Game Theory
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Lecture - 9
Combinatorial Games: The Sylver Coinage Game

In the sylver coinage game, two players alternatively announce different numbers. But they are not allowed to name any number which is a sum of previously named ones. For example, if the number 3 is announced already, then all multiples of 3 are not allowed. If 3 and 5 are announced, then all numbers of the form $3x + 5y$ for $x \geq 0, y \geq 0$ are excluded. Note that if a player announces 1, then the game ends immediately. Thus, we consider the Misère play of this impartial game. Therefore, whoever announces 1 or forced to announce 1 is the loser of the game.

How long does this game continue? The simple answer is as long as one wants. It can go e.g., 1,00,000 rounds if the players name $1, 00, 000, 99, 999, \cdots, 3, 2, 1$ sequentially. Another possibility is that first several rounds can be $2^{1000}, 2^{999}, \cdots, 2^2, 2$ and the rest of the game can be as long as you like: $1000001, 999999, 999997, \cdots, 5, 3, 1$. Thus the length can be anything. Thus this game is unboundedly unbounded.

Nevertheless the game can't go forever. It will end due to a result by Sylvester and hence the game is coined Sylver Coinage.

Theorem 1 (J.J. Sylvester)**.** *Let $a, b \in \mathbb{N}$ and $(a, b) = 1$. Then $ab - a - b$ is the largest integer not representable as $xa + yb$ for some $x, y \geq 0$.*

Proof. Let $n \geq (a-1)(b-1)$. Choose x_0, y_0 be integers such that $ax_0 + by_0 = 1$ (exists because $(a, b) = 1$). Therefore there are integers x_1, y_1 such that $ax_1 + by_1 = n$. Without loss of generality, we can assume that $x_1 \geq 0$. Therefore the solutions of $ax_1 + by_1$ are given by $x = x_1 - tb, y = y_1 + ta, t \in \mathbb{Z}$.

Let t be the smallest such that $y_1 + ta \geq 0$. We have

$$a(x_1 - tb) + b(x_1 + ta) = n \geq (a-1)(b-1).$$

Since t is smallest, we must have $y_1 + ta \leq a - 1$. Thus

$$a(x_1 - tb) \geq (a-1)(b-1) - b(a-1) = -a + 1.$$

Therefore $x_1 - tb \geq -1 + \frac{1}{a}$, which implies that $x_1 - tb \geq 0$.

Next we show that $ab - a - b$ is not representable. Suppose

$$ab - a - b = ax + by$$

for some integers x and y. Then clearly $-b \equiv by \mod a$ as well as $-a \equiv ax \mod b$. Therefore $y \equiv -1 \mod a$ and $x \equiv -1 \mod b$. Hence $y \geq a - 1$ and $x \geq b - 1$. Now

$$ax + by \geq a(b-1) + b(a-1) = 2ab - a - b > ab - a - b$$

which gives a contradiction. □

Remark 1. *The theorem with multiple numbers is known as Frobenius problem. Frobenius problem is still unsolved. This problem is also related to the theory of numerical semigroups.*

We will now analyse this game for few moves. The game is still unsolved. If a player names 2, then the other player can name 3 (if available) to win the game. Similarly if a player names 3, other player's best response will be 2. Thus anyone who names a number from $\{1, 2, 3\}$ first is going to lose the game.