will discuss another combinatorial game called Nim game.

Nim Game

The normal game is between two players, say blue player and red player, and played with two heaps of any number of coins. The two players alternate taking out any number of coins from any one of the heaps. The one who picks the last coin will be the winner of the game.

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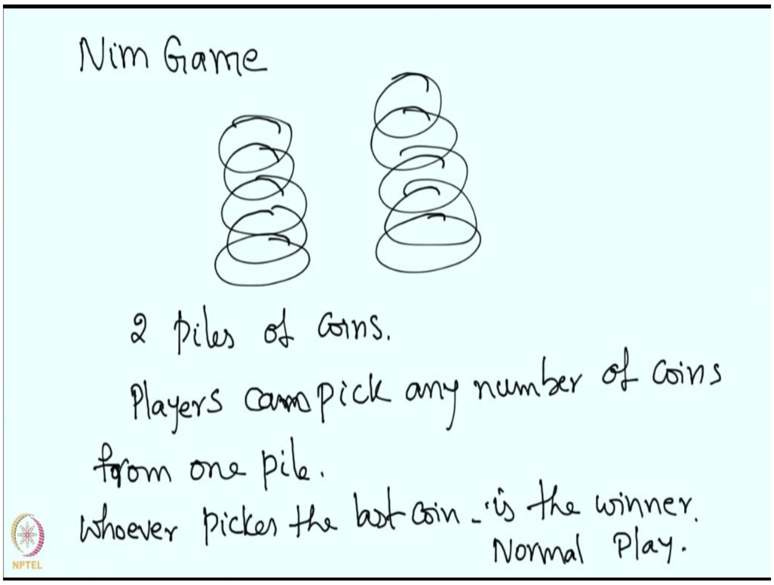

If in above game, there is only one pile then what happen? Will it be interesting? No. Because, first player starting the game will pick out all the coins and will win the game.

There are games which we play with single pile in which players can pick more than one coin but not whole pile and these games are called take away game.

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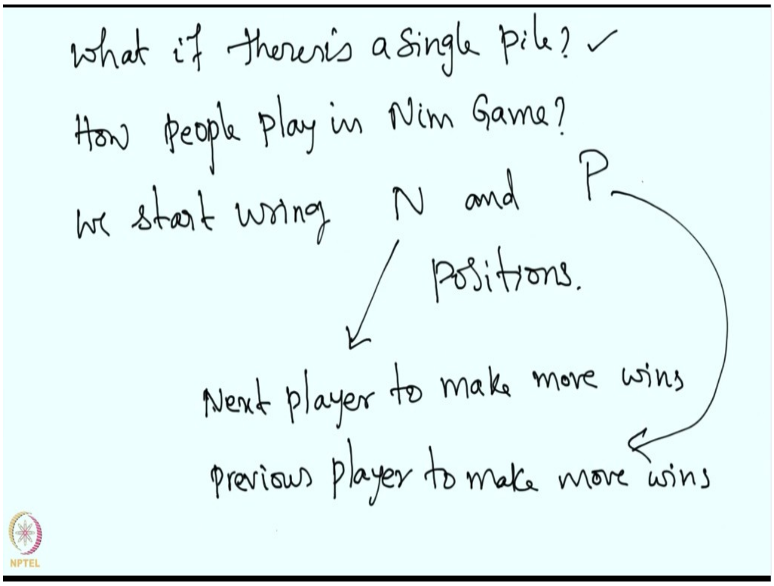

But our focus is Nim game and how to play this game? And who will be the winner?

We will use following two positions, N and P position, where N is the next player to make move, wins and P is the previous player to make move, wins. Now, we will see who will win this game.

In order to see who is going to be the winner we will require “Nimber arithmetic”, or specifically Nimber addition (Nim-Sum) .

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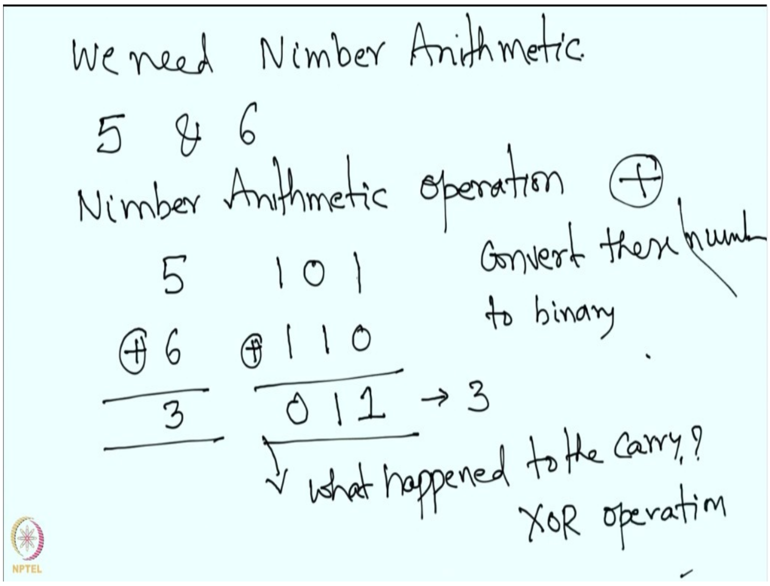

Nimber addition is binary addition without carrying. For example, if we apply Nimber addition operation on 5 and 6, we will get 3.

First, we will convert these numbers in to binary. So, 5 is going to be 101 and 6 is going to be 110 and then binary addition of 101 and 110 without carrying yields 011, which is 3 in decimal form.

If you notice, this is nothing but a XOR operation. Thus, we can conclude that after applying Nimber arithmetic addition on odd number of ones (even number of ones), we will get 1(0). For any number x, x Nim sum x is always 0. Nim sum helps us in providing strategy in Nim game.

Theorem : Winning strategy in Nim game is to finish every move with Nim sum of zero.

The proof of above theorem depend on following two lemmas.

Lemma 1 : After a player's turn, if the Nim sum is zero, the next player to make move must change it to non-zero.

Proof of this lemma consists of two steps. In first step, we need to prove that if the Nim sum is 0 after a player's turn, next player's move is to changes it to a non-zero and in the second step is that how can a player make Nim sum to 0.

Assume that there are n heaps, say $x_1, x_2, \ldots, x_n$. Let s is the nim sum of $x_1, x_2, \ldots, x_n$.

After a move, let $y_1, y_2, \ldots, y_n$ are heap sizes and t is the nim sum of $y_1, y_2, \ldots, y_n$. Since we know a player will remove coins from one heap only, thus $y_i = x_i$, expect for one i for all i = 1,2,….,n. And after applying nim sum and doing some arrangements we can show that if s=0 then t cannot be zero (**Refer Slide Time: 10:11,13.02,14:23**). Which completes our proof.

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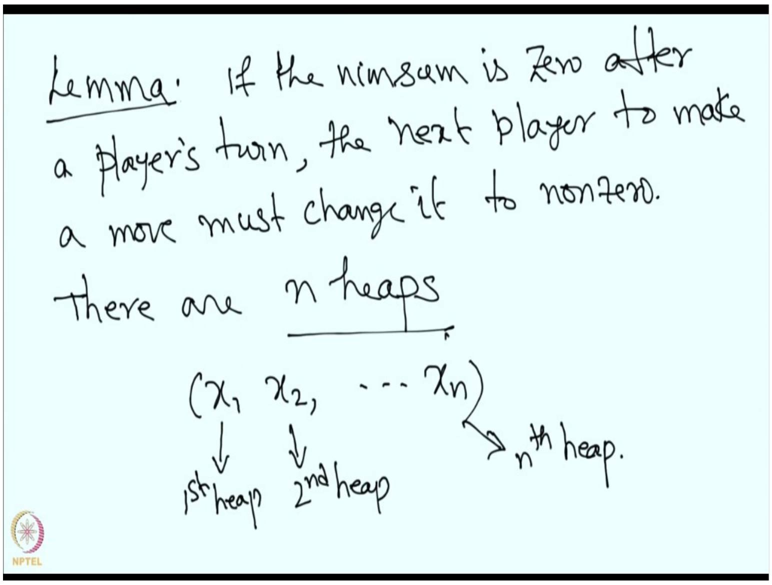

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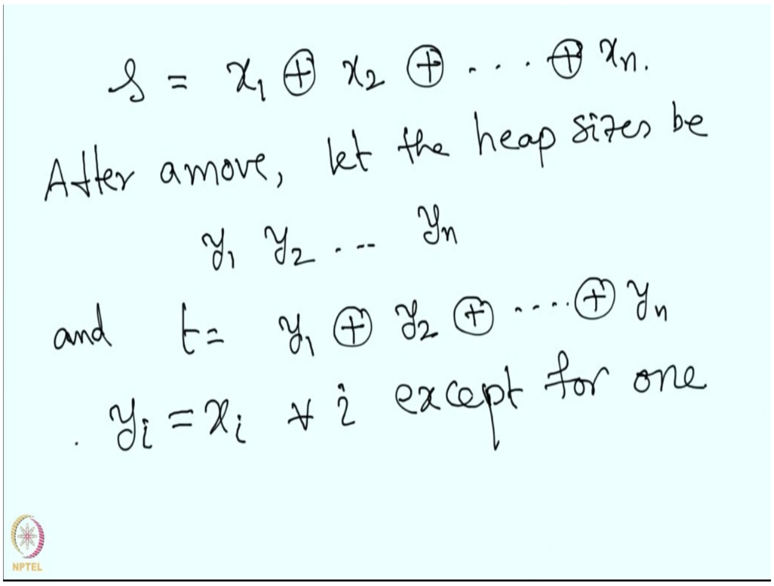

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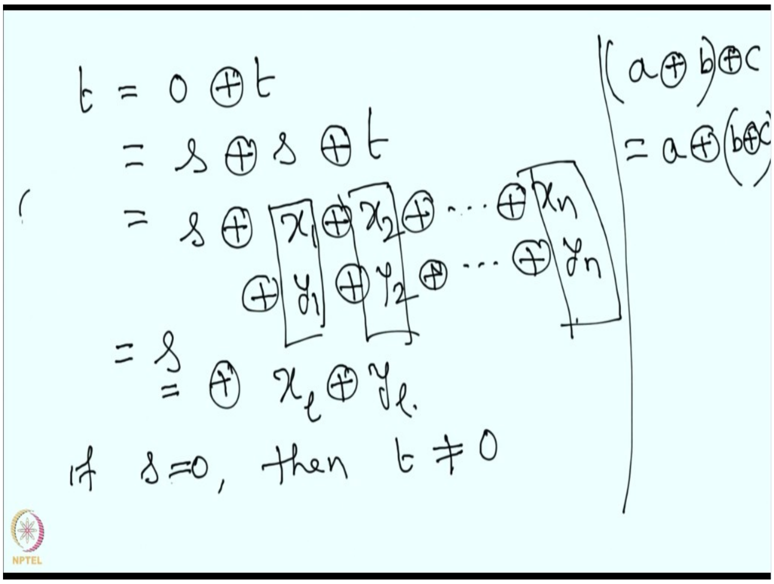

The next question that we address is that at the beginning of a player turn if nim sum is non-zero, then is it possible for a player to make nim sum to be zero after removing coins from the heap?

Lemma 2 : It is always possible to make the nimsum to be zero on your turn, if it was non-zero at the beginning of your turn.

Proof: Let s is the current nim sum which is the nim sum of $x_1, x_2, \ldots, x_n$. Assume s is non-zero. Let d be the position of the leftmost (most significant) nonzero bit in the binary representation (i.e., 1) of *s*, and choose *k* such that the d^{th} bit of x_k is also 1. (Such a *k* must exist, since otherwise the d^{th} bit of *s* would be 0.)

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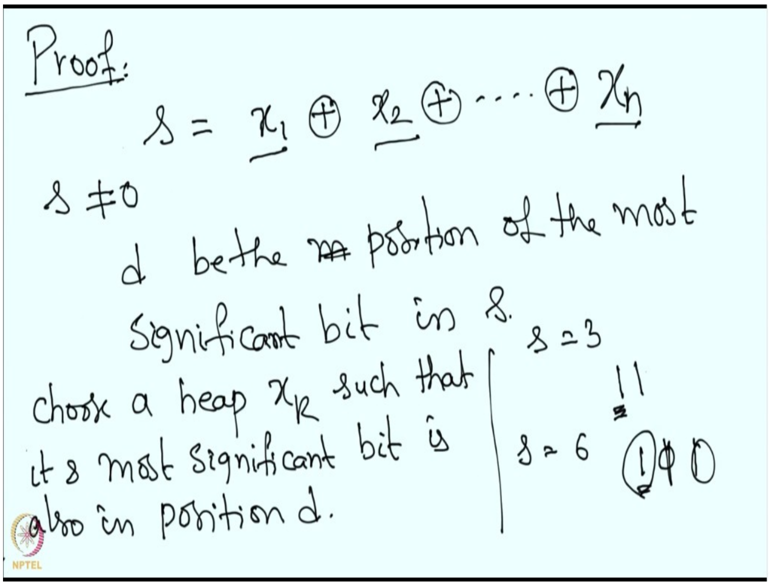

Choose the heap x_k and a player want to remove from this heap. Let the heap contains y_k coins, where y_k is equals to x_k nim sum s. So, we are basically removing x_k minus y_k .

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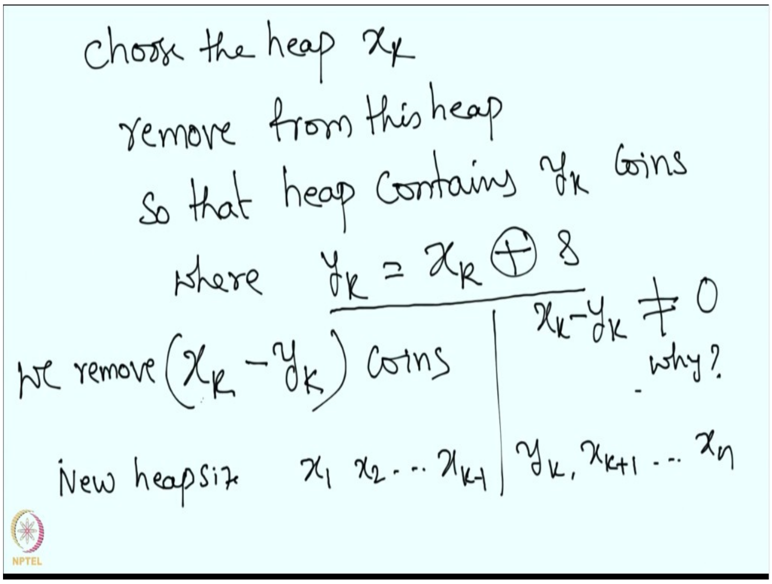

After a move, nim sum becomes

$$t = x_1 \oplus x_2 \oplus \ldots.. x_{k-1} \oplus y_k \oplus x_{k+1} \oplus \ldots.. \oplus x_n$$

$$= x_1 \oplus x_2 \oplus \ldots.. x_{k-1} \oplus x_k \oplus s \oplus x_{k+1} \oplus \ldots.. \oplus x_n$$

$$= s \oplus s = 0$$

Which completes our proof.

From the above two lemmas we can conclude that "Winning strategy in Nim game is to finish every move with Nim sum of zero".

Nim game is analyzed by Bouton and these type of games are called impartial games, in which the allowable moves depend only on the position and not on which of the two players is currently moving. Whereas there are also games where the position depends on the player. A classical example is a chess game. And we will discuss these games in upcoming lectures. They are known as partisan games. In the next lecture we will study about Sprague-Grundy theorem.