Advanced Financial Instruments for Sustainable Business and Decentralized Markets

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Week 4
Lecture - 12

In this lesson, we will discuss single and multi-index models. We will start the discussion with the rational and intuition behind these models, which leads to a simplifying correlation structure. First, we will discuss the expected return and risk characteristics of individual securities. Subsequently, we discuss the portfolio characteristics in the presence of single-index models.

We will also understand the index models with the help of a simple example. We will examine the role of β that is sensitivity of the portfolio towards these indices. Next, we will introduce multi-index models. We will compute the expected returns and risk for individual securities. And then we will derive these for portfolios.

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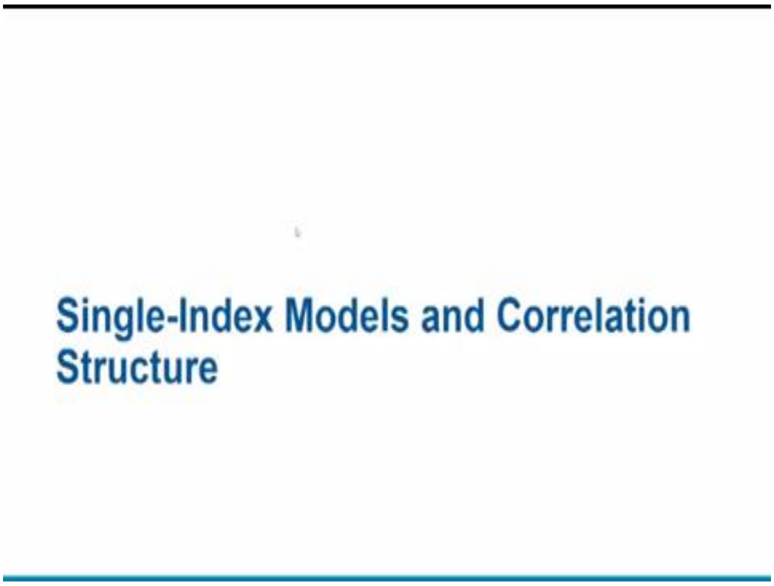

Single-index models and correlation structure. In this video, we will introduce single-index models, their correlation structure, and compare them with a situation where there is no simplifying structure is assumed.

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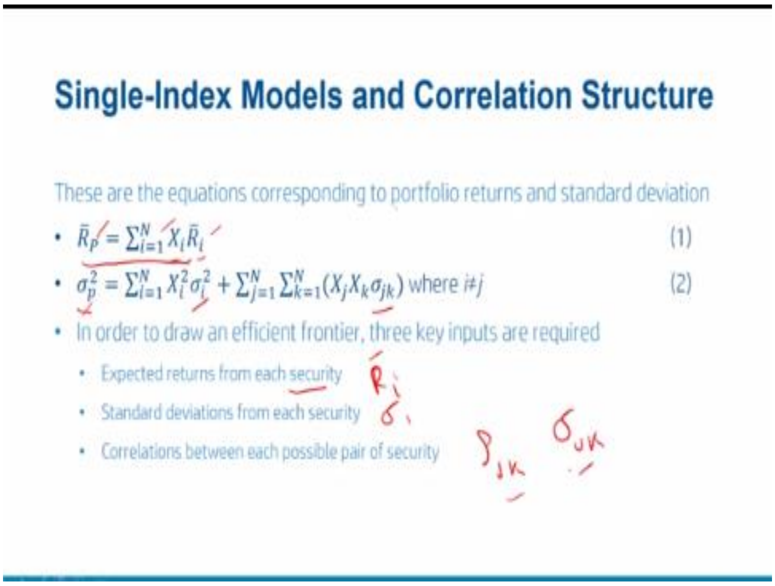

We will begin with the two equations that provide the base to all the analysis in portfolio management. These are the equations corresponding to portfolio returns, which is this one, portfolio returns $\bar{R}_p = \sum_{i=1}^{N} X_i \bar{R}_i$ and standard deviation. These notations are familiar to us. $\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{i=1}^{N} X_j X_k \sigma_{jk}$

Here $\bar{R}_p$ is the expected return on portfolio. X_i is the weight. $\bar{R}_i$ is the individual security expected returns. Σp^2 is the variance of portfolio. Σi^2 are the variance of individual securities and σ_{jk} is the covariance between security j and k. Now for the analysis of portfolio and examination of an efficient frontier, three key inputs are required here in these equations one and two.

First is the expected returns for each security which is a$\bar{R}_i$. Standard deviation for each security which is σi and correlations across each pair of security which is either you can see ρ_{jk} correlation or covariance σ_{jk}. Correlation is just the standardized form of covariance.

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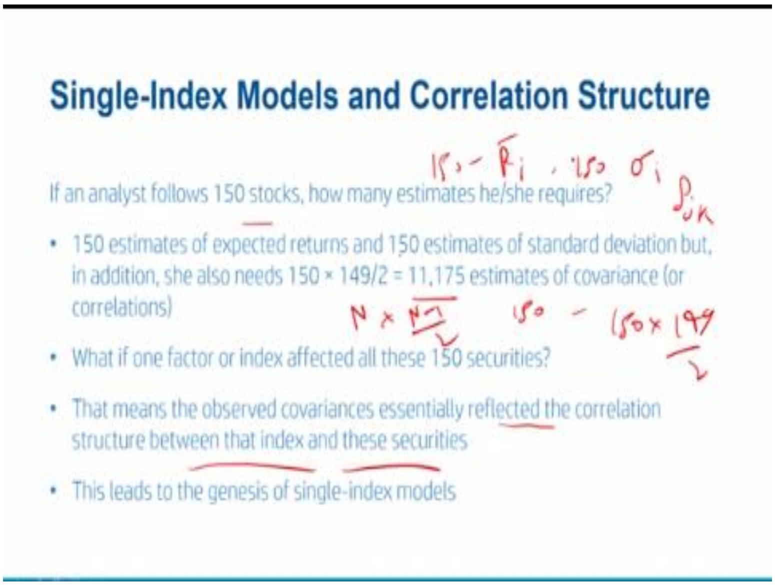

So now if an analyst is following 150 stocks, they need to prepare 150 estimates of expected returns. So 150 estimates of expected return $\bar{R}_i$, 150 estimates of standard deviation, that means σ_i's. But the correlation structure and correlation requirements are much intensive. That means ρ_{jk}'s and for example, if you have n stocks you need n into n minus 1 upon 2 correlations.

For example here you have 150 stocks, so you need 150 into 149 upon 2 correlations, which is 11,175. So we need 11,175 inputs for correlation coefficients. This kind of mammoth nature of correlation coefficients has motivated researchers to look for correlation structure across securities to simplify the analysis. How? Consider N variables to track.

It is difficult, but if one finds that 90% portion of these variables is explained by one single variable or one single index in this case, then you only need information about that one variable to predominately define or get information about all the N variables. And this leads to the genesis of single-index models. That means, the observed covariances essentially reflected the correlation structure between that index and these securities.

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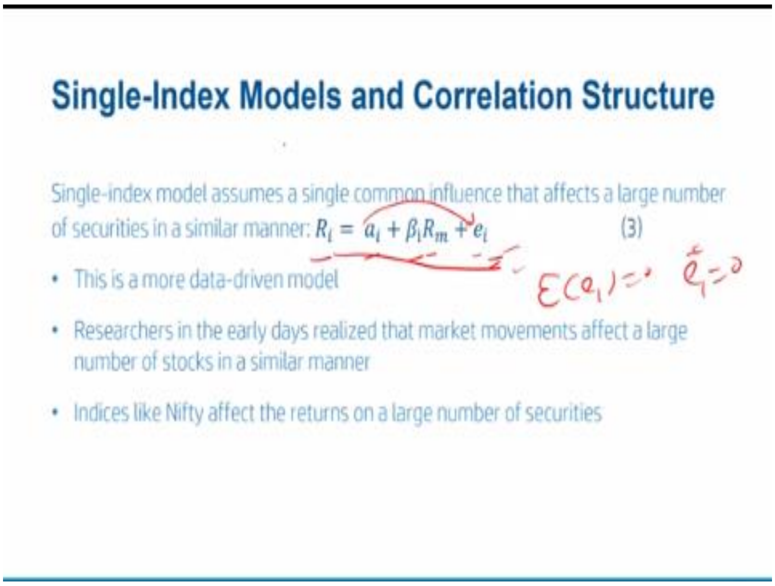

This search for correlation structure across securities has led to an important set of models called single-index models. These models assume a single common influence that impacts or affects large number of securities simultaneously in a similar manner. The single common factor is responsible for the core movement of large number of securities, thus simplifying the correlation structure to a large extent.

The model is based on evidence that most of the stocks follow market movements to a certain degree. For example, when market goes up, a large number of stocks go up and vice versa. This leads to the genesis of single-index models, where market index such as nifty or S&P 500 acts as the proxy of the single-index that is market index.

Therefore and hence this returns on the security, returns on a security can be described by a single-index model with this equation. Here R_m is the return on market index. R_i is the return on the security, e_i is the error term which is specific to the security, β is the constant which measures the return in R_m for a given change in R_i. For example, a β_i of 2 would indicate that 1% change in market would cause 2 percentage change in stock's return.

Again, this a_i is also the part of return that is unrelated to market performance. So this a_i and e_i are two components that are not related to market. e_i can be written as or said to be the random probabilistic element of the stock return with the expected value of 0. So expectation of e_i is 0 or $\bar{e}_i$ is assumed to be 0 here.

To summarize, in this video, we examined how the assumption of single-index models simplifies the correlation requirements considerably. We also discussed how these single-index models help us define the expected returns from a security in the form of a single influence or single factor like market in a more simplistic manner.

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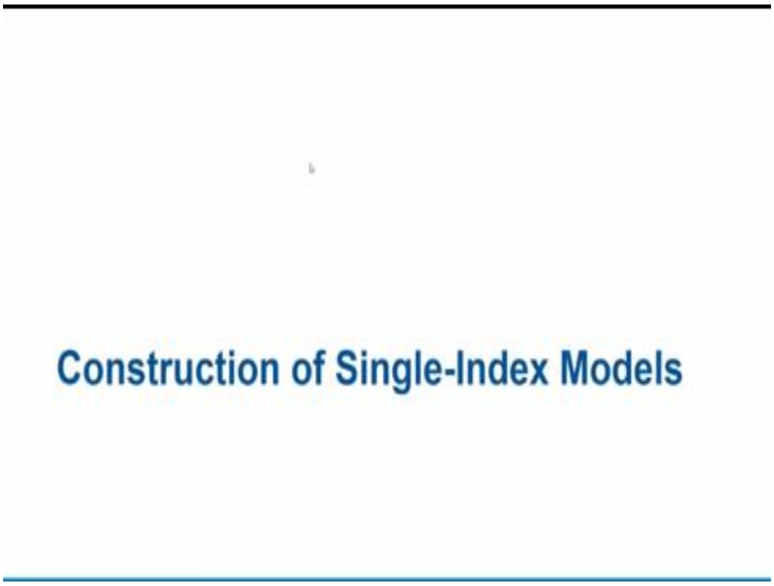

Construction of single-index models. In this video we will provide the definition of single-index models and various aspects related to that definition.

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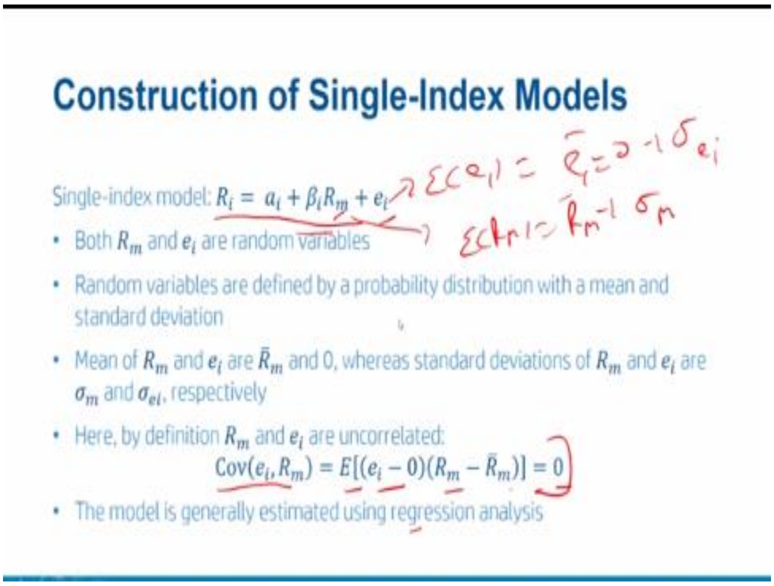

The basic equation of single-index models can be simply written as $R_i = \alpha_i + \beta_i * R_m + e_i$. Important to note that R_m and i are random variables. This R_m and e_i are random variables. Often random variables are defined by a probability distribution like a normal distribution with a mean and standard deviation. In this case, consider the mean of R_M and e_i as 0.

So mean of e_i as 0 as we discussed earlier. Expected value of e_i or mean of e_i as 0 and mean of R_m or expected value of R_m as $\bar{R}_m$. And their standard deviations as σ_m and σe_i. We have also said that R_m and e_i are uncorrelated and therefore if they are uncorrelated, therefore the covariance between e_i and R_m should be 0. Or in other words, the correlation between e_i and R_m should be 0.

That is E ((e_i – 0) *(R_m - $\bar{R}_m$)) should be equal to 0. Generally, this equation, this model is estimated with the regression analysis. In regression analysis by design to a great extent it is guaranteed that e i and R m are correlated, not correlated over the period of analysis. So over the period of analysis, this assumption to a great extent is held by the model itself by the design or construction, we can say.

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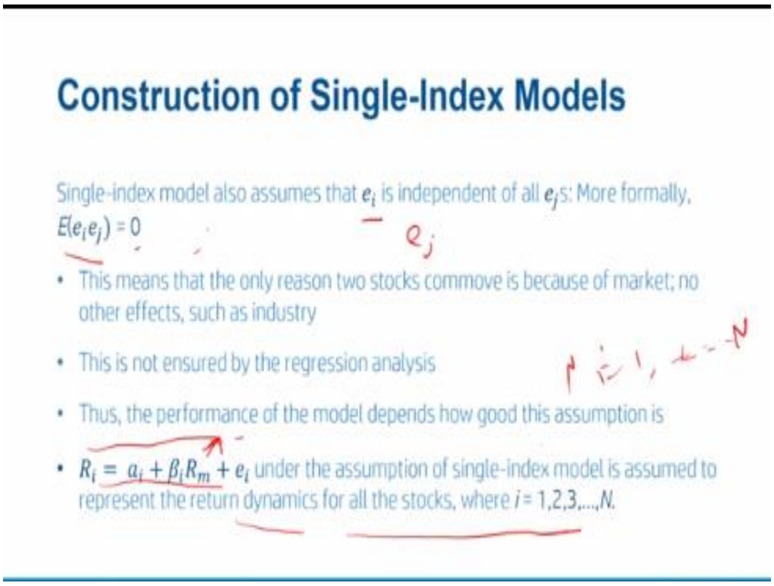

However, there is one additional assumption of single-index model that is not guaranteed by regression analysis. This is as follows. Single index model assumes that e_i, the error e_i is independent of e_j for on the security. Or more formally one can say the expectation of e_i e_j 0 or correlation between errors e_i e_j is 0. This requires that the only recent stock should commove is because of the market movement.

There are no other effects like industry effects, etc., that can cause correlation across stocks. So there is nothing in regression analysis that ensures it. And therefore the performance of the model depends on how close this assumption is to reality. This model $R_i = \alpha_i + \beta_i * R_m + e_i$ under the assumption of single-index model is assumed to represent the dynamics of all the stocks in the market.

That is, if there are N stocks i equal to 1 and so on 2, 3 and up to N. All those stocks the return is expected to be defined by the single-index model. Generally, it is given that or assumed by default that the single-index would be market factor because market factor there is a wide evidence that market factor affects most of the stocks in the market systematically.

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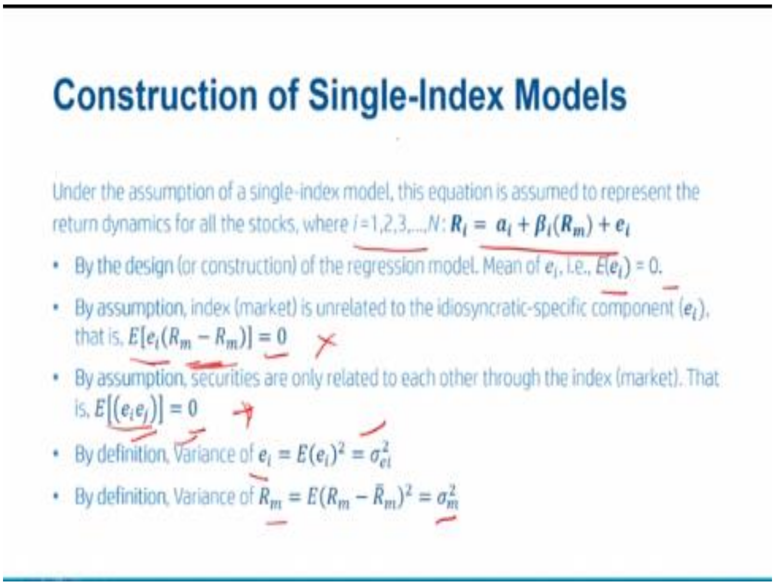

Let us now define the single-index model. Basic question of single-index model we have already seen. This is the basic index equation. Under the assumption of single-index model, this equation is assumed to represent the return dynamics of all the stocks in the market. Now, by design or you can say by construction of the regression model, the mean of error term or residual term is 0.

This is by design insured by regression model. Then by assumption, the index is unrelated to the idiosyncratic component which is e_i or the error term that is expectation of e_i and market is 0. That means market returns and error terms are not correlated. There is no correlation between error and market or residual or stock specific term and market. Also securities are only related to each other through the index which is market.

That is, there is no correlation across securities between two stocks. Between two stocks, the residual terms or stock specific term, there is no correlation or covariance in other words. However, this assumption is not insured by regression model as we discussed earlier. Also by definition will define the variance of error term or stock specific residual or stock specific component of return as σe_i^2 and the variance of market is defined as σ^2_m.

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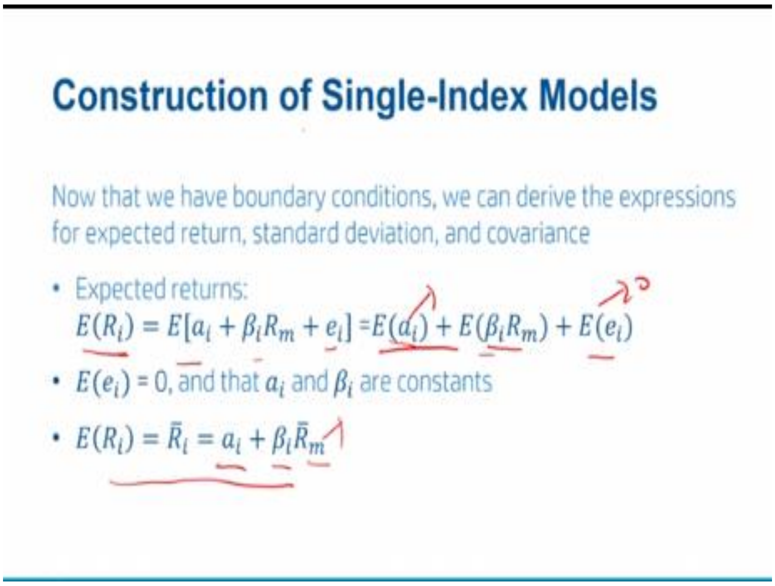

Now that we have boundary conditions, we can derive the expressions for expected return and standard deviation and covariance easily. Expected returns under the single-index model can be easily obtained by this expectations operator on $E(R_i) = E[\alpha_i + \beta_i R_m + e_i] = E(\alpha_i) + E(\beta_i R_m) + E(e_i)$

Now expectations of e_i is 0, we already discussed that. Since α is constant time, so expectations of α_i will be α itself and expectations of $\beta_i * R_m$ should be β_i because β_i is a constant and $\bar{R}_m$. $\bar{R}_m$ is the expectation of market return. So this is our single-index model, expected returns definition for single-index model.

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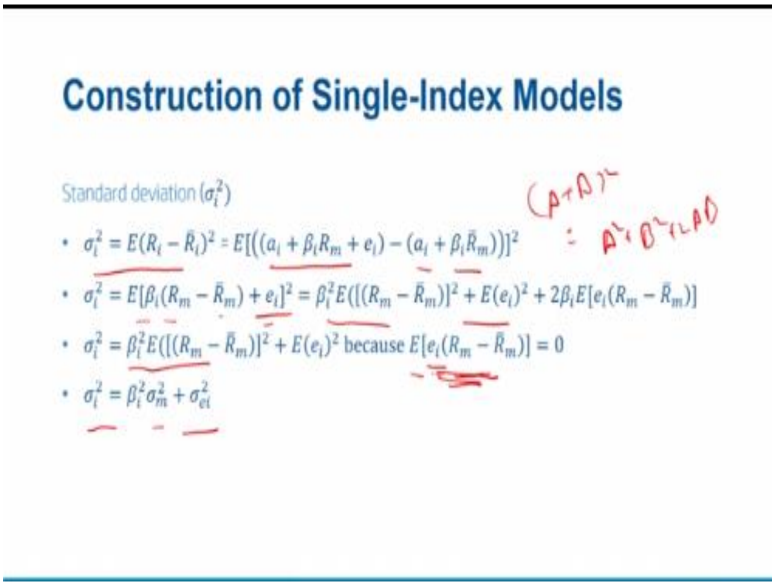

Coming to standard deviation, standard deviation of return can be easily given by this formula, $\sigma_i^2 = E(R_i - \bar{R}_i)^2$. Now we can extend the expression for returns, the return definition here, which is $(\alpha_i + \beta_i R_m + e_i)$. And $\bar{R}_i$ we already know $\alpha_i + \beta_i \bar{R}_m$.

Simplifying this expression, we get expectations of β_i times R_m - $\bar{R}_m$ + e_i^2. This can be expanded in the form of A plus B whole to the power 2 equal to A square plus B square plus 2AB. In the similar manner, we can expand this expression.

And here please note we will obtain this expression β_i^2 expectations of R_m minus $\bar{R}_m$ raised to the power 2 plus expectations of e_i^2 because, the third term which is expectations of e_i times R_m minus $\bar{R}_m$ will be 0 because by assumption there is no correlation between e_i error term and market term and to a great extent this is ensured by the regression model as well.

So the resulting expression for standard deviation becomes σ i square equal to β i square times σ m square plus σ e i square. Notice it does not include a common term which is the error and R m because by assumption as well as to great extent by the design and construction of the regression model, there is no correlation between error and market returns.

$$\sigma_i^2 = E(R_i - \bar{R}_i)^2 = E\left[\left((\alpha_i + \beta_i R_m + e_i) - (\alpha_i + \beta_i \bar{R}_m)\right)\right]^2$$

$$\sigma_i^2 = E[\beta_i(R_m - \bar{R}_m) + e_i]^2 = \beta_i^2 E\left((R_m - \bar{R}_m)\right)^2 + E(e_i)^2 + 2\beta_i E[e_i(R_{m-}\bar{R}_m)]$$

$$\sigma_i^2 = \beta_i^2 E\left((R_m - \bar{R}_m)\right)^2 + E(e_i)^2 \ beacause \ E[e_i(R_{m-}\bar{R}_m)] = 0$$
$$\sigma_i^2 = \beta_i^2\sigma_m^2 + \sigma_{ei}^2$$

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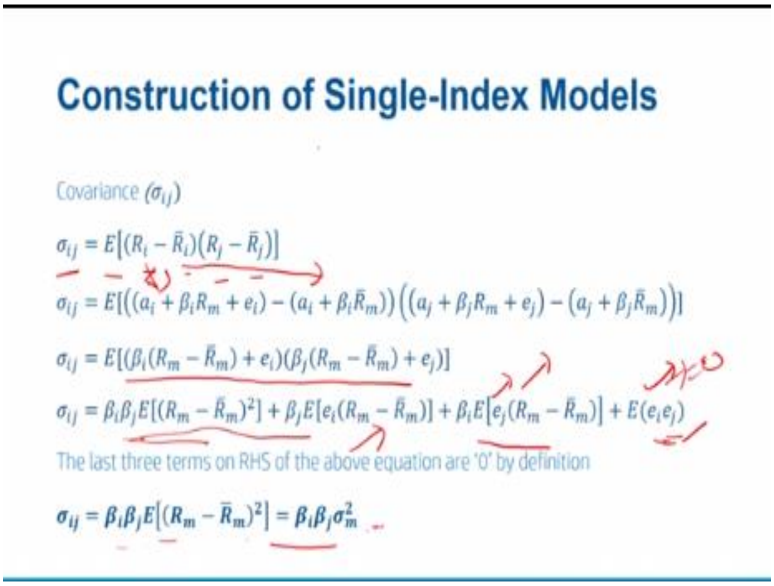

Lastly, coming to the covariance part. The covariance between two securities i and j can be easily defined as $E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)]$. Further, again expanding for Ri, Ri can be expanded in this form by the single-index model definition. Similarly, $\bar{R}_i$ can also be expanded with the definition of single-index model. So we get the resulting term like this.

This term can be further opened and simplified in these terms. Please notice across these terms the expectation of ei ej is 0. This is by assumption. Regression model does not exactly hold this. But by assumption, since market is the only influencing factor, therefore there should be no correlation across error terms or idiosyncratic part of return. So this is 0.

By design of regression model and by assumption error terms are not correlated with market. So this correlation or covariance is also equal to 0. And same goes for this. So we have only this term remaining which is $[\left((\beta_i(R_m - \bar{R}_m) + e_i)(\beta_j(R_m - \bar{R}_m) + e_i)\right)]$. This is the expression for covariance.

To summarize, in this video, we discussed how the assumption of single-index models results in the definition of return and expected return for a security. We also discussed how we can compute the risk or standard deviation and covariance between two stocks in the presence of or the assumption of single-index models.

And we saw how this kind of analysis considerably simplifies the correlation structure and results in a model where computation of risk and return is very simplified.

$$\sigma_{ij} = E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)]$$

$$\sigma_{ij} = E[((\alpha_i + \beta_i R_m + e_i) - (\alpha_i + \beta_i \bar{R}_m))\left((\alpha_j + \beta_j R_m + e_j) - (\alpha_j + \beta_j \bar{R}_m)\right)]$$

$$\sigma_{ij} = E[\left((\beta_i (R_m - \bar{R}_m) + e_i)(\beta_j (R_m - \bar{R}_m) + e_i)\right)]$$

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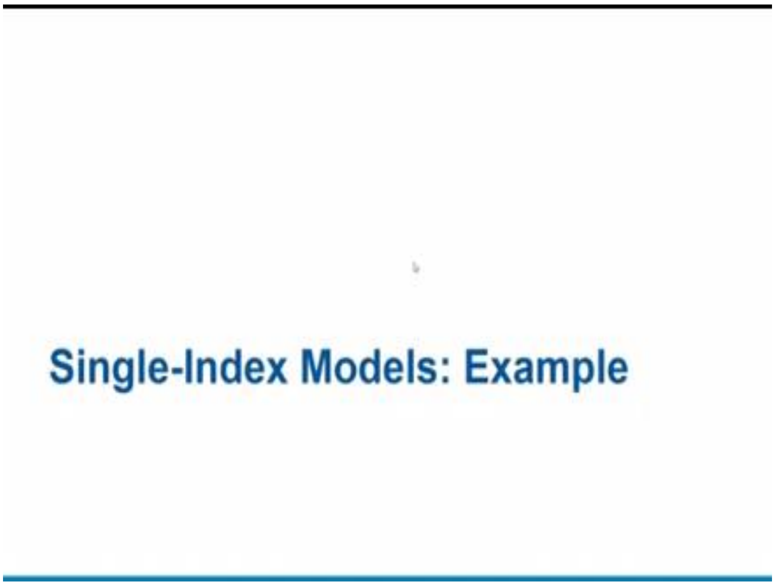

Single-index models with an example. In this video, we will examine the application of single-index models with the help of a numerical example.

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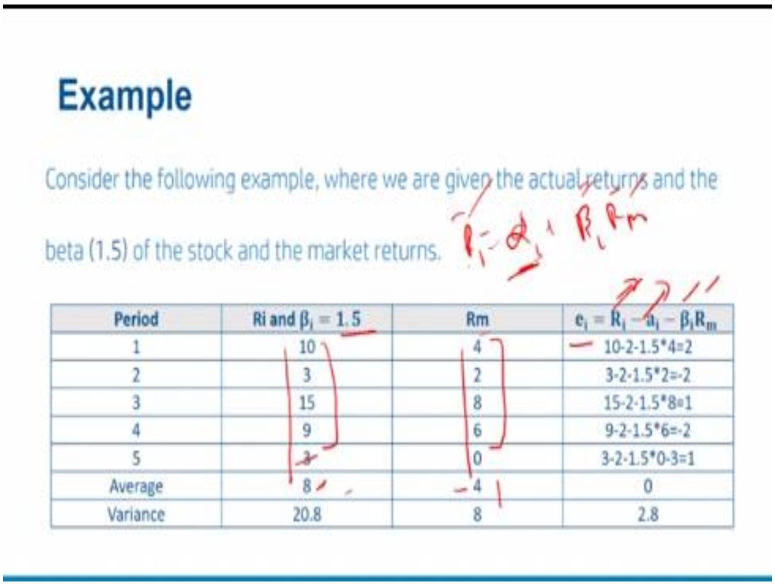

Period	Ri and β_i = 1.5	Rm	$e_i = R_i - a_i - \beta_i R_m$
1	10	4	10-2-1.5*4=2
2	3	2	3-2-1.5*2=-2
3	15	8	15-2-1.5*8=1
4	9	6	9-2-1.5*6=-2
5	3	0	3-2-1.5*0-3=1
Average	8	4	0
Variance	20.8	8	2.8

Consider the following example, where we are given the actual returns and the β for a stock and market returns. So you have β equal to 1.5 and the actual returns are given here and the returns from market are given here. To perform this numerical, first we need to compute the average returns for the stock which are computed here. The average of these stocks to get the average return.

And we also compute the average returns for the market here. Once we have the average return for stock and market here, we can solve the model which is the single-index model $\alpha_i + \beta_i * \bar{R}_m$. We can solve this equation, because we know here R_i, we know here R_m, we know the β, we can solve this to get the estimate of α_i.

Once you have the estimate of α_i, you can then have all the entities known here in this model, which is the actual returns R_i or predicted returns R_i - a_i – β_i*R_m. You have R_i known. This α_i or alternatively a_i, what we are referring to is also known. B is known and R_m is known. So you can compute the residuals also and define the model completely for each period.

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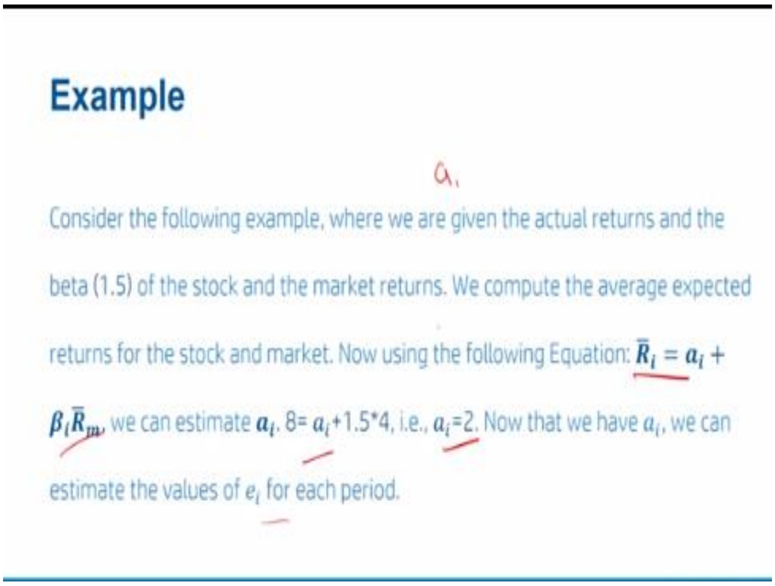

So using this equation, we can solve for a_i to get a_i as 2% or α_i. I am referring to a_i or α_i interchangeably. Now that you have α you can estimate the values of a_i as we discussed.

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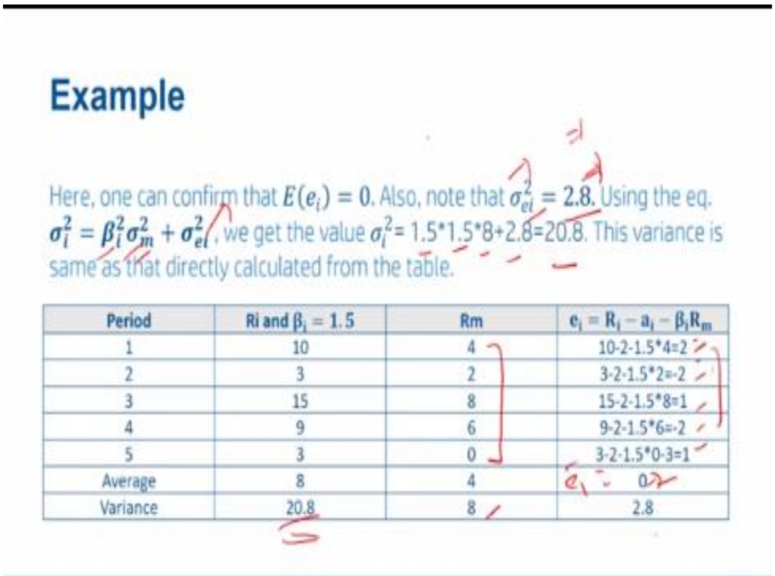

Period	Ri and $\beta_i = 1.5$	Rm	$e_i = R_i - a_i - \beta_i R_m$
1	10	4	10-2-1.5*4=2
2	3	2	3-2-1.5*2=-2
3	15	8	15-2-1.5*8=1
4	9	6	9-2-1.5*6=-2
5	3	0	3-2-1.5*0-3=1
Average	8	4	0
Variance	20.8	8	2.8

So while computing these values, we get these values of e_i for each of the periods using these e_i's and we can check here the average of e_i is 0, as we have assumed. A lot of it is ensured by regression model. You get average of e_i or $\bar{e}_i$ as 0. You can compute the variance of e_i as 2.8. So we have the variance here. Using these values, we can compute the variance of e_i which is σ square e_i.

This is also the residual component of risk in the model. Now that you have the residual component of risk, you can either compute separately the overall variance which is 20.8, which is the total risk which includes the market part of risk as well as idiosyncratic part of risk. Also you can cross check it by computing the idiosyncratic part which is $\sigma^2 e_i$, 2.8 separately and market part of risk.

Market part of risk is $\beta_i^2 * \sigma_m^2$, which can also be computed for σ^2. And you can compute the variance of market using these values which we can compute variance of market, which is 8 here. And then you can multiply it by β_i^2, which is 1.5 into 1.5 into 8 plus 2.8, which is 20.8, which again we indirectly compute it and directly also we compute it and the numbers are matching.

To summarize, in this video, we understood the application of single-index model for a given security. We also cross checked the implications of applying single-index model by directly and indirectly computing the values for risk and expected return.

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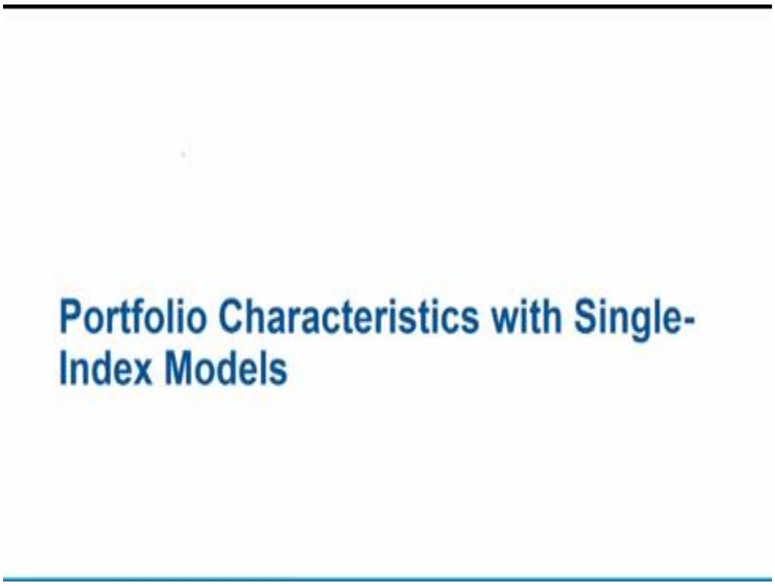

Portfolio characteristics with single-index models. In this video, we will discuss various portfolio characteristics such as expected return and risk in the presence of single-index models.

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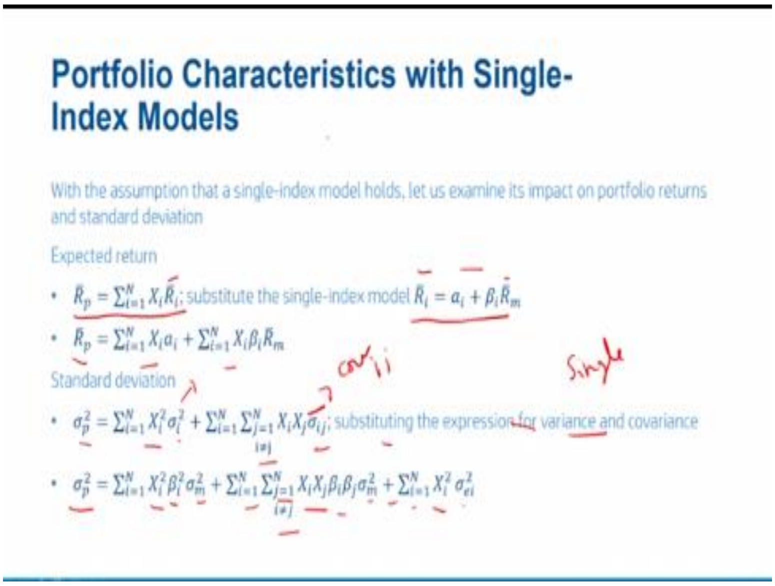

Now with the assumption that the single-index model holds, let us examine its impact on portfolio returns and standard deviation. The expression for expected returns on portfolios is provided by this equation $\bar{R}_p$ equal to summation X_i $\bar{R}_i$, i equal to 1 to N. Assumption here is that this kind of single-index models which is $\bar{R}_i$ equal to a_i +β_i *$\bar{R}_m$ holds where R_i is the expected return on security and $\bar{R}_m$ is the expected return on market. $\bar{R}_p = \sum_{i=1}^{N} X_i \bar{R}_\iota$

$$\bar{R}_p = \sum_{i=1}^{N} X_i a_i + \sum_{i=1}^{N} X_i \beta_i \overline{R_m}$$

Substituting for this expression, the portfolio return expression becomes $\bar{R}_p$ equal to summation Xi ai plus summation X i β i R m bar. Also the variance of the portfolio can be written as X i square σ i square summation i equal to 1 to N. And submission i and j over X i σ ij, i not equal to j where σ ij is the covariance term and σ i square is the variance of security i. $\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{i=1}^{N} X_j X_k \sigma_{jk}$

Σ ij is the covariance between security i and j. Now, substituting the expression for variance and covariance in the presence of single-index models as we saw earlier, the resulting expression becomes σ p square equal to summation i equal to 1 to N, x i square β i square σ m square plus summation i equal to 1 to n and j equal 1 to n where i not equal to j X i X i β i β j σ m square plus i equal to 1 to N x i square σ e i square.

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{j=1}^{N} \sum_{i=1}^{N} X_i X_j \beta_i \beta_j \sigma_m^2{}_{i \neq j} + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$$

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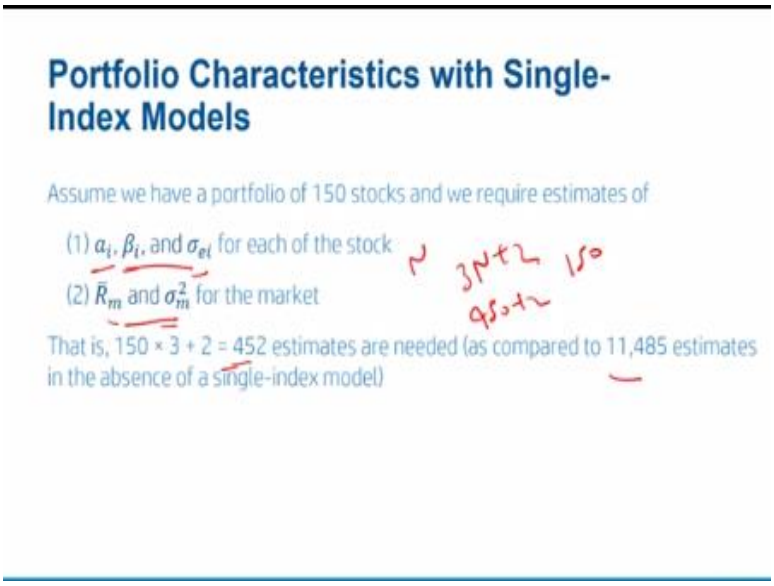

Looking at these equations of portfolio risk and return, it should be clear that we require estimates of αi, βi and σei for each of the stock and estimates of Rm bar and σm square for market. Now if you need N securities, then 3N estimates for these three terms and 2 estimate for these two terms. Overall 3N + 2. So if you believe that there are 150 securities that you want to try, you need 450 + 2, 452 estimates.

As compared to the 11,485 estimates, when we did not assume any kind of correlation structure like single-index or multi-index model, this is a very small requirement, a very simplifying structure that we have assumed here.

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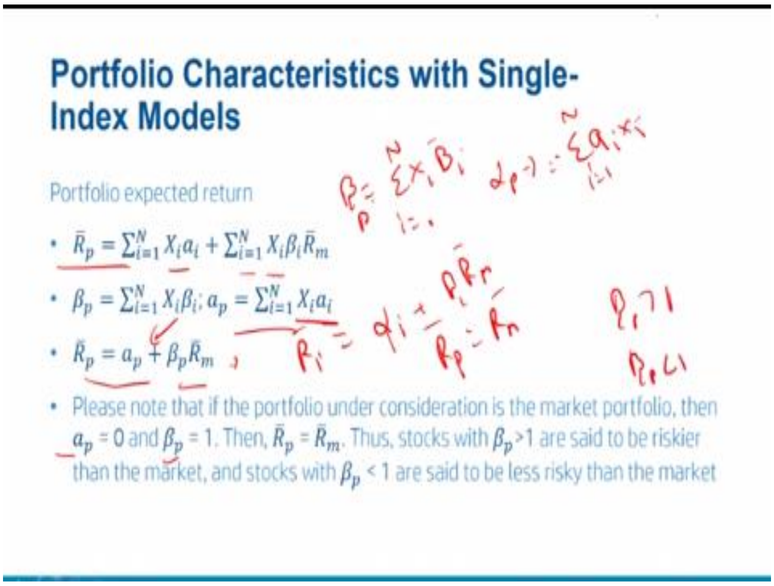

Let us further simplify the portfolio expected return in these expressions. We can define the β of a portfolio which is β p as weighted average of β i over all the securities where i equal to 1 to N. Similarly, α p can also be defined as summation a i into X i, i equal to 1 to N. And therefore, the resulting expression for portfolio becomes X i α i summation over i equal to 1 to N and X i β i R m bar summation i equal to 1 to N.

And therefore, if I put this expression as α p and this expression as β p, again R p equal to α p plus β p times R m bar, which is very similar to the equation that we already saw, which is α i plus β i times R m bar. Please note here in this equation, that if the portfolio under consideration is market portfolio, then α p equal to 0 and β p equal to 1.

$$\bar{R}_p = \sum_{i=1}^{N} X_i a_i + \sum_{i=1}^{N} X_i \beta_i \overline{R_m}$$

$$\beta_p = \sum_{i=1}^{N} X_i \beta_i; \quad a_p = \sum_{i=1}^{N} X_i a_i$$

$$\bar{R}_p = a_p + \beta_p \bar{R}_m$$

That means, R p bar is nothing but R m bar. Also the stocks with β p greater than 1 are said to be riskier than the market. And stock with β p less than 1 are said to be less risky than market.

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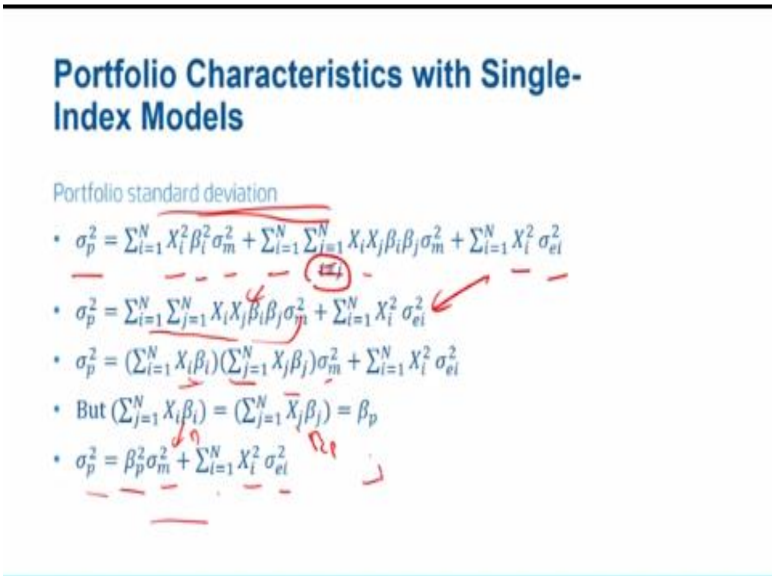

Now let us examine the risk of a portfolio using the equation we saw earlier, σ p square equal to X i square β i square times σ m square plus i equal to 1 to N and j equal to 1 to N summation X i X j β i β j σ m square plus summation i equal to 1 to N X i square σ i square. Let us simplify this equation a little bit.

We can combine these two terms so that this i is not equal to j expression will be removed and this resulting expression is received which is just a combination of these two expressions and this expression remains same. Now again here we can replace this expression or divide this expression into these two components X i β i summation 1 to N and X j β j summation 1 to N times σ m square.

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{j=1}^{N} \sum_{i=1}^{N} X_i X_j \beta_i \beta_j \sigma_m^2{}_{i \neq j} + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$$

$$\sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$$

$$\sigma_P^2 = (\sum_{j=1}^{N} X_i \beta_i) (\sum_{i=1}^{N} X_j \beta_j) \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$$

$$(\sum_{j=1}^{N} X_i \beta_i) = (\sum_{i=1}^{N} X_j \beta_j) \sigma_m^2 = \beta_p$$

$$\sigma_P^2 = \beta_p^2 \sigma_m^2 + \sum X_i^2 \sigma_{ei}^2$$

Now please note that these are nothing but β p. So this is also β p, this is also β p. And therefore, the resulting expression is σ p square times β p square σ m square plus X i square σ e i square summation i equal to 1 to N. This is a very interesting expression of standard deviation for portfolio in the presence of single-index models.

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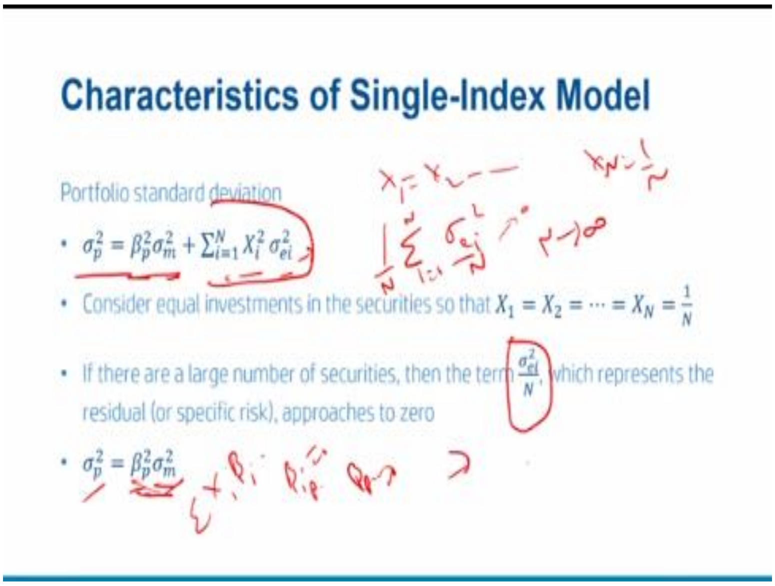

Let us carefully examine this expression of portfolio standard deviation. The expression says X σ p square is a function of β p σ and X i square σ a square. If you assume that there are large number of securities and equal amounts are invested in them that is X 1 equal to X 2 and so on equal to X n equal to 1 upon N, then I can further simplify this expression as σ 1 upon N into σ e i square upon N i equal to 1 to N.

$$\sigma_P^2 = \beta_p^2\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{ei}^2$$

Now in this expression if N is very large, then this expression tends to zero. This expression tends to zero. And therefore, the stock specific or the residual component of risk which is this, it approaches to zero and only this expression remains. What it means is that the remaining risk which is β p square σ m square, it is only on account of that single-index or market.

Irrespective of the risk of securities, the risk of a well-diversified portfolio with a large number of securities is only determined by this market risk and the sensitivity of portfolio to this market risk. Here β p is just a weighted average of individual's security sensitivity that is β i X i. So it is just a weighted average, this β p.

That means, for a well-diversified portfolio, what matters is the contribution of securities to the portfolio risk which is how sensitive are the securities in the portfolio to the market movement. For example, consider the hypothetical situation in a well-diversified portfolio, all securities have β equal to zero, let us say β is equal to zero.

Then the risk of the portfolio will also be zero in the assumption of single-index model. So β p is also built up i's are zero then β p is also zero and therefore, this σ p square becomes zero. Therefore, we are able to say that because we know that each securities contribution to market risk in this portfolio is zero and residual risk, which is the diversifiable risk which is also zero.

Here β p square σ m square is the measure of portfolios non diversifiable or systematic risk. And therefore σ p, σ m the σ m is independent of securities and only the term β p is dependent on the securities and on average reflects the sensitivity of the portfolio constituent towards market risk.

To summarize, in this video, we discussed the expected return computation and risk computation of this security and portfolios in the presence of single-index model. We found that the expression for expected return is quite similar to the one with single-index model which is R p equal to α p plus β p times R m bar, which is very similar to the one we saw earlier R i bar equal to α i plus β i times R m bar.

We also saw that the risk of a portfolio the center division has two component and if the portfolio is well diversified, the idiosyncratic component becomes zero. And the only component of risk that matters is the non-diversifiable risk, which depends on the market risk and the sensitivity of individual securities that contribute to β p or sensitivity of the portfolio to the market risk.

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Single-Index Models: Market Model. In this video, we will discuss special class of single-index models, which is market model.

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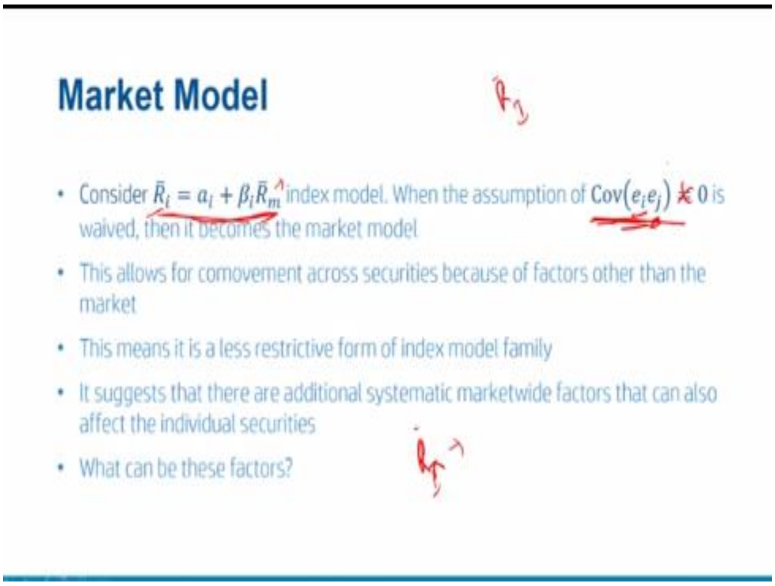

A less restrictive form of single-index model known as market model has found wider acceptance. The only difference here is less restriction. Consider a single-index model like R i bar equal to α i plus β i times R m bar. Here the assumption that covariance between e i e j equal to zero, which was a very integral and subtle assumption of the index model is relaxed or waived, and it becomes the market model.

$$\bar{R}_i = a_i + \beta_i \bar{R}_m$$

This simply means that there may be some co-movement across securities that are not due to the market. This is in contrast to the single-index model. The relationship between the expected returns on the security and the market remains the same which is this relationship. However, since the assumption of covariance is not the same as the single-index model, we do not get the same simplified portfolio expression.

What are the additional factors that may affect the returns of a security in addition to market index? In addition to this R m bar, there can be industry specific factors for example, industry or factors for example, international factors related to some international markets specific to regulatory changes or certain niche factors, which are not part of this market factor, but still can affect the returns of a security.

Market model allows the presence of these factors and says that there can be possibly these factors which can cause correlation between the error term or auto correlation between the error term, between two errors like e i e j, because of these factors like industry factor. And they will not be on account of market factor. So this is market model which is a special class of single-index model which assumes the covariance between e i and e j is not equal to zero, it can be nonzero.

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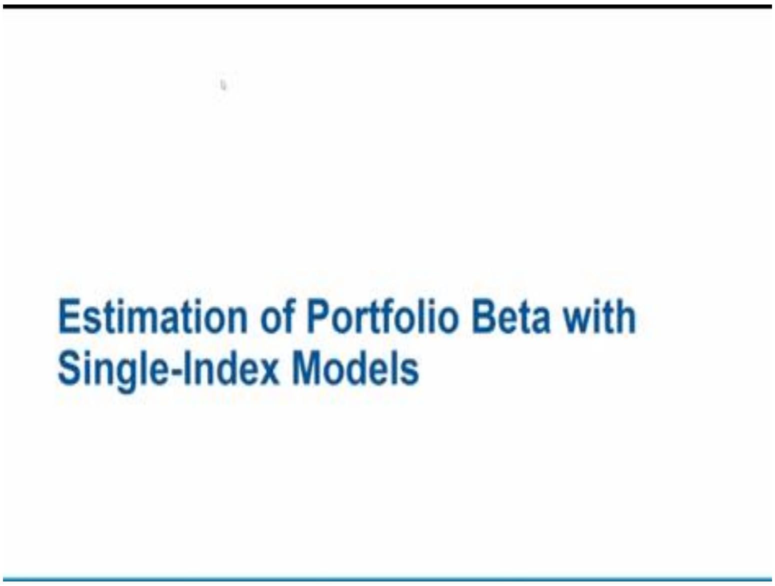

Estimation of portfolio β with single-index models.

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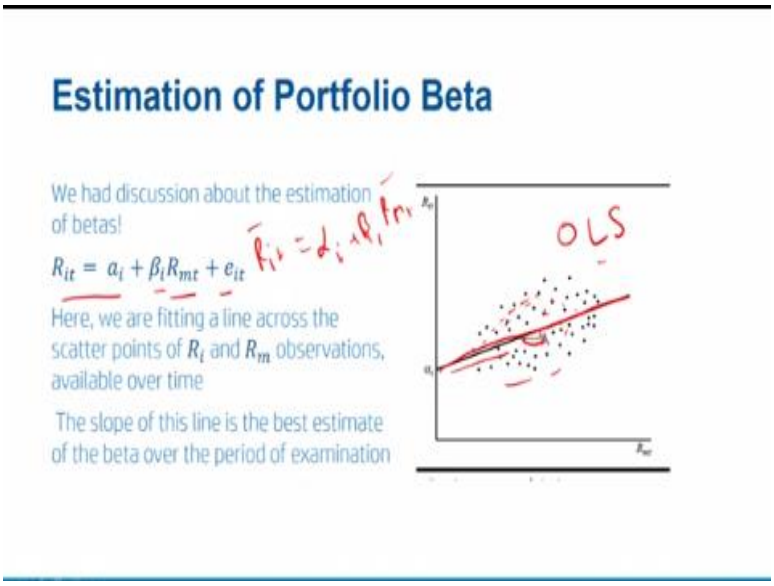

The discussion on single-index model makes it amply clear that one needs to estimate β. One of the simple and most commonly employed methods relies on the estimation of historical βs. Here we make use of this very simple index model R it equal to α i plus βi times Rmt plus e it. And we try to fit this regression model. These are the scatter points, these are our observations.

And we try to fit this model R it bar equal to αi plus βi times Rmt bar. We fit this regression model. The model can be represented in the form of this straight line. The slope of this line which is βi is our estimate of β. So we are fitting a line across these scatter points with simple often used OLS ordinary least square method of regression.

So this is called method of ordinary least squares. And this slope of this line is the best estimate of β over the period of examination.

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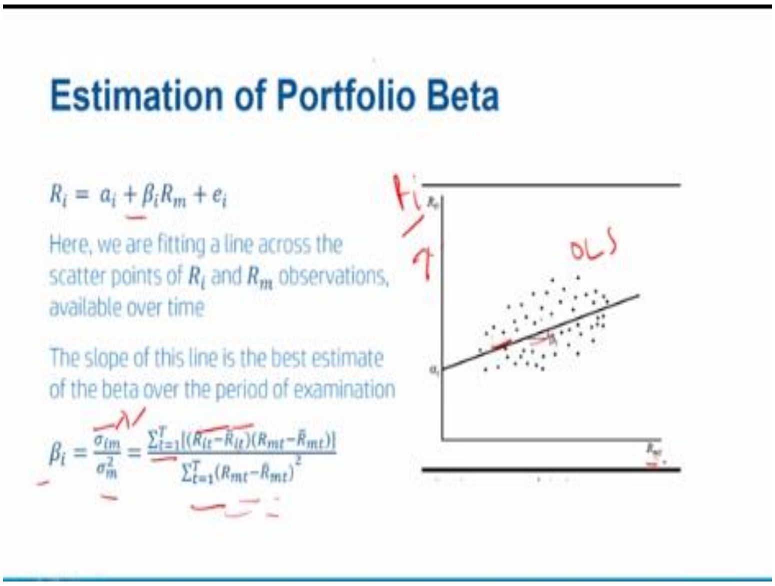

Now, here since we are fitting the line across scatter points of Ri, this is the R it or often called Ri. So these are the security i's written over different times and Rmt which is the market return or different time t. We are fitting this line and this expression we can observe Ri and Rm from the various databases which are directly available from exchanges or databases like Bloomberg.

$$R_i = a_i + \beta_i R_m + e_i$$

This plot of Ri and Rm will result in this kind of figure, the line fit between Ri and Rm observations available over time. And for drawing the straight line like I said, we often use a measure called ordinary least square measure. The slope of this line is the best estimate of β over the period of examination.

Now, once you have this kind of expression, the value of β is very easily computed in the form of this expression, which is nothing but σ m which is the covariance between security i and m and divided by σ square m which is the variance of market. This is nothing but this R it minus R it bar into R mt minus R mt bar summation over t equal to 1 to T and R mt divided by R mt minus R mt bar raised to the power 2 summation over t equal to 1 to T. So this is the expression which is driven by this regression model.

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{\sum_{t=1}^{T}[(R_{it} - \bar{R}_{it})(R_{mt} - \bar{R}_{mt})}{\sum_{t=1}^{T}(R_{mt} - \bar{R}_{mt})^2}$$

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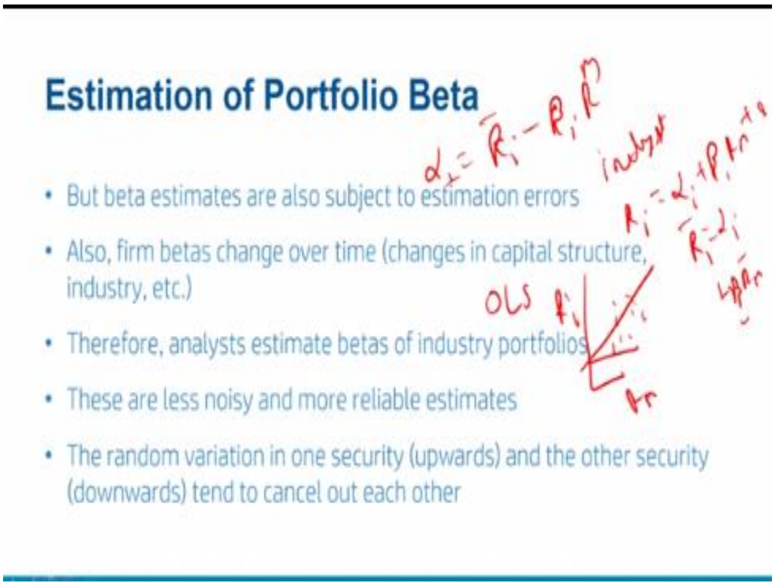

Now once your β is estimated, you can also estimate the α i simply by this equation α i equal to R i bar minus β i times R m bar, we can estimate α i from here. But these β estimates are also subject to errors. Also the fact that firm characteristics such as capital structure, industry etc., change over time. Hence firm β is also expected to change.

$$a_i = R_i - \beta_i R_m$$

Generally to make more accurate estimates of β analysis compute β of portfolio. So we tend to work with β of portfolios, not single securities, where portfolios are selected from specific industry. So these industry βs or industry portfolio βs are less noisy and more informative or more informationally efficient. So portfolio βs are less noisy and subject to less estimation error.

Generally these are computed for industry specific portfolios. This happens because the randomness that is specific to security averages out in the portfolios. For example, there may be some random variation that is upward in one security and downward in the other. Overall, this effect is canceled out and therefore, historical βs of portfolios are often employed for future production of security βs.

To summarize, in this video, we discussed the estimation of β through ordinary least square method where we fitted the scatter points. We fitted a straight line here, the slope of this fitted line between Ri, which is a security return and market return is often referred to as β which represents this kind of model Ri equal to αi plus β times Rm plus error term.

The fitted line equation is Ri bar equal to αi plus β times Rm bar and β is our slope estimate.

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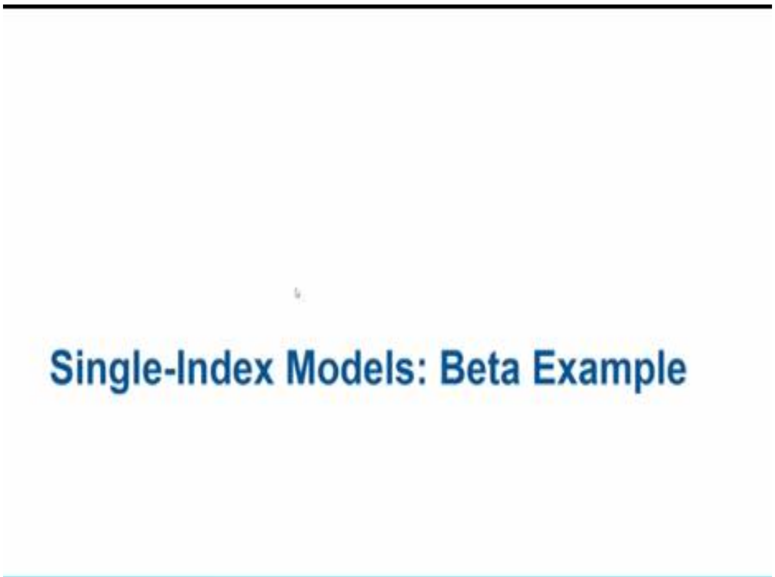

Single-index models, β example. In this video, we will try to understand computation of β in the presence of single-index models.

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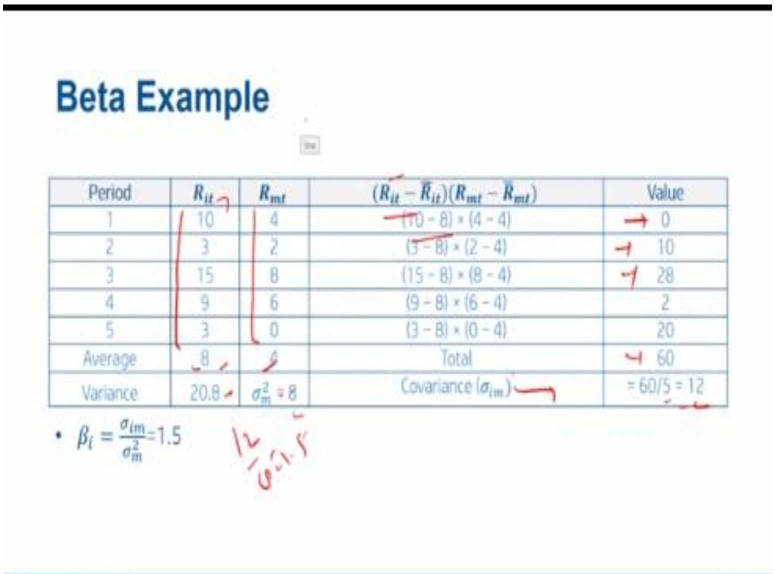

In our previous discussion on the similar example, we assumed β to be given, but let us try to examine β now. In the example, we have been given values of R it which is for these five periods, we have return values, market returns for these same periods. So first we can compute the average values of market return which is 4% and security return 8%, their variances also σ m square and σ it square which is 20.8 σ m square is 8.

We can also compute the covariances with this expression R it into R it bar times R mt minus R mt bar. So this is R it minus R it bar times R mt minus R mt bar for which the values are computed here and we can see the summation is 60 when we divided by 5 we average it out to get the covariance number σ m which is 12. Now the β is nothing but the ratio of this 12 divided by 8.

So 12 upon 8, 1.5. This is my β for the returns that are given here. So this is how we compute β for any security where returns for the security and returns on the market are given to us.

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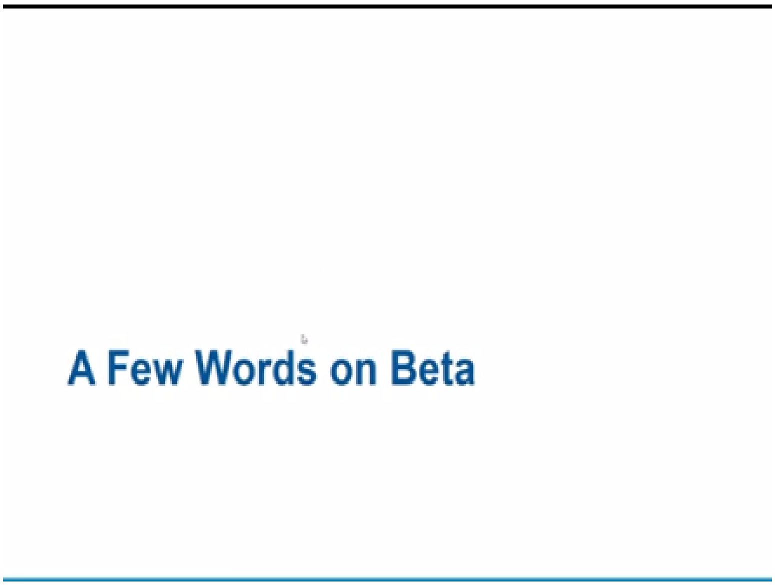

A few words on β. In this video we discuss some of the very important properties of β.

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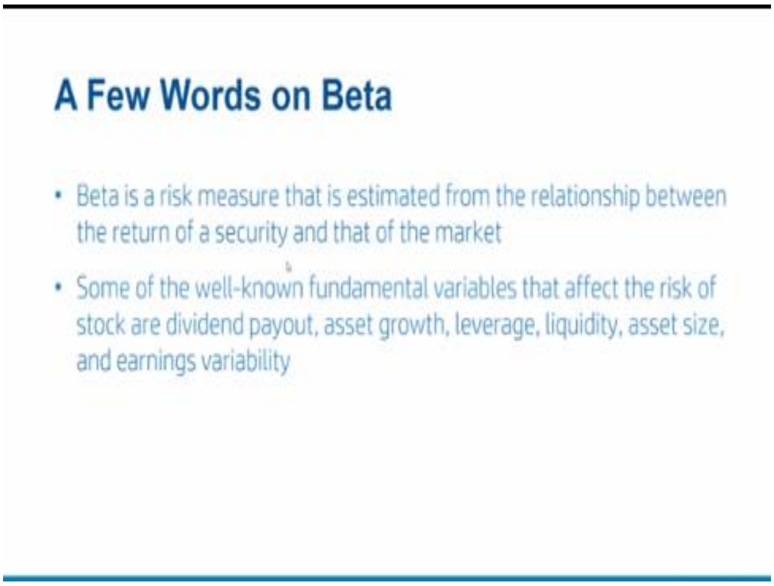

Essentially β is a risk measure that is estimated from the relationship between the returns of a security and that of the market. Since it is the risk measure, then mathematics apart it should be linked to the firm characteristic and the risk profile of the firm. There are some of the very well-known for variables that affect the risk of a stock, for example, dividend payout, asset growth, leverage, liquidity, asset size, earnings variability, accounting data and so on. We will discuss some of these properties here.

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It is assumed that firms that pay more dividends have positive future expectations and these firms are considerably less risky and therefore, have no β. These are called low β firms and therefore, we can expect a negative correlation between the firm β and dividends, firm β and dividends. High growth firms are generally young firms with high capital requirements.

These firms are considered risky and therefore, these firms have a positive relation with βs. So the firms with high growth have positive relation with the β. Firms with high liquidity are also considered to be less risky and therefore, liquidity has a negative relation with β. So firm β with liquidity.

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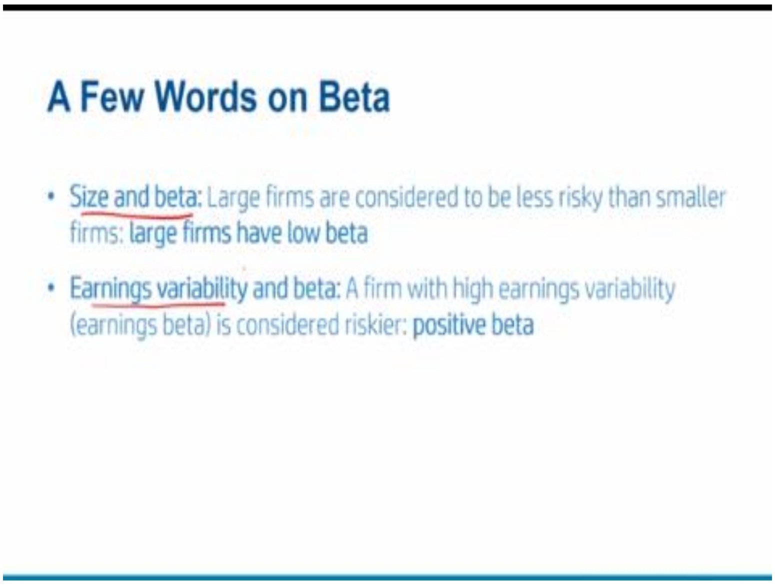

Next, large firms are considered to be less risky than the small firms and therefore, size is considered to be negatively related to β. Finally, a firm with high earnings variability is considered riskier and therefore, earnings β is expected to have a positive relationship with the security β. To summarize, in this video, we discussed some of the properties of security that are linked to β.

These include relationship between dividend payments and β, growth and β, liquidity and β, size and β and finally earnings variability with β.

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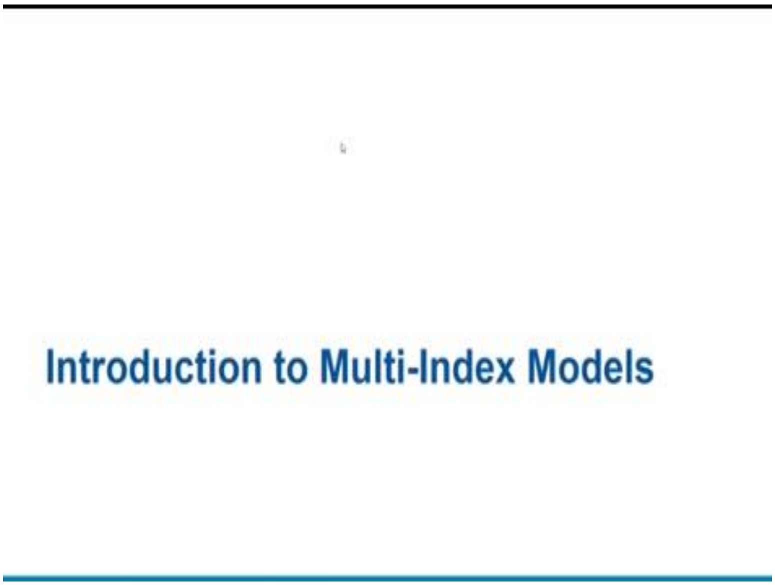

In this video, we will introduce multi-index models. We will discuss some of the basic properties of these models.

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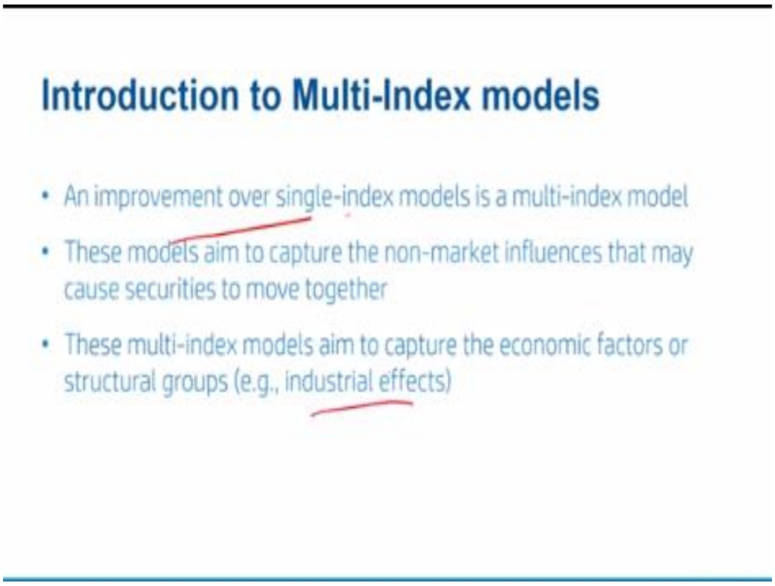

An improvement over single-index model is multi-index model. These models are often employed to explain and estimate the correlation structure of security returns. These models aim to capture the non-market influences that may cause securities to move together. That is the movement or correlation across securities that cannot be accounted for by the market index itself.

These multi-index model aim to capture economic factors for structural groups for example, industrial effects.

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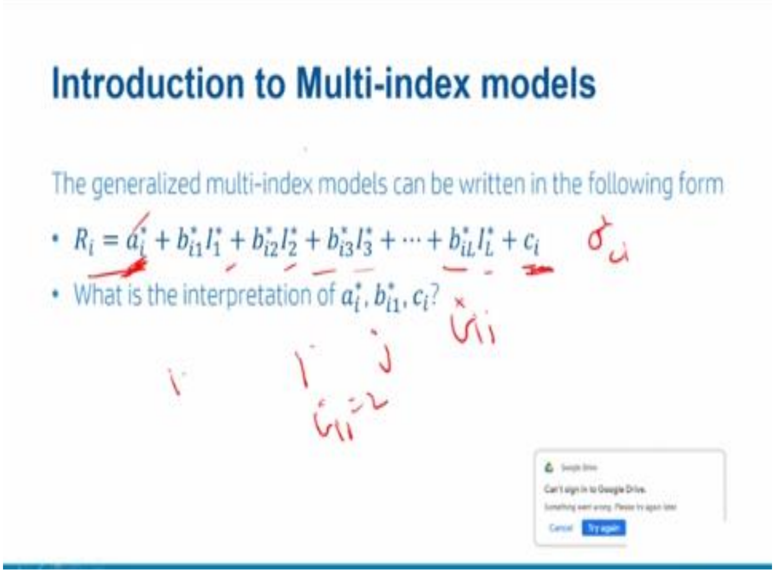

The generalized multi-index model can be written in the form of this equation R i equal to a i star plus b i1 into i 1 star plus b i 2 star into i 2 star and so on up till b iL star into iL star plus c i. What is the interpretation of these a i star b i1 star and c i1? In this equation returns on security i is experienced as a function of indices i 1 star i 2 star and so on up till i L star.

$R_i = a_i^* + b_{i1}^* l_1^* + b_{i2}^* l_2^* + b_{i3}^* l_3^* + \cdots + b_{iL}^* l_L^* + c_i$

B ij star is a measure of sensitivity of stock i to changes in index j. And that is why it is written as or noted as b ij star. b ij star has the same interpretation as that of β i. b ij star is a measure of sensitivity of return of stock i to changes in index j and it has the same interpretation as that of β i in case of single-index models.

For example, bij star of 2 means, b ij star of 2 means that if the index changes its direction in a particular direction by 1% whether increase or decrease, the stock return is expected to change by 2% in the same direction. This is similar to single-index models. The stock specific component again similar to simulate single-index models the stock specific components, this ai1 star and c i these are stock specific components.

Here a i star is the value, expected value of the unique return or constant, this is the constant term, usually one would expect this term to be zero. That is not significantly different from zero in the regression model output. However, short term investors, day traders **are** often lookout for securities with this kind of positive or negative α s or α i star. Often I refer to this αi or ai.

The securities are considered to be underpriced or overpriced. However, one cannot expect a very high and significant α over long durations only to the extent trading costs prevent traders arbitrage from taking the long or short position in that security the nominal returns as indicated with this α i star or a i star that can sustain for a short term.

That is only to the extent trading cost prevent traders and arbitrages from taking a long or short position in that security the abnormal returns indicated by this α can sustain. Any excess α over the trading costs will be swiftly exploited by traders and arbitrages. This α may also include risk free component of expected returns.

And therefore advanced models adjust the expected returns with a risk free rate to obtain or estimate this α that is a more precise measure of risk adjusted or nominal returns. C i is the random variable with a distribution mean of zero and standard deviation of σ square c i. So this is sort of stochastic variable.

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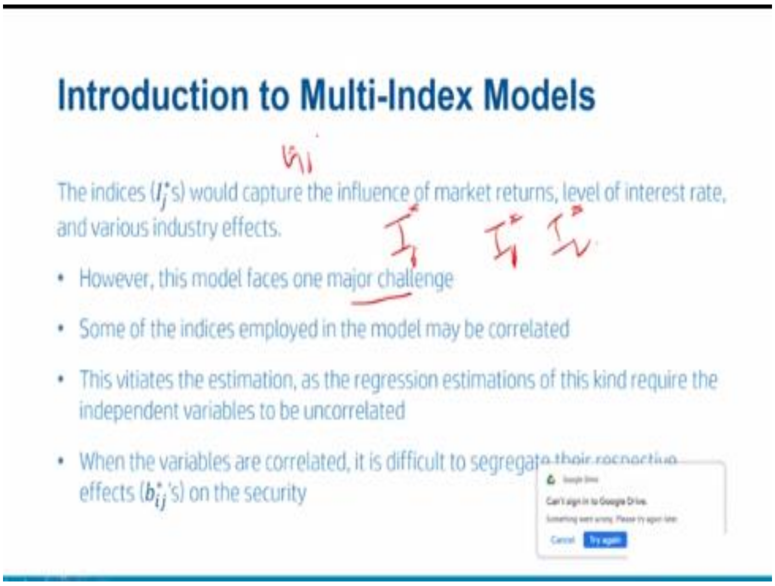

The indices ij stars, these ij stars the indices, they would capture the influence of market returns, level of interest rates and various industry effects. While a model of this kind is often employed in the literature, it faces one major challenge. Some of the indices employed in the model may be correlated.

This vitiates the estimation as the regression estimations of this kind require the independent variables that is all the independent variables ij like i1 star i2 star and so on to be uncorrelated.

However, when the variables are correlated, it is difficult to segregate their respective effects that is b ij stars on the security.

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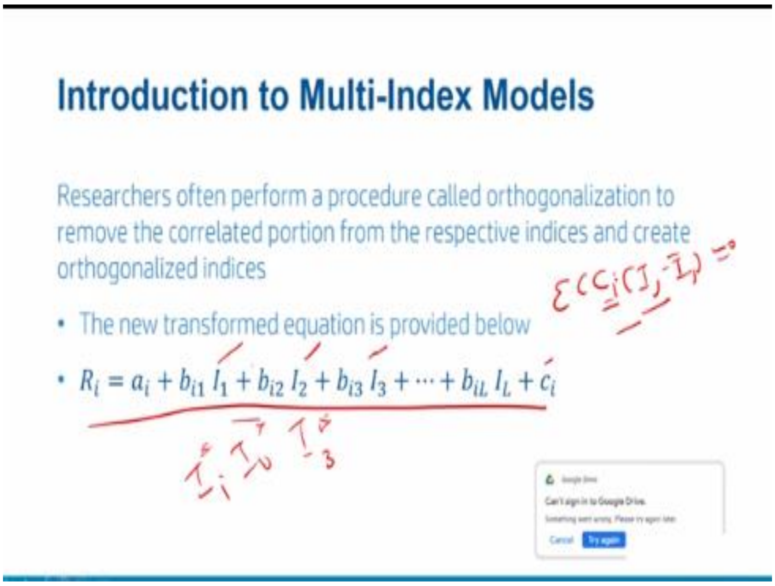

And therefore, researchers often perform a procedure called orthogonalization to remove the correlated portion from the respective indices and create orthogonalized indices that is the original equation which is this equation through orthogonalization procedure, which is a mathematical procedure is transformed into this new equation.

The important point to note about this new equation is that unlike the original I 1, I 2 stars which there was a possibility of these origins indices to be correlated, the new indices that are created that is I 1, I 2, I 3 and so on, these are supposed to be uncorrelated. That means, these new indices that are constructed, they are so constructed as to have no correlation with each other and the error term.

$$R_i = a_i + b_{i1}l_1 + b_{i2}l_2 + b_{i3}l_3 + \cdots + b_{iL}l_L + c_i$$

So first and foremost, we can say that error term is not correlated with these indices and that can be represented as this ij minus ij bar. So the expectations of c i into ij minus ij bar is zero, which suggests that there is no correlation between indices and this error term. And also these indices are so designed that do not have any correlation with each other as well. So the new equation, this is our new modified equation.

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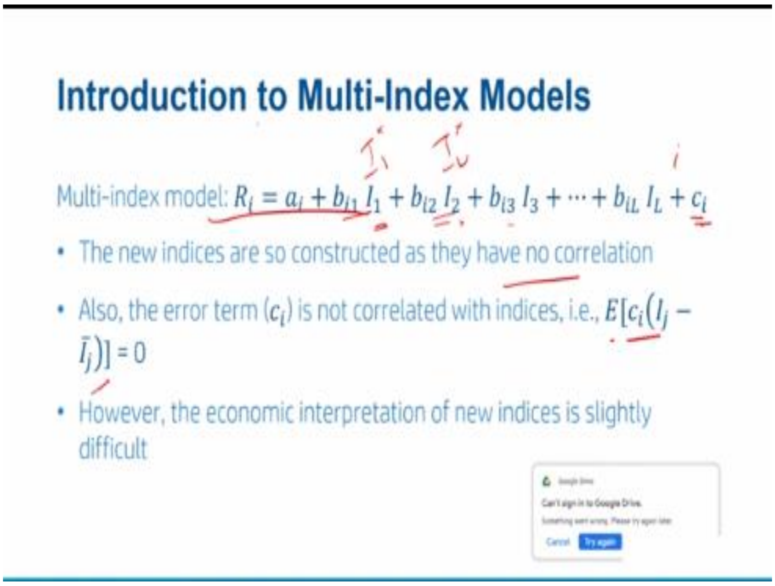

This is our modified equation. In this equation, all these I 1, I 2's are expected to be uncorrelated with each other. And they are also expected to be uncorrelated with the error term which is c i. So for example, for any security i this error term plus dual term c i is expected to be stock specific and therefore, not correlated with indices I j's.

$$R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + b_{i3}I_3 + \cdots + b_{iL}I_L + c_i$$

However, for this new equation, unlike the original equation, where we would have said that I 1 star is probably the market influence or I 2 star is probably the influence of broad interest rate, here it is difficult to economically interpret the new indices. So the exact economic interpretation is difficult. To summarize, in this video, we introduced multi-index models.

$$E[c_i(I_j - \bar{I}_j)] = 0$$

We also discussed some of the basic properties. We found that these indices can be correlated, these economic indices when they proxy economic variables like market influences like interest rate or broad market wide movements, they can be correlated. Through orthogonalization procedure, one can make these influences or indices uncorrelated and also uncorrelated with the stock specific components like residuals.

However, the newly designed indices may not have exact economic interpretation and it may be difficult to interpret them.

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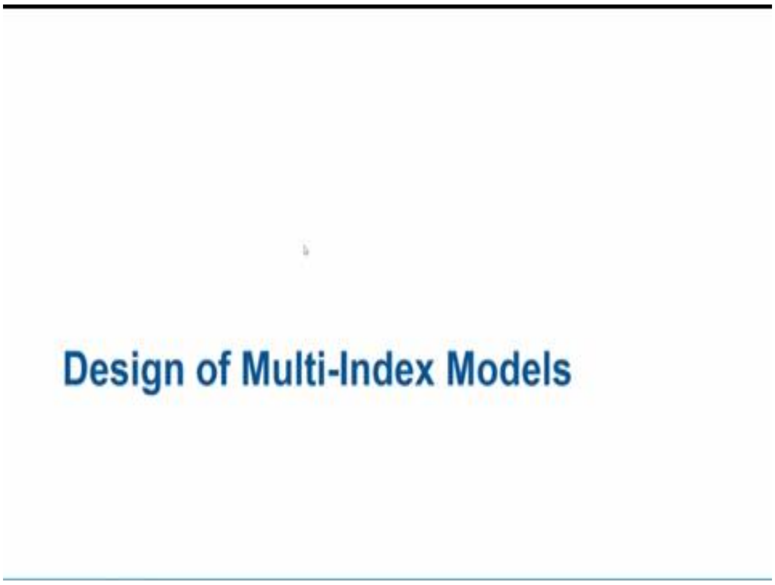

Design of multi-index models. In this video, we will discuss some of the basic definitions, assumption and construction of multi-index models.

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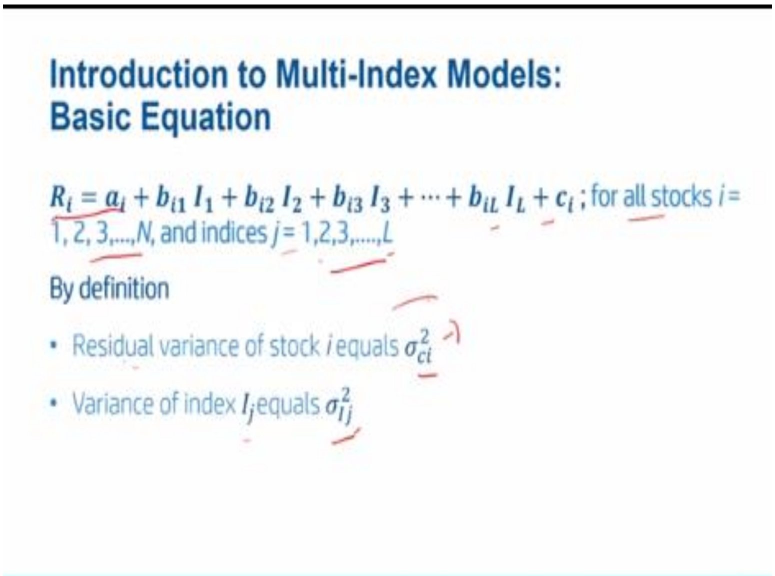

The final general multi-index model can be defined as follows, this equation R i equal to α i or a i plus b i1 into I 1 plus b i2 into I 2 and so on up till b iL into I L plus c i for all stocks i from 1 to N if there are N stocks and j indices 1, 2, 3 and so on up till L. So there are total N indices. And here let us define the model. By definition, the residual variance of the stock i equal to σ c i square.

$$R_i = a_i + b_{i1}l_1 + b_{i2}l_2 + b_{i3}l_3 + \cdots + b_{iL}l_L + c_i$$

So this is a stock specific component. The variance of index ij is equal to σ ij square which is the variance of index. So this is by definition.

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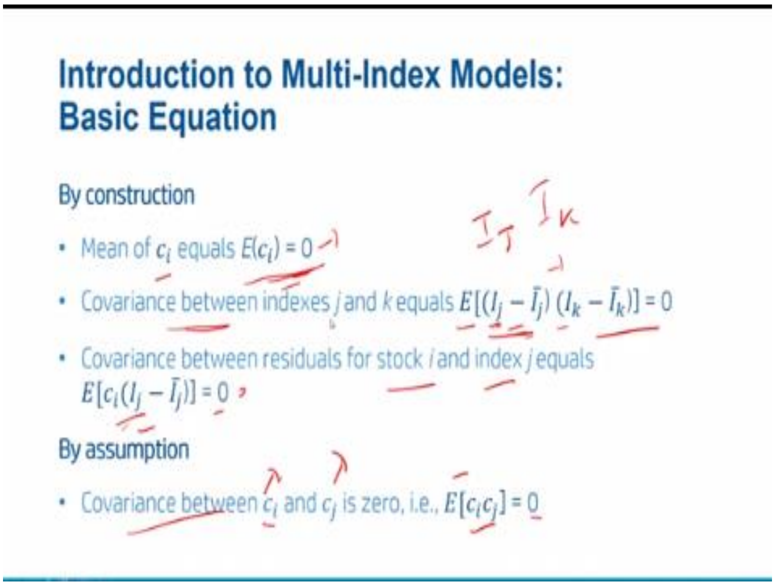

Next, by construction we say that the means of c i the error terms in the model the residual, this term, the mean of this term is 0 by construction. So regression model to some extent, to a good extent in fact, regression model is to ensure that this property is held. We also assume that covariance, by construction covariance between indices j and k that is ij and ik are not correlated.

Or the covariance between them is 0 which is this expectation of ij minus ij bar into ik minus ik bar. Remember, we discussed the orthogonalization procedure in the previous video where we said that the indices are so constructed that they do not have any correlation with each other. We also say that stock i and index j do not have any covariance or correlation.

$$E\left[\left(I_j - \bar{I}_j\right)\left(I_k - \bar{I}_k\right)\right] = 0$$

So the expectations of c i into ij minus ij bar equal to 0. So this is also by construction. The way these in indices are generated not only they have no correlation with each other they also do not have any correlation with error term or residual in the model. Last and a slightly more difficult term that covariance between error terms c i here for security i and c j for security j are not correlated.

$$E\left[c_i\left(I_j - \bar{I}_j\right)\right] = 0$$

That means expectation of c i into c j is 0. Now this is a slightly tricky term as we will see.

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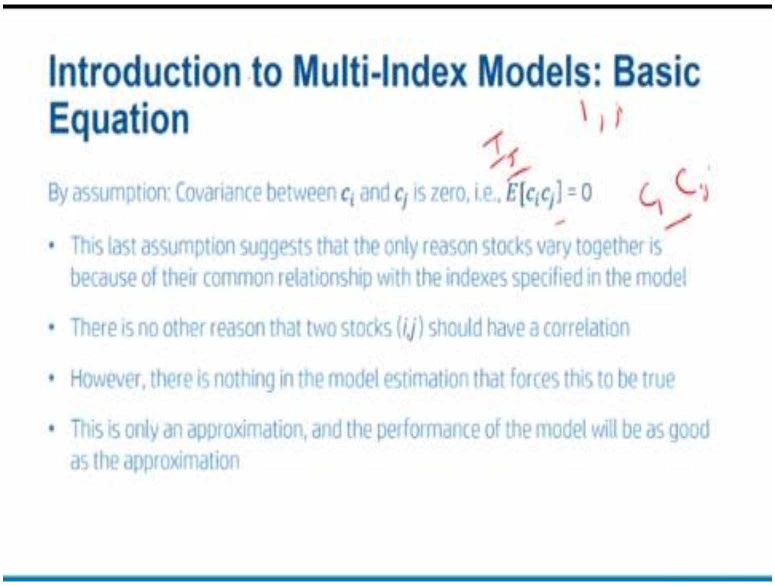

This is the last assumption that the covariance between c i and c j is zero is very important for multi-index models. It suggests that the only reason stocks vary together is because of their common relationship with the indices specified in the model that is I j's. And therefore, there is no other reasons two stock i and j should have a correlation because all that is common to them is contained by these indices that we have modeled.

So the remaining error term that is c i c j that are residuals or stock specific terms should not have any correlation. However, there is nothing in the model that forces this to be true including the regression model that also does not ensure that this relationship is held. And therefore, this is only an approximation or sort of assumption, and the performance of the model will be as good as this approximation.

To summarize, in this video, we discussed some of the basic properties of multi-index model. We discussed some of the definitions and also some of the properties that are held by construction. And lastly, some of the properties that are just assumptions only. And the performance of the model is as good as these assumptions are held in practice.

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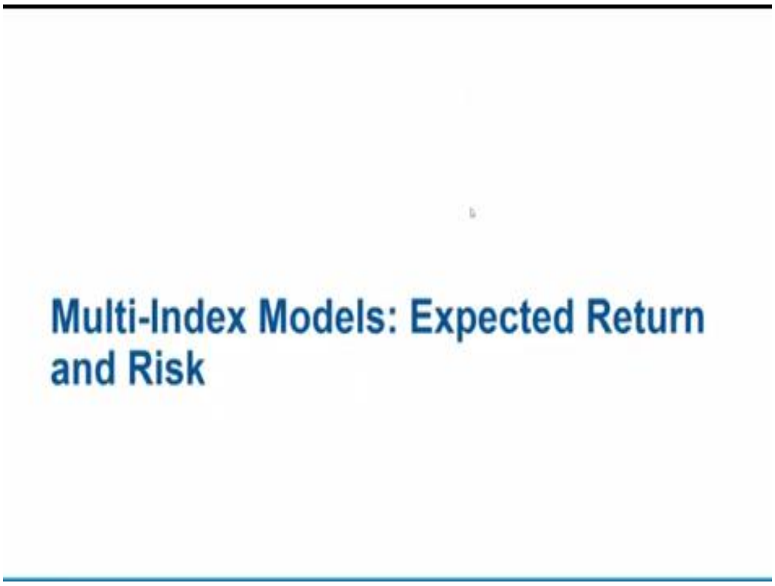

Multi-index models expected return and risk. In this video we will introduce the expressions for expected return variance and covariance for multi-index models.

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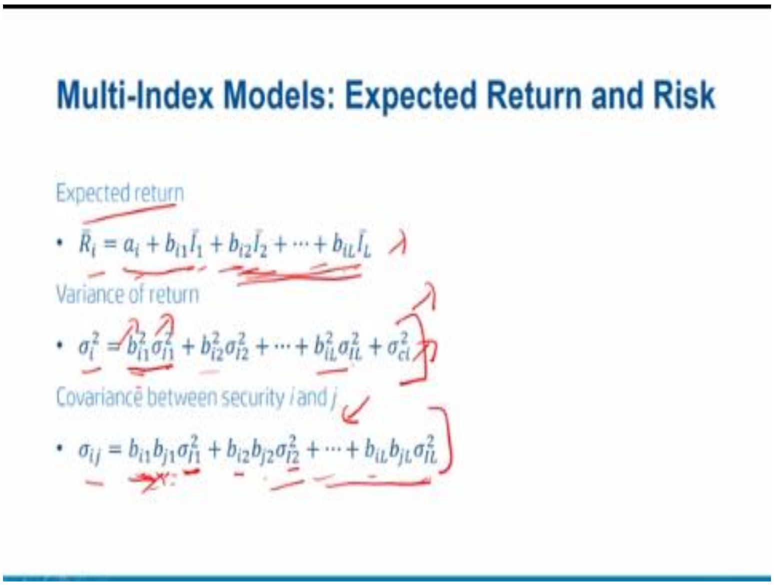

Similar to single-index models, the following equations provide the expected returns, variance of returns, and covariance between securities in the context of multi-index models. We can straightaway translate those results. For example, for expected return R i bar equal to α i plus b i1 into I 1 plus b i2 into I 2 and so on up till b iL into I L bar.

$$R_i = a_i + b_{i1}\bar{I}_1 + b_{i2}\bar{I}_2 + \cdots + b_{iL}\bar{I}_L$$

While the expression up till here is very similar to single-index model in a very identical fashion we introduce other indices and their sensitivities with respect to security i that is b i2 b iL and so on. So this is our expected return expression. Again similar to the single-index model, we introduced variance of return as σ square equal to b i1 square into σ i 1 square and so on until b iL square into σ i square plus σ c i square.

$$\sigma_i^2 = b_{i1}^2 \sigma_{I1}^2 + b_{i2}^2 \sigma_{I2}^2 + \cdots + b_{iL}^2 \sigma_{iL}^2 + \sigma_{ci}^2$$

This σ c i square represents the residual or stock specific idiosyncratic component of risk, which is specific to security i. The remaining components include one variance terms for the index which is included and the sensitivity of the security. So it is also very similar to the single-index model where we had just one term which is b i1 square into σ iL square.

$$\sigma_{ij} = b_{i1} b_{j1} \sigma_{I1}^2 + b_{i2} b_{j2} \sigma_{I2}^2 + \cdots + b_{iL} b_{jL} \sigma_{iL}^2$$

Now that we have L indices we have b i1, b iL square into b i σ L square so up till L terms. But again very similar to the variance of single-index model. Lastly, we have covariance between security ij which is given as σ ij equal to b i1 into bj 1 into σ i1 square plus b i2 two into bj 2 into σ i 2 square and so on up till b iL into b jL into σ L square.

And again it is also very similar to the single-index model where we had only one term which is b i1 into b j1 into σ L square. And in a very identical and intuitive manner, we have introduced the covariance transfer other securities. Here for example, b i1 into b j1 are the sensitivities of security i and j with respect to this index I capital I and so on. So security i, security j and their inference with respect to this index i1.

Similarly, i2, j2 with respect to index I 2. And similarly, we have all the other covariance terms. So this is how we define covariance terms.

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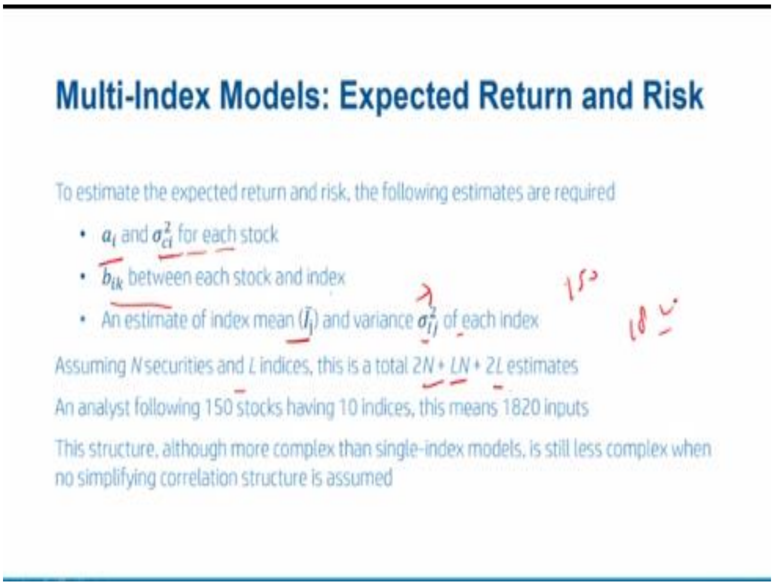

Here we can clearly see that the expected return and risk can be estimated for any portfolio. If we have the estimates of α i this one and for each stock b ik between each stock and the index and an estimate of σ c i square for each stock, which is the variance of the stock. And finally, an estimate of ij bar, which is index specific, mean of the index and variance of σ ij square which is the variance of index ij for each index.

Assuming there are N securities and L indices that results in total 2N + LN +2L estimates. For example, as an analyst if you are following, let us say 150 stocks and 10 indices, then this means 1820 inputs. Generally, analysts start with the single-index model, which is usually the market index and then keep on adding industry indices.

The structure although it is slightly more complex than single-index models, but is still less complex when no simplifying correlation structure is assumed.

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Please note that these industry indices are so constructed that they are uncorrelated with each other as well as with the market. The basic assumption here is that a firm's return can be affected by market and some other industries. Researchers often derive indices from the available data using quantitative techniques, for example principal component analysis and factor analysis.

One can add more and more indices to increase the explanatory power of the model. However, with more indices, the model becomes less efficient and more complex. So it is a tradeoff between complexity, efficiency and explanatory power of the model. To summarize, in this video, we discussed three important properties of multi-index model including expected return, variance of return and covariance between securities i and j.

Lastly, we also discussed that in a multi-index model addition of new indices can be done on purely on mathematical basis. However, there is a tradeoff while you add more and more indices you add to the explanatory power of the model but at the same time your model becomes complex and probably less efficient.

So while adding more or less indices and optimizing the model, there is a trade off in terms of complexity, efficiency and explanatory power of the model.

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Multi-index models: 3-Factor Fama-French Model. In this video we will discuss a very important model which is part of the multi-index model family. This is 3-Factor Fama-French Model.

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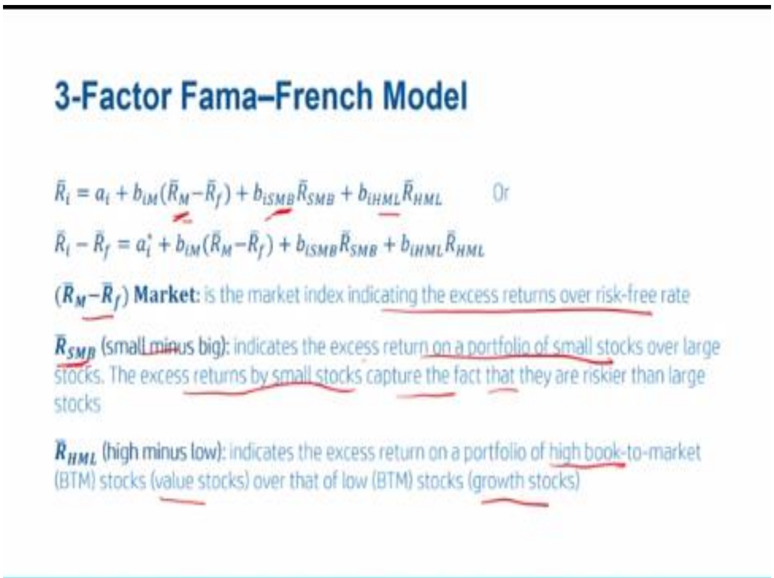

To begin with, one very important model needs a special mention here, which is Fama-French 3-Factor Model. In addition to market, which is this factor R M minus R f, R M bar is the market factor. Fama-French found that the size which is proxied by market capitalization and ratio of book to market have explanatory power on the cross section of expected returns for common stocks.

They documented that cross section of expected returns are negatively related to size and positively related to book to market. So this small -b is this second factor which is size that is small stocks have higher returns as compared to large stocks and high minus low, high minus low book to market value. So here first factor is R M minus R f which is the market index indicating the excess return over this period.

The second factor is R bar SMB, which is small minus b indicates the excess return on a portfolio of small stocks or large stocks. The excess returns by small stocks capture the fact that they are riskier than large stocks. So that is R bar SMB and then R bar HML high minus low book to market ratio, which indicates the excess return of portfolio of high book to market stocks or that of low book to market stocks.

$$\bar{R_\iota} = a_i + b_{iM}\overline{(R_M} - \overline{R_f)} + b_{iSMB}\bar{R}_{SMB} + b_{iHML}\bar{R}_{HML}$$

That is generally these high book to market stocks are also referred to as value stocks and low book to market stocks are referred to as growth stocks. So they also, the value stocks offer higher return as compared to growth stocks. Now, the idea is that for example, in the context of size factors, SMB factor, it appears that these indices the size as well as HML, they represent some sort of risk factor that is small firms are argued to be riskier than large firms.

$$\bar{R_\iota} - \overline{R_f} = a_i^* + b_{iM}\overline{(R_M} - \overline{R_f)} + b_{iSMB}\bar{R}_{SMB} + b_{iHML}\bar{R}_{HML}$$

Also firms with high book to market ratio are considered to be undervalued or value stocks like we discussed. Maybe market is more cautious about these value firms and prices are repressed could be due to some significant changes happening in the organization. In contrast, firms with low book to market values are valued by market at a premium. For example, tech firms like Google, Microsoft, they often traded a premium as market is upbeat about them.

Therefore, on average over long horizons high book to market firm offer higher returns as compared to low book to market firms. To construct the size factor analyst compute the difference between small and large firms based on market capitalization. So that is our R bar SMB small minus big factor.

The second variable defined as high minus low is constructed using the difference in returns between the high and low book to market firms, which is value versus growth factor. To summarize this lesson, introduction of single and multi-index models, considerably simplifies security analysis. In particular, the complex correlation structure between two securities is replaced by the common influence of the index on each of the security.

With the application of these index models, portfolio analysis is considerably simplified. Portfolio βs are often less noisy and more informationally efficient than βs of individual securities. Construction of multi-index models broadly employ similar theoretical underpinnings except that they employ multiple indices. Construction of these index models requires certain assumptions, some of which are held by the assumption, design or definition of the respective model.

Some of these key assumptions in these index models are as follows. First, idiosyncratic error terms are not correlated with indices that are more systematic influences that is expected value of c i into ij minus ij bar equal to zero. Next, these indices are not correlated across each other. That means expected value of ij minus ij bar into ik minus ik bar equal to zero. The error terms are not correlated with each other that means expected value of c i into c ij equal to zero.