Game Theory

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Lecture 06
Combinatorial Games: Sprague-Grundy Theorem I

In this lecture, we continue our study on impartial games. We start discussing the Sprague-Grundy theory of impartial games. For this, we need to introduce the notion of graphical games. A game consists of a graph $G = (X, F)$, where

- X is the set of all positions.
- F is a function that gives $\forall x \in X$, a subset of possible positions to move to, called followers. If $F(x)$ is empty, then x is a terminal position.
- The start position is $x_0 \in X$. Player 1 moves from x_0.
- Players alternate moves. At position x, the player chooses some $y \in F(x)$.
- The player confronted with empty set $F(x)$ loses.

Note that the last point says that if a player confronts with an empty set $F(x)$ he loses. In other words, we are in a normal play setting.

Let us see some examples. We will start with a very simple game called Chomp which we have come across already. Consider the 2×2 square collection of cells. As we have seen before, the bottom right cell is the poisonous cell. This gives the initial position from where the game starts.

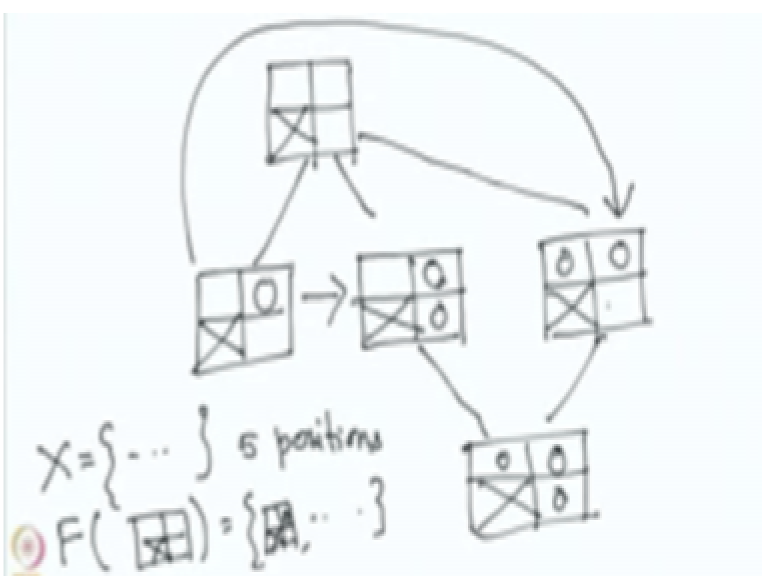

Figure 1: Refer Slide Time: 07:24

Refer to the above slide for the game. From this position, the game can go to three positions depending on the cell, Player 1 picks. The slide above depicts the game tree which contains all the possible positions of the game including the terminal position, which will be one with only the poison cell remaining. The set X consists of these positions. Including the initial position, there are a total of 5 positions. This is one example of how impartial games can be denoted by graphs.

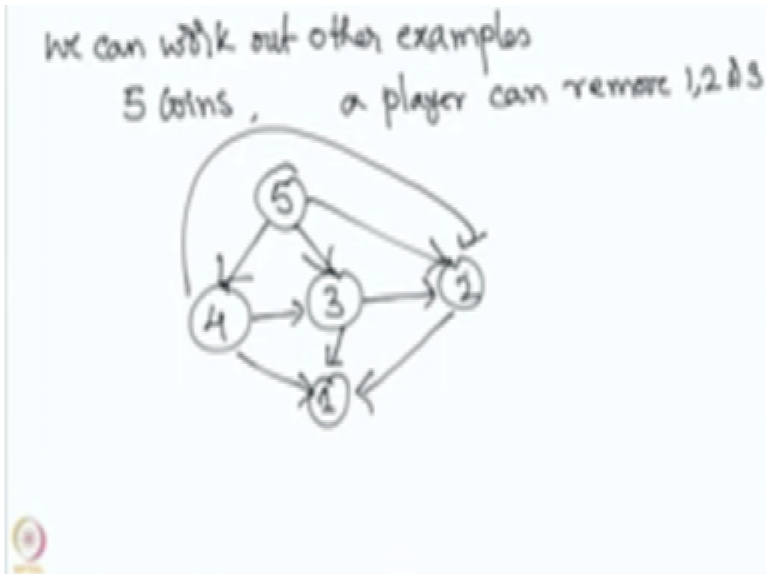

Figure 2: Refer Slide Time: 09:05

We can think of more such examples. Let us now consider the Take Away game with 5 coins. In this game, a player can remove 1,2 or 3 coins from the pile. It is simple to see that the positions of this game would be $\{0,1,2,3,4,5\}$. Refer to the slide above for a pictorial view of the game. From 5, the game can go to 4,3 or 2. From 4, the game can go to 3, 2 or 1.. and so on. This is another such example.

We are only interested in games which are *progressively bounded*.

Definition. A game is called progressively bounded if its graph has the property that the length of any path from any position is finite.

This means that if the game starts from some initial position and players make moves alternatively, then a terminal position is reached in finite number of moves. This is also known as the length. So, the maximum possible moves to reach a terminal position is called its length and the length of the game has to be finite.

Note that, it also immediately implies that the graph has no cycles. If there is a cycle, the game can keep cycling around it thus giving rise to an infinite path. So, having no cycle is an important property of these graphical games which are progressively bounded.

So, we now consider progressively bounded impartial games. Let us recall impartial games. In an impartial game, the positions available to players are the same, that is one cannot distinguish that a particular position can only be played from by a particular player, like in Chess.

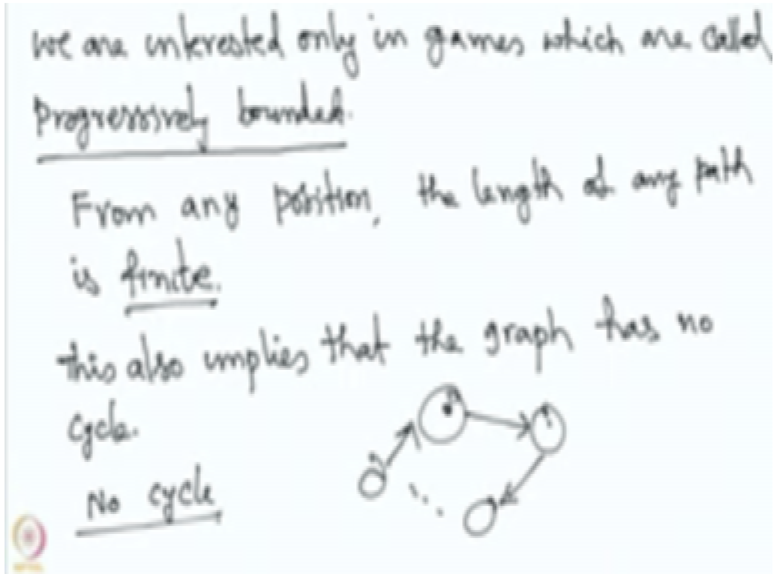

Figure 3: Refer Slide Time: 10:29

Now, we would like to introduce the Sprague-Grundy function. The Sprague-Grundy function is defined on X which takes non-negative integers.

$$g : X \to \{0, 1, 2, ..\} \tag{1}$$

such that

$$g(x) = min\{n \geq 0 : n \neq g(y) \;\; \forall \;\; y \in F(x)\} \tag{2}$$

It is exactly like what we have defined earlier when we were proving the NIM game. So, $g(x)$ is the smallest integer which is not equal to any of $g(y)$ for $y \in F(x)$. In fact, this is defined in a recursive fashion and we have to introduce this definition recursively.

First, we have to define $g(x)$ for the terminal nodes. From there, we define $g(x)$ for the other nodes, inductively. So, if x is a terminal position, then $g(x) = 0$, because there is no following position from x and $F(x)$ is empty. This means that the set over which we take the minimum consists of all non-negative integers and minimum of them is going to be 0. Once we know the Grundy values for the terminal positions, we define the same for the previous nodes, recursively.

In fact, as we have defined earlier, $g(x)$ can also be defined as:

$$g(x) = mex\{g(y) : y \in F(x)\} \tag{3}$$

Recall the definition of *mex* from the section about NIM games. So, let us say let us take a simple game which will take, this is the position that it takes.

Now, notice the classification of positions into N and P positions for the given example in the slide above. If we work from the bottom, we see that the bottom most positions, which are P

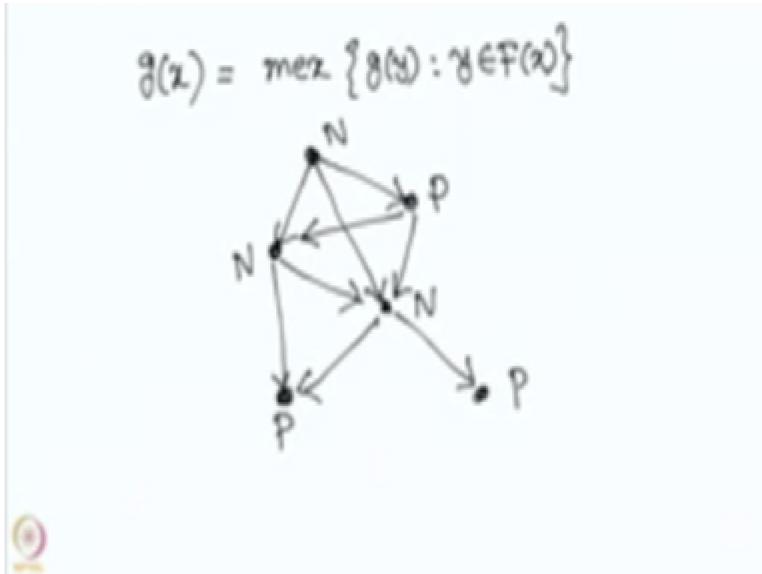

Figure 4: Refer Slide Time: 16:52

positions, will have $g(x) = 0$. The ones above that are N positions and will have Grundy values 2 and 1, respectively. The P position above that will have Grundy value 0 as both its followers are N positions and both have non-zero Grundy values, thus making the minimum excluded value zero. The position on the top, being an N position, has followers with $g(x) = 0, 1, 2$ and hence, has a Grundy value of 3.

Note that, all the P positions in this example have Grundy value zero. We can safely say, that if the Grundy value of a position is 0, then it is a P position. Moreover, for N positions, the Grundy value is non-zero.

Now, we introduce *Adding Games*. Recall the NIM game with 2 heaps. At any point of time when a player makes a move, she can remove the coins from any of these two heaps. When there are 2 heaps at any point of time the player has a choice of choosing which heap and how many coins to choose. This is essentially the idea of Adding games.

Let us consider two games given by $G_1(X_1, F_1)$ and $G_2(X_2, F_2)$. We are interested in defining $G_1 + G_2$.

$$G(X, F) = G_1 + G_2 \tag{4}$$

For G, we need to define X and F, which gives the corresponding graph structure. X has to specify all possible positions of the game and is given by,

$$X = X_1 \times X_2$$

This is similar to the idea of a NIM game with 2 heaps. Next, we define F. Consider the NIM game with two heaps. If the game is at a position (x_1, x_2) and from there, goes to (y_1, x_2), then $y_1 \in F_1(x_1)$. Similarly, if the game goes to (x_1, y_2), then $y_2 \in F_2(x_2)$. Hence, using this idea, we define F.

$$F(x_1, x_2) = F_1(x_1) \times \{x_2\} \cup \{x_1\} \times F_2(x_2)$$

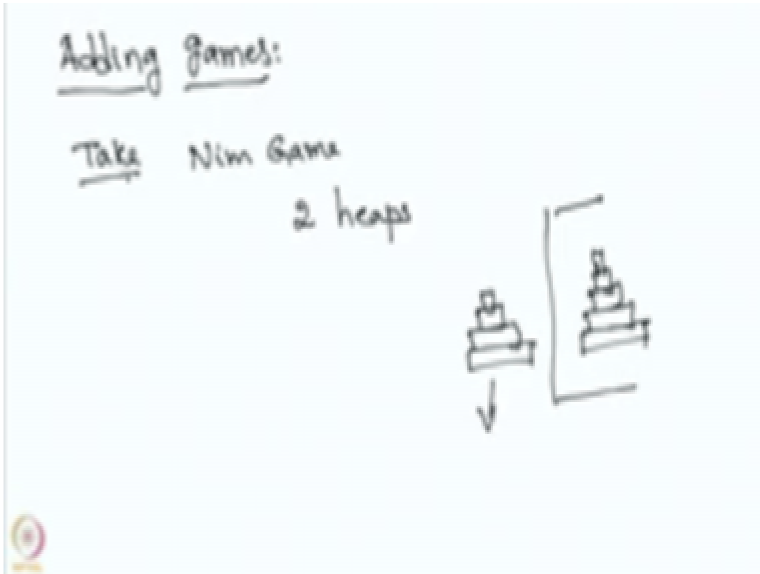

Figure 5: Refer Slide Time: 20:56

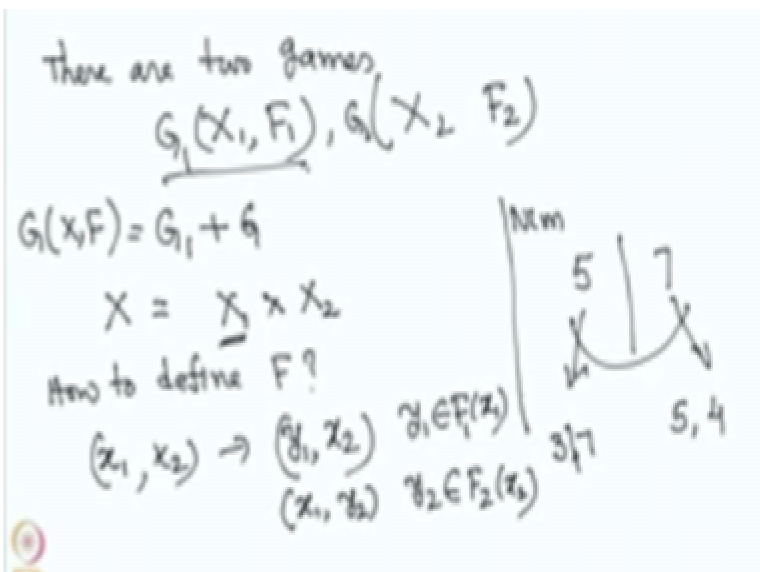

Figure 6: Refer Slide Time: 22:19

Next, we define this addition for n number of games. Given, n games, $G_1(X_1,F_1)$, $G_2(X_2,F_2)$, .., $G_n(X_n,F_n)$, we have:

$$G = G_1 + G_2 + .. + G_n$$

where,

$$X = X_1 \times X_2 \times .. \times X_n$$

and

$$F(x_1,x_2,..,x_n) = F_1(x_1) \times \{x_2\} \times .. \times \{x_n\} \cup \{x_1\} \times F_2(x_2) \times .. \times \{x_n\} \cup .. \cup \{x_1\} \times \{x_2\} \times .. \times F_n(x_n)$$

Now we need to define the terminal positions. So, a terminal position is a position from where, in any of the games, you cannot move to any other position. This will be simply the Cartesian product of the terminal positions in the individual games G_1, G_2,.., G_n.

In the next lecture, we will look at how to find the Sprague-Grundy function of the above. We also want to understand how the Sprague-Grundy functions of G_1, G_2,.., G_n are related. This leads to the Sprague-Grundy theorem.