Game Theory
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Lecture 08
Combinatorial Games: Sprague-Grundy Theorem-III

In the previous lecture, we have proved the Sprague Grundy theorem. Recall that the Sprague Grundy theorem identifies any impartial game with a Nim game. We will now see how this happens.

In any Nim game with a single heap, let us denote the value of the heap by $*n$, known as *nimber*. In any impartial game G with Grundy value g, we have that

$$G \cong *g(G)$$

Consider the Nim game with k piles of sizes $n_1, n_2, .., n_k$. This is a P position if and only if the Nimsum of the individual pile sizes is 0. We have proved this in the earlier lectures on Nim games. We assert that a Nim game of multiple piles is equivalent to the sum of the individual games. This comes from the Sprague-Grundy theorem. We define the equivalence of two games as follows:

Definition. Let G and H be two impartial games. We say that $G \approx H$ if, for any other impartial game K, we have the same outcomes in $G+K$ and $H+K$.

This means that if $G+K$ is at an N position, then $H+K$ will also be at an N position. Similarly, if $G+K$ is at a P position, then $H+K$ will also be at a P position. We look at some important properties:
If G is an impartial game, then

- $G+G$ is always at a P position initially. This means that the second player can force a win, in this game. The proof of this is intuitive. We use a similar argument, we used in the proof of Hex being a first player win. If Player 1 has a winning strategy in one, then the second player just copies her strategy in the second game. Thus, by this symmetry argument, we can see that $G+G$ is always at a P position.

- If K is P, then
$$G \approx G+K$$
To see why this happens, we state that the sum of two P positions will also be a P position. The remainder of the proof of how this leads to $G \approx G+K$ is left to the reader as a simple exercise.

The following two statements also hold and their proofs can be solved as simple exercises by the reader:

- The sum of two P positions is always a P position.

- The sum of an N position and a P position is an N position.

Coming back to the equivalence of G and $*g(G)$, note that the Grundy value of $G + *g(G)$ is $g(G) \oplus g(G)$. This implies that $G + *g*(G)$ is always a P position. By the second property mentioned above,

$$G \approx G + (G + *g(G))$$

By associativity, this is the same as

$$(G+G) + *g(G) \approx *g(G)$$

Therefore, any impartial game is equivalent to a Nim game. This is a consequence of the Sprague-Grundy theorem.

So, this implies that if we know how to play a Nim game, we can always play any impartial game by using this consequence of the Sprague-Grundy Theorem. This tells us why this Sprague-Grundy theorem is an important theorem and in fact, characterizes every impartial game. Let us look at one more example.

Consider a one pile Nim game with the following change of rules:

1. Any even number of coins can be removed provided it is not the whole pile.
2. The whole pile can be removed provided that it has an odd number of coins.

We want to find the terminal positions in this game. Note that, trivially 0 is a terminal position. Other than this, 2 is also a terminal position. This is so because, the whole pile has 2 coins and hence one cannot pick the whole pile. Moreover, 2 being the smallest even number, one cannot pick an even number of coins. Hence, the game ends if there are only 2 chips remaining.

Let us see the grundy values of some possible positions.

$$x : 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 ...$$
$$g(x) : 0 \;\; 1 \;\; 0 \;\; 2 \;\; 1 \;\; 3 ...$$

In fact, we can calculate that, for $k \geq 1$

$$g(2k) = k - 1$$
$$g(2k-1) = k$$

Now, as an example, let us look at this game played with three piles of sizes $10, 13, 20$. Therefore, $g(10) = 4$, $g(13) = 7$ and $g(20) = 9$.

$$4 \oplus 7 \oplus 9 = 10$$

The Nimsum of the above is 10, which is not equal to 0. Therefore, this is an N position. Recall that if a player is at an N position, the winning move for the player is to make a move in such a way that the Nimsum becomes 0. Hence, the Sprague-Grundy value is 9. We want to make it 3. Hence, we can remove 12 coins from the pile of 20 leaving 8. Since, $g(8) = 3$ and we want 3 there, instead of 9, as

$$4 \oplus 7 \oplus 3 = 0$$

And as $g(8) = 3$, we want there to remain 8 coins in the heap of 20. This is another example of how the Sprague-Grundy theorem can be used.

So far, we have seen impartial games with normal play. Now we would like to see a game with misére play. Recall, under normal play, the player who makes the last legal move is the winner. Under misére play, the player who makes the last legal move loses the game.

Let us look at an interesting example known as Sylver Coinage. The Sylver Coinage is named after the famous mathematician J.J. Sylvester. In this game, the players alternatively pick natural numbers according to the following rule:

A player cannot pick a number which is the combination of any previously used numbers.

We illustrate this with an example given in the slide below:

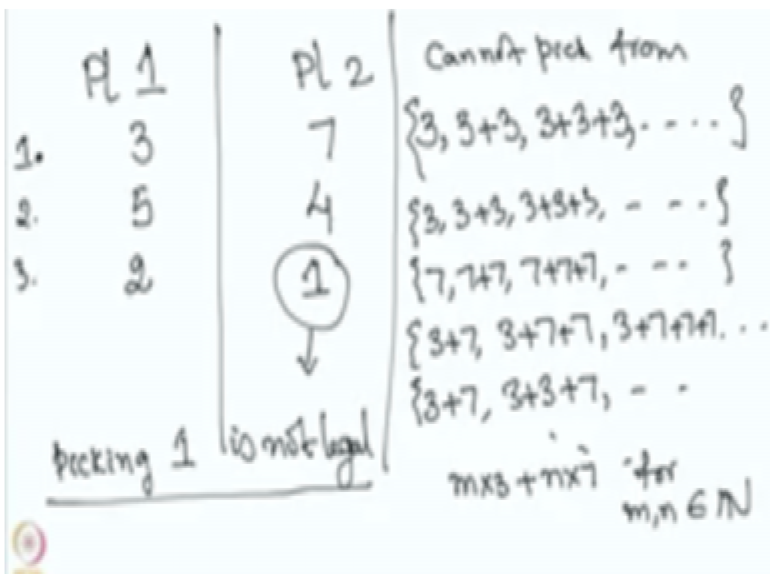

Figure 1: Refer Slide Time: 20:29

Let the first player pick 3. Then, the second player cannot pick 3 or any multiples of 3. In the above example, the second player picks 7 in the first round. Then, in the second round, Player 1 cannot pick any multiple of 3 or 7 or any number which is the sum of multiples of 3 and 7. For example, 20 can be expressed as $23 = 3 \times 3 + 2 \times 7$ and hence, is an invalid move. Player 1 chooses 5 followed by the choice of 4 by Player 2. Player 1 then chooses 2. This leaves only the number 1 to be picked, as every positive number greater than 1 can be expressed as a combination of multiples of 2 and 3.

And once Player 2 picks a 1, there are no more moves. Therefore, whoever picks 1, because any number is a multiple of 1, the game ends. An interesting question would be if normal play would make sense here. If 1 is a choice for a player, they naturally will pick at the very first move and win the game because the second player will not be left with any legal moves. Therefore, this game under normal play makes no sense.

Hence, picking 1 is not legal move here. If a player picks one then he rules out every other possibility. Therefore, the rule should dictate that whoever picks one loses. Now, let us look at a few interesting points about this game. How long can this game go on?

For example, any large number can be picked by players and can be reduced one by one alternatively by each player. Therefore, the game can take any amount of time. Nevertheless, the game can't go on forever. This is due to a result by Sylvester which says that if Player 1 picks a and Player 2 picks b, then every number $c \geq (a-1)(b-1)$ can be represented as $xa + yb$ for some $x, y \in \mathbb{N}$.

For example, if Player 1 picks 3 and Player 2 picks 7, then any number $c \geq 12$ can be written as a combination of 2 and 7 and hence, is not a valid move. This imposes a finiteness on the length of the game. We will formally state and prove the above theorem by Sylvester, in the next lecture.