Game Theory
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Lecture 07
Combinatorial Games: Sprague-Grundy Theorem-II

In the previous lecture, we introduced graphical games as another way of looking at impartial games and we also discussed the addition of these games. We also introduced this Sprague-Grundy function of an impartial game. In this session, we are interested in proving the Sprague-Grundy theorem.

Let's recall the Sprague-Grundy function. Let $G(X,F)$ be the given game. The Sprague-Grundy function g of this game is defined as:

$$g(x) = mex\{g(y) : y \in F(x)\} \tag{1}$$

This is the minimum non-negative integer that is not already given to the followers from this position x. Now, we move on to the *Sprague-Grundy theorem.*

Theorem. *Let us consider n impartial games G_1, G_2, .., G_n and let g_1, g_2, .. g_n be their Sprague-Grundy functions respectively. Let g be the Sprague-Grundy function of $G_1 + G_2 + \ldots + G_n$. Then,*

$$g(x_1, \ldots, x_n) = g_1(x_1) \oplus g_2(x_2) \oplus \ldots \oplus g_n(x_n)$$

Proof. Let $x \in X$. Recall that x is n-dimensional, i.e.

$$x = (x_1, x_2, \ldots, x_n)$$

And, define b as follows:

$$b = g_1(x_1) \oplus g_2(x_2) \oplus \ldots \oplus g_n(x_n)$$

We need to show the following two things:

1. For every $a < b$, there is a follower of $(x_1, x_2, .., x_n)$ that has g value a.

2. No follower of x has g value b.

These two statements together will imply that $g(x) = b$.

proof of (1): So, the idea is essentially to follow the NIM game proof done earlier. Let us take $d = a \oplus b$. Let k be the number of digits in the binary expansion of d, so that,

$$2^{k-1} \leq d < 2^k$$

and d has a 1 in the kth position. So, we are choosing this in such a way that there is a 1 in the kth position of d. This is in order to have the largest a which is less than b. Now, we have:

- $a < b$
- b has 1 in the k^{th} position.
- a has 0 in the k^{th} position.

$$b = g_1(x_1) \oplus g_2(x_2) \oplus \ldots \oplus g_n(x_n)$$

This implies that at least 1 x_i is such that the binary expansion of $g_i(x_i$ has a 1 in the kth position. This follows because of the definition of b above.

Moreover, there must be odd number of ones in the kth places of these x_j's to make b's kth place 1 (One can refer to the proof of NIM to have a clearer understanding of this). For simplicity, let's assume this to be x_1. Therefore, $g_1(x_1)$ has a 1 in its kth position.

Now, $d \oplus g_1(x_1) < g_1(x_1)$. Because, both of their kth positions have 1, the sum $d \oplus g_1(x_1)$ will have 0 in its kth position. Now, therefore, there is a move from x_1 to some x_1' with

$$g_1(x_1') = d \oplus g_1(x_1)$$

where this $g_1(x_1')$ has to be smaller than $g_1(x_1)$. This comes from the definition of the Grundy function and the Nimber arithmetic. This means that the move from $(x_1, x_2, ..x_n)$ to $(x_1', x_2, .., x_n)$ is a legal move in the game G and further,

$$\begin{aligned} &g_1(x_1') \oplus g_2(x_2) \oplus .. \oplus g_n(x_n) \\ &= d \oplus g_1(x_1) \oplus g_2(x_2) \oplus .. \oplus g_n(x_n) \\ &= d \oplus b \\ &= a \end{aligned}$$

This proves the the first part.

Now, we prove the second part.

proof of (2): We need to show that no follower of x has g value b. Suppose, to the contrary, $(x_1, x_2, .., x_n)$ has a follower with g value b. And suppose, without loss of generality, this involves a move in the first game, from $(x_1, x_2, .., x_n)$ to $(x_1', x_2, .., x_n)$. Hence, our assumption says that this is the move that gives the same g value. Then,

$$\begin{aligned} &g_1(x_1') \oplus g_2(x_2) \oplus .. \oplus g_n(x_n) = g_1(x_1) \oplus g_2(x_2) \oplus .. \oplus g_n(x_n) \\ &\Rightarrow g_1(x_1') = g_1(x_1) \end{aligned}$$

This comes from a simple application of the cancellation law. We can keep adding(xor) $g_i(x_i)$ to the right of both sides for $i \in \{2,3,..,n\}$ in descending order, which will result in the equality above. But, this is a contradiction. This means that a follower of x_1 has the same g value as x_1, which is not possible. Hence, whatever we have assumed in the beginning cannot hold and therefore, the second statement must hold. This concludes the proof.

Therefore, $g(x) = b$. This implies that $g = g_1 \oplus g_2 \oplus .. \oplus g_n$, i.e. the Sprague-Grundy value of the addition of these games satisfies the addition principle. This is a very important result in this theory of impartial games. This works with games which are progressively bounded. Any impartial game's Sprague-Grundy value give kind of an identification with a NIM game with the same Sprague-Grundy value.

Lemma: Any impartial game is equivalent to a NIM game where the Sprague-Grundy value is the corresponding Nimber, the number associated with that NIM game.

We will see some applications of this theorem. Consider the subtraction game where G_m is a one pile subtraction game.

So, in a sense, if there is a pile of coins and in a player's turn she can remove any number of coins between one to m. So, the Sprague-Grundy function of this game, denoted by g_m of x is given above, where x is the starting size. To see this, we start from the terminal position, as before.

$g_m(0)$ is 0 as it is a terminal position. Similarly, $g_m(1) = 1$ and till $g_m(m)$ all the values will be 1. If there are $m+1$ coins, then the player to move next can pick a minimum of 1 coin and a maximum of m coins and hence, is going to lose. This being a P position, as we have seen before, will have $g(x) = 0$. This is, in fact, just $m+1$ mod $m+1$. Proceeding inductively, we can see that $g_m(x)$ is x mod $m+1$. Therefore, we have the following:

$$
\begin{aligned}
&g_m(0) = 0 \\
&g_m(1) = 1..g_m(m) = m \\
&g_m(m+1) = 0 = m+1 \;\; mod \;\; m+1 \\
&. \\
&. \\
&g_m(x) = x \;\; mod \;\; m+1
\end{aligned}
$$

where $0 \leq g(x) \leq m$.

Now, let us consider the sum of three subtraction games. For the first game, let $m = 3$ starting with 9 chips. For the second game, we take $m = 5$ and there are 10 chips. The third game has $m = 7$ and the pile has 14 chips. Therefore, the initial position is $(9, 10, 14)$, the Cartesian product of all three positions. Therefore, we are actually considering the game $g_3 \oplus g_5 \oplus g_7$. We calculate the value of $g(9,10,14)$ given by,

$$\begin{aligned} g(9,10,14) &= g_3(9) \oplus g_5(10) \oplus g_7(14) \\ &= 1 \oplus 4 \oplus 6 = 3 \end{aligned}$$

Therefore, by looking at this Grundy value, one optimal move is to change the position in game g_7 to have Sprague-Grundy Value 5. This can be done by removing one coin from the pile of 14, leaving 13 coins. Apart from this there are other optimal moves as well.

So, the importance of this Sprague-Grundy theorem is that when we have an addition of multiple games, one can start looking at each game individually, find their Sprague-Grundy values and then use them to see which game the player needs to play.

Note that, we can have multiple optimal moves in these games. It would be a nice exercise to find the other optimal moves in the above game.