Decision Modelling
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Lecture 14
M/M/1 Queuing Model

Today we are going to discuss the MM 1 queuing model in more detail, if you recall in our previous class we have seen what is known as the birth and death process and in while discussing birth and death process we have also discussed MM 1 queuing model in some detail but you know let us look at this once again.

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The most when we talk about MM 1 we really tell about MM 1 FCFS infinity/infinity model. The M is that arrival process is Markovian, Second M service process is Markovian, 1 is the number of servers, FCFS which is called the queue discipline which is first-come first-serve and infinity the first infinity is the system size that means infinite waiting line right, there is no restriction on the queuing anyone who comes can join without any problem and the last infinity is the population size that means input population size which is also called the calling population right. So these are the simplest assumptions, MM1, FCFS, infinity, infinity.

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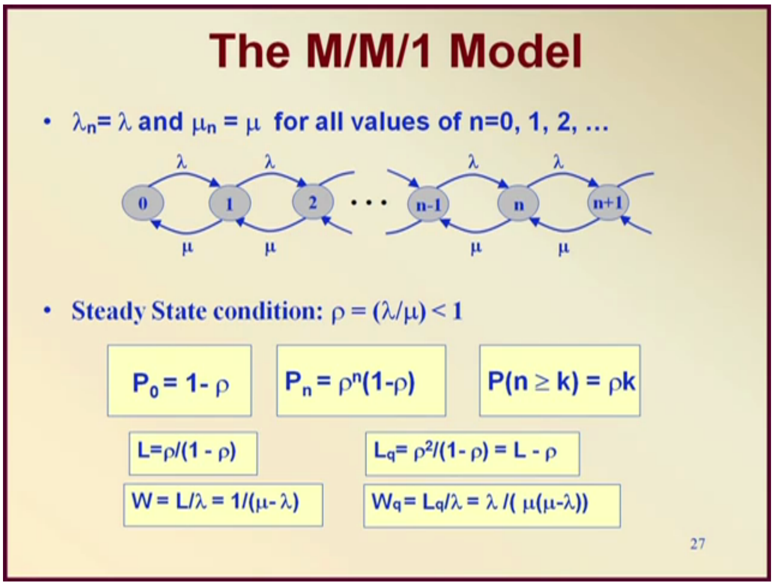

In our previous class we have already seen that, in an typical MM 1 model the birth and death, the rate diagram will look exactly like this that means these other states 0, 1, 2, then n – 1, n, n + 1, every time an arrival comes the Lambda the state changes right, with every arrival the state changes but when a person serviced then the state changes to 0. Now this service will be always Mu because it is a single server system right, in higher-order system there will be maybe 2 Mu, 3 Mu or something like this depending on the situation. Now under steady-state condition we had got certain relationships and these relationships are very important so let us write them down right.

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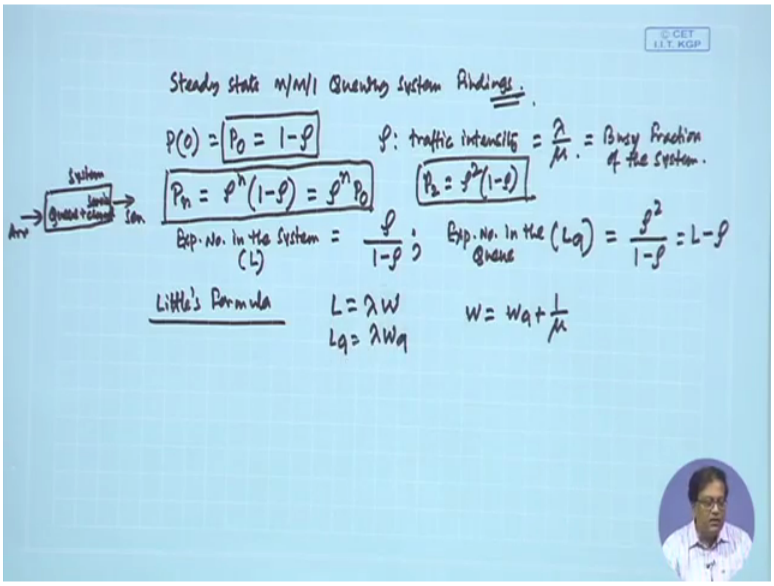

So steady-state MM1 queuing system findings, right. First of all P 0 you can write this way also in a very simple way you can write this. The P 0 comes out to be 1 – Rho, what is Rho? Rho is the traffic intensity equal to Lambda by Mu. Something about traffic intensity Rho, the traffic intensity Rho is basically the busy percentage, right? The Rho basically tells that this percentage that is Lambda by Mu suppose it is 70 that means 70 % of the time the system is busy, right? So it can also be called busyness or busy fraction, it is a busy fraction of the system right that is Rho.

Now one thing must be remembered that when you tell Rho equals to Lambda by Mu, we should remember that this busy fraction Lambda by Mu will not be always Lambda by Mu, for MM1 situation it is Lambda by Mu must be remembered. For example, for MMS system it is Lambda by S Mu right, so it is like total arrival by total service usually so but for all MM1 cases this Lambda by Mu is a usual formula and we shall continue to use them, but this point must be remembered very much. Now, the other results that we have seen in our previous day that probability n that means n % in the system is given by Rho to the power n then 1 – Rho, basically it is Rho to the power n P 0 right so this is the probability equation.

There are a few more results they are also very important that expected number in the system, please recall what is a system? A system means the, let us draw very quickly here this is the system, the system includes the queue and the service channel, this is system. Must be remembered, arrival is here, the service is here, arrival service. So this is a simple diagram that is there is a system queue the queue portion and a service channel portion, the people are arriving, people are going out, are being serviced, the queue + service channel the total thing is called system. Now expected number in system, what is it? This is called L, in the system mind you not in queue that means queue + the people who is getting service. The n is given by the formula Rho by 1 – Rho that is very important must be remembered right.

And what is the expected number in queue? Expected number in the queue that is known as L q, the L q is given by Rho square by 1 – Rho or in other words it is L – Rho right. So these are the formula about L and L q once you know that, but at least you can calculate anyone of them. Look here, very simple you have to just remember this one P 0 equals to 1 – Rho, you have to remember that P n is obtained by Rho to the power n P 0, in simple what is P 2 then? The P 2 is nothing but you know Rho square 1 – Rho, why 1 – Rho? Because P 0 equals to 1 – Rho right, so P 3 Rho cube 1 – Rho so this way you can find all the probabilities.

What is the (())(7:55) you have to remember just one formula other than that that is either you remember L equals to Rho by 1 – Rho or you remember L q equal to Rho square by 1 – Rho which is nothing but L – Rho because expected number of people in the service is Rho, why Rho? Because the busy period is Rho right, at any point of time it is the Rho percentage people who are you know the time the system is busy. So in another words you can tell that Rho number of people are there in the service, so how many people are there in the queue, it is L – Rho right so that is L q.

So this and then obviously remember little’s formula L equals to Lambda W, L q equals to Lambda W q and W equals to W q + 1 by Mu. So this formula if you also remember, all these little’s formula you can calculate one of the L or L q or W or W q then you can get the others right. So really there is no need to remember all the other formula, you know that W is the expected waiting time in the system, W q expected waiting time in the q right, so all these formulas are going to be very much required in our MM 1 queuing system.

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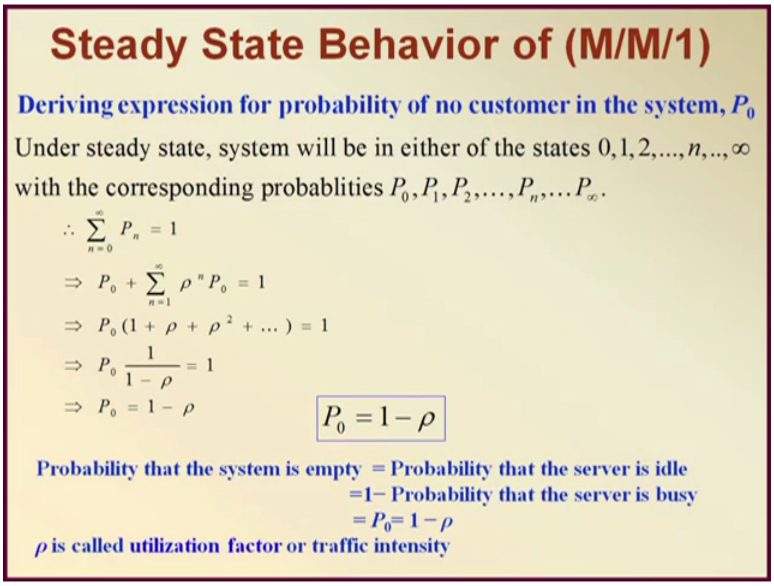

So once again quickly the proof is shown that you know Sigma P n equals to 0 so you can say that P 0 + Sigma Rho n P 0 equals to 1 so P 0 1 + Rho + Rho square etc, etc, equal to 1 so that sum is 1 by 1 – Rho so P 0 is 1 – Rho right so this is one calculation that we can do. Rho is called the utilisation factor or the traffic intensity.

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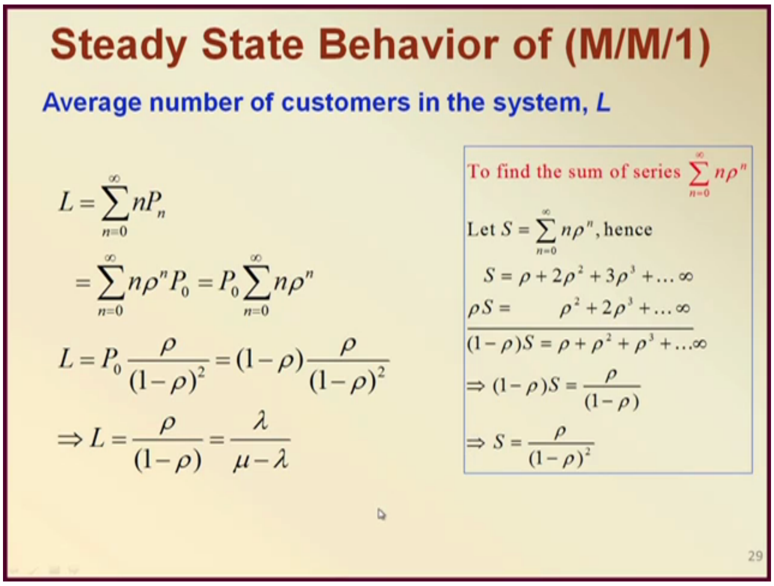

Then let us look at this how L is obtained right, so these derivations are you know to understand really because the average number of customers in the system is you know this is an expected value so number of people let us say is you know what is that expected number of people? It could be anything between 0 to n, probability of 0 is P 0, probability of n is P n, so since it is an expected number, it should be the sum of all of these right, so it is 0 P 0, 1 P 1, 2 P 2, 3 P 3, 4 P 4 and then finally n P n. So it is n equal to some over n equal to 0 to Infinity N p n right. Now it comes out that you know you can since P n equals to Rho to the power n P 0 already we have seen so we can write down L equals to P 0 Sigma n equals to 0 n Rho n right, so this n Rho n here how the sum is obtained?

Look here S equals to Rho + 2 Rho square + 3 Rho cube + up to this Infinity. If you multiply this S by Rho then you get Rho square + 2 Rho cube + Infinity right. So and then if you did one from the other then 1 – Rho into s equals to Rho + Rho square + Rho cube + dot dot dot Infinity. In other words 1 – Rho multiplied by S equal to Rho by 1 – Rho right so what is the sum? Sum S is Rho by 1 – Rho square so it is a creek that is used to really find the S quickly right in a involved manner. Now S is known that is that sum, so now put that sum here L equals to P 0 Rho by 1 – Rho square but P 0 equal to 1 – Rho so you can therefore 1 – Rho and 1 – Rho from you can remove from numerator and denominator and therefore L equal to Rho by 1 – Rho, which is Lambda by Mu – Lambda but this formula is easy to remember L equal to Rho by 1 – Rho right.

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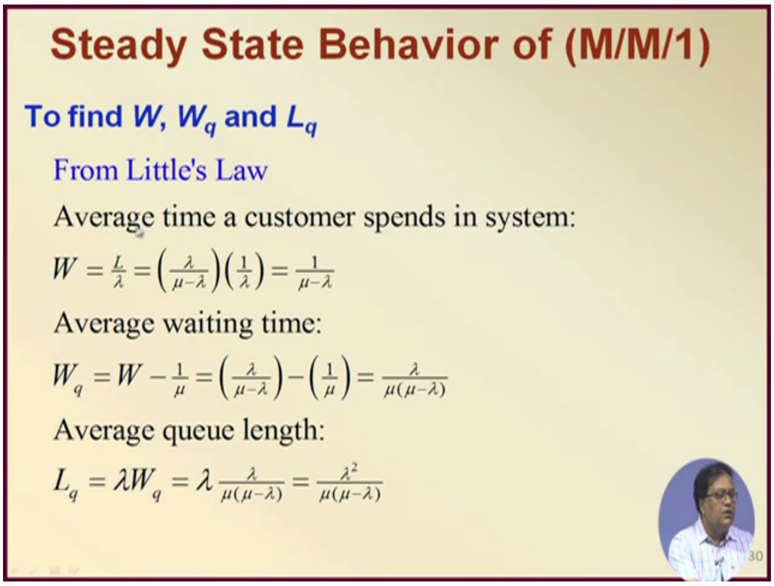

So how is W? W is L by Lambda, W q, W – 1 by Mu, L equals to Lambda W q, so all of these things can be very easily obtained you have to simply remember little's formula right. So if you look at the slide once again, so we know L equal to Rho by 1 – Rho, L q equal to L – Rho or Rho square by 1 – Rho and you can use all these little's formula right, now let us look at some problem.

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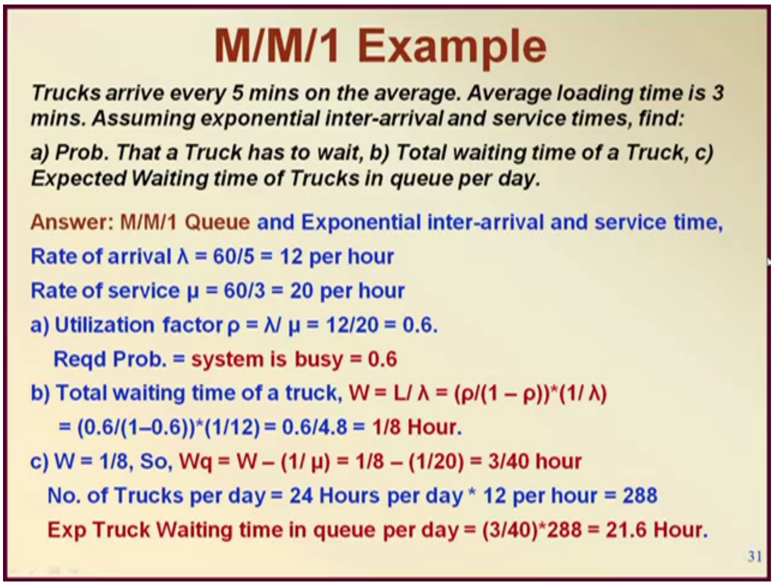

The problem is like this, that trucks let us take one MM 1 problem. The trucks are arriving every 5 minutes on the average, the average loading times 3 minutes. Assuming exponential Inter arrival or service time, look here most of the time we shall use what is known as Poisson arrival and exponential service time. But Poisson and exponential distribution they are interchangeable how? If the number of arrivals are Poisson then inter-arrival time is exponential, this point has to be remembered right. So sometimes we shall write number of arrivals are Poisson, service time is exponential, sometimes we shall write the exponential inter-arrival and service time, one has to know that both are same right. So find out what is known as expected waiting time of trucks in queue right.

So in this problem it is an MM 1 queue and exponential inter-arrival and service time. What is the rate of arrival Lambda is 12 per hour, rate of service Mu 20 per hour so utilisation factor Rho Lambda my Mu is 0.6. See once you get Lambda equals to 0.6, utilisation factor then system is busy because the question was probability the truck has to wait. So when truck has to wait? The truck has wait when the system is busy right, and what is the probability, 0.6 why, because utilisation factor is 0.6. So I told you before that utilisation factor is nothing but the busy percentage right, so if we know that what is the % of time that particular you know truck is to wait then all we have to do, we have to really find out how much time the you know the is the system is in busy or it is equivalent to the utilisation period.

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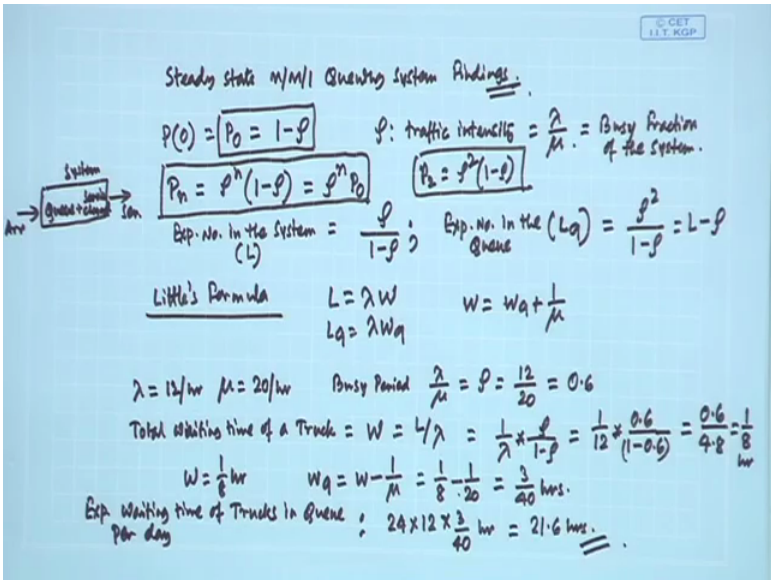

Now if we have to find out what is the total waiting time of the truck, the waiting time of the truck W is nothing but L by Lambda that little's formula we are using right, so let us calculate this also here let us calculate. In this case we have known that Lambda equals to 12 per hour, Mu equals to 20 per hour, so busy period or traffic intensity 0.6. So the total waiting time of a truck W is given by L by Lambda right, little's formula L equals to Lambda W so same formula we are using W equals to L by Lambda, what is L? L is Rho by 1 – Rho so it is one by Lambda times Rho by 1 – Rho right. So Rho is 0.6, Lambda is how much 12, so equal to 1 by 12 into 0.6 by 1 – 0.6 right so this is coming out to be 0.4, this is 12 so 4.8 so 0.6 by 4.8 equal to 1 by 8 right, so we get the total waiting time of a truck is 1/8.

What is the unit? Unit is hour because everything is mentioned in hour, so W is one eighth of an hour. Suppose now we have to find out, out of this one eighth hour, what is the time that the truck has to wait in the queue, that will be found out as W q, look this formula W q + 1 by Mu equals to W, so what is W q, waiting time in the q, it should be W – 1 by Mu right so we know W which is one eighth hour, we know Mu which is 20, so it is nothing but 1 by 8 – 1 by 20. So how 1 by 8 – 1 by 20 how much it will become, suppose this you can make 40 so 5 – 2 so it is 3 by 40 hours so that is the waiting time in the queue.

Now there is a third question, what is the expected waiting time of trucks in queue per day? Let us look at it here, this is what we have to compute now, expected waiting time of trucks in queue per day, so what is the expected waiting time of trucks in queue per day. You see this calculation would require really computing first of all what is the waiting time of a queue of a expected waiting time of a given truck, it is 3 by 40 hours. Now that is for one truck right, when we talk about W q that is that 3 by 40 hours that we have got that is for a given truck that means every truck has to wait a certain time that is 3 by 40 hours. So if that is what happens then what will happen to all the translate come. Now if we assume 24 hours working hours and there are 12 per hour that is average trucks are coming then how many trucks are coming in here in a particular day right.

See, 24 hours into 12 that many trucks are coming, so 24 hours into 12 because 12 trucks that is the arrival rate, 24 is the hours in a day so that many trucks are coming and each track is on the average waiting for 3 by 40 hours right, so that is the expected waiting time of trucks in queue per day. So please remember that W q which you have got 3 by 40 hours that is the expected waiting time in a queue forgiven truck but we need to find out expected waiting time of trucks in queue per day. So in a day there are 24 into 12 trucks that is coming, so if you multiply all of these then the total value is obtained which is coming to approximately 21.6 hours right, so this is the problem that we have solved, let us look at another example.

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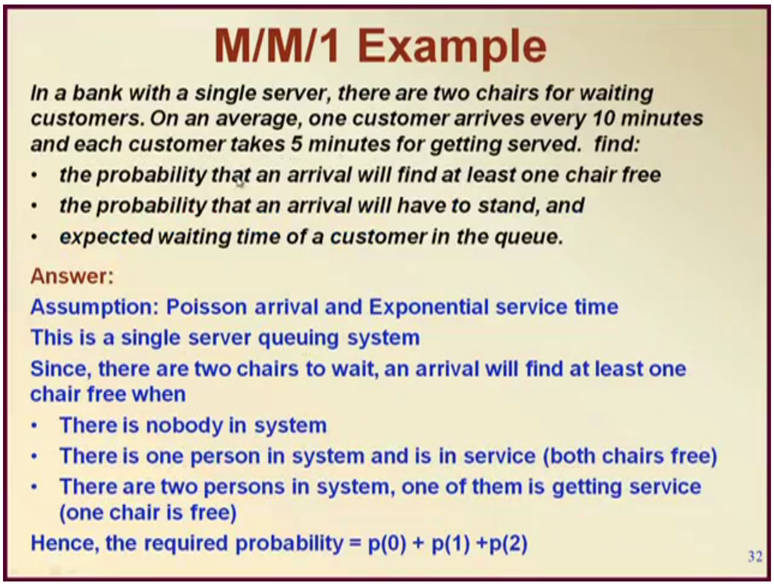

So in this example it is given that in a bank with single server there are 2 chairs for waiting customers. On an average 1 customers arrive every 10 minutes and each customer takes 5 minutes for getting served right, find the probability that arrival will find at least one chair free, is it alright at least one chair free? The probability that an arrival will have to stand and the expected waiting time of a customer in the queue right. So in this case again it is an MM 1 queue because there is a single server but there are 2 chairs for waiting customers, right. So we are assuming MM1 Infinity that means waiting queue space is infinite, queue space is infinite but only 2 chairs so what will happen?

Supposing if another person comes then he will not get a chair, is it alright? So let us look here, at any point of time if the system has got nobody that means free, you simply go and you can get the service right so this is P 0. What is the probability that there is only one person in the system that means there is a person, the person is getting service right so this is P 1 both the chairs are free. There is a third situation, there are 2 persons in the system, one is getting service and another one is sitting on the chair so second chair is free, so what is therefore it means, it means that the probability of that an arrival will find at least one chair free is equal to P 0 + P 1 see in this case it is like this.

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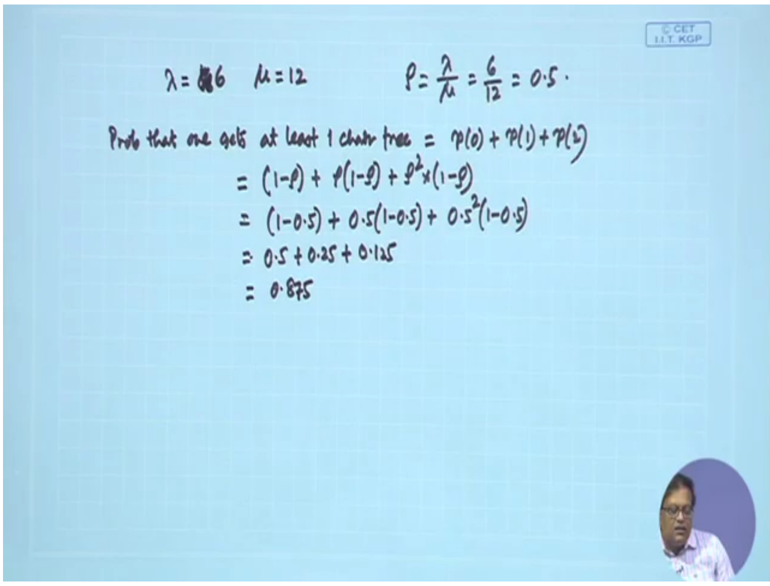

The Lambda is given equal to every 10 minutes, customers are arriving every 10 minutes so what will be Lambda? It will be 1 by 6, what is Mu that and each customer takes 5 minutes per day sorry not 1 by 6, it is customers arrive every 10 minutes so Lambda equals to 6 at each customer takes 5 minutes for getting service so Mu equals to 12 right, so Lambda equal to 6 and Mu equal to 12 then probability that you get at least one gets at least one chair free equal to P 0 + P 1 + P 2 right, so probability that one gets at least one chair free is P 0 + P 1 + P 2. P 0; nobody in the system both chair free, P 1; one person getting service both chairs free, P 2; one person getting service, one person sitting on one chair second chair free so that is what we need to calculate.

So what will be P 0? What is Rho first of all? Rho equal to Lambda by Mu equals to 6 by 12 equals to 0.5. What is this value? P 0 + P 1 + P 2, this is P 0 is 1 – Rho, P 1 is Rho into 1 – Rho and P 2 Rho into 1 – Rho, so 1 – 0.5 + 0.5 into 1 – 0.5 + 0.5 square 1 – 0.5, so this is how you can calculate 0.5 + 0.25 +0.25 into 0.5 equals to 0.125, so 0.875. So probability that one gets at least one chair free comes out to be 0.875 so that is what is calculation is also shown here.

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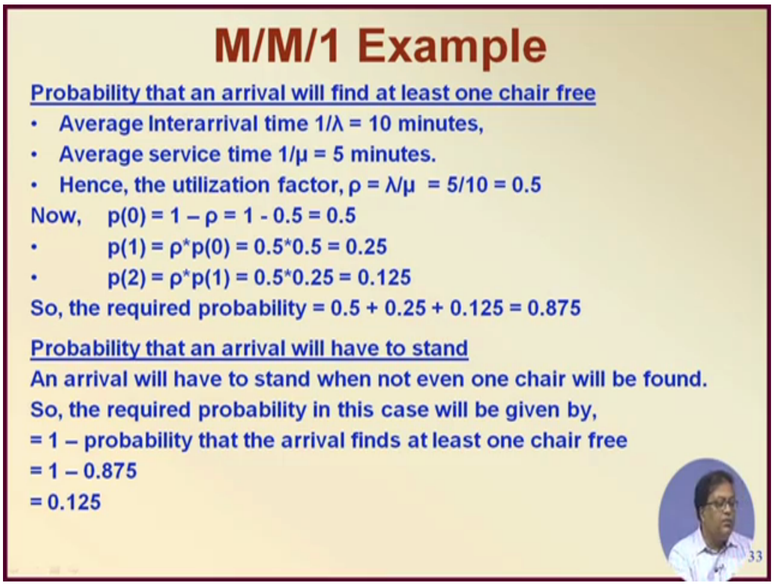

What is the probability that an arrival will have to stand? Very simple, arrival will have to stand no when not even 1 chair will be found so that require probability will be 1 – probability that the arrival finds at least one chair free. So you see sometimes you have to get it from the other side right. You have to find the probability that the arrival will have to stand, where what happens when there are more than 2 people in the system that means 1 – P 0 – P 1 – P 2 alright. But P 0 + P 1 + P 2 already we have found 0.875 so we deduct it from 1 equals to 0.125 that will be the probability that arrival will have to stand.

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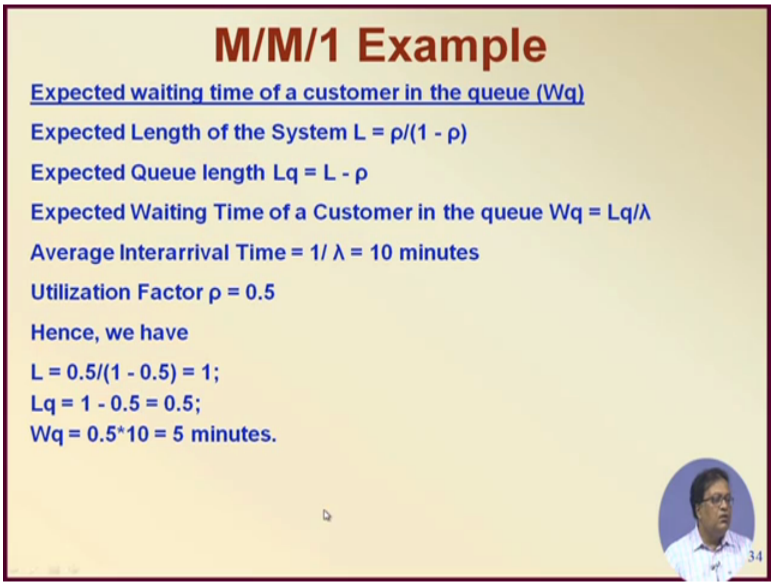

Third one is very simple because we already know L equals to Rho by 1 – Rho, L q equals to L – Rho and expected waiting time of a customer in the queue W q is L q by Lambda,

Lambda equal to 6 so 1 by Lambda will be 10 minutes so utilisation factor Rho is 0.5, so L is Rho by 1 – Rho it comes out to be 1, so L q is 0.5 so W q will be you know L q divided by Lambda but 1 by Lambda is 10 minutes so it will come to 5 minutes right so that is the example.

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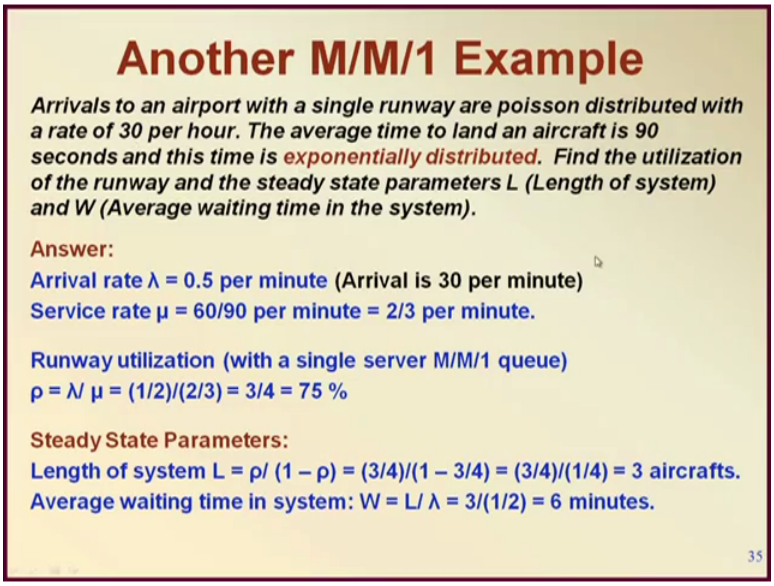

Let us look at another example, in this case arrival to an airport with a single runway are Poisson distributed with a rate of 30 per hour. The average time to land an aircraft is 90 seconds and this time is exponentially distributed, find the utilisation of the runway and steady state parameters L and W right. So again a simple problem, Lambda is 0.5 per minute that is 30 per minute, service rate Mu 60 by 90 that is 2 by 3 per minute right, and runway utilisation Rho Lambda by Mu half by 2 by 3 comes out to be 75 % and then steady-state parameter L Rho by 1 – Rho is 3 aircrafts and average waiting time in system W is L by Lambda comes out to be 6 minutes right. So as you can see that most of these problems in MM 1 queue are very simple problems really and you have to really remember all the important formulas which we have already shown.

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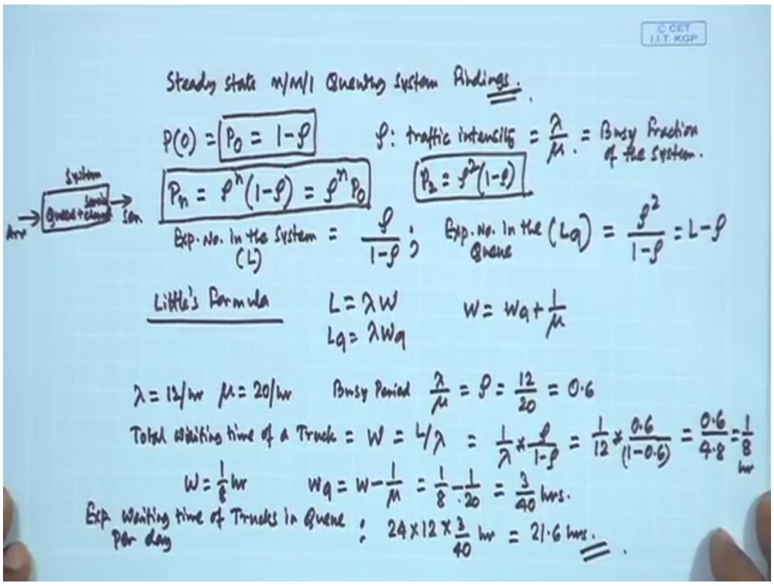

Look at the slide one more time that is P 0 equal to 1 – Rho, P n equals to Rho to the power n 1 – Rho or Rho to the power n P 0 so you can therefore easily calculate P 1, P 2, P 3 and all those things then L equal to you know the expected number of in the system Rho by 1 – Rho, then L q is the expected number in the queue that is L – Rho right and then the little's formula. So if you have this much knowledge, you can calculate almost all the parameters of MM 1 queue only you have to just understand the problem right. So that is about it, we shall solve more problems in our next section before we move over to other things right, so thank you very much.