Decision Modelling

Professor Biswajit Mahanty
Department of Industrial and Systems Engineering
Indian Institute of Technology, Kharagpur

Lecture-16

Queuing Examples (Contd)

Now today we shall continue the queuing problems that we are discussing in our previous class.

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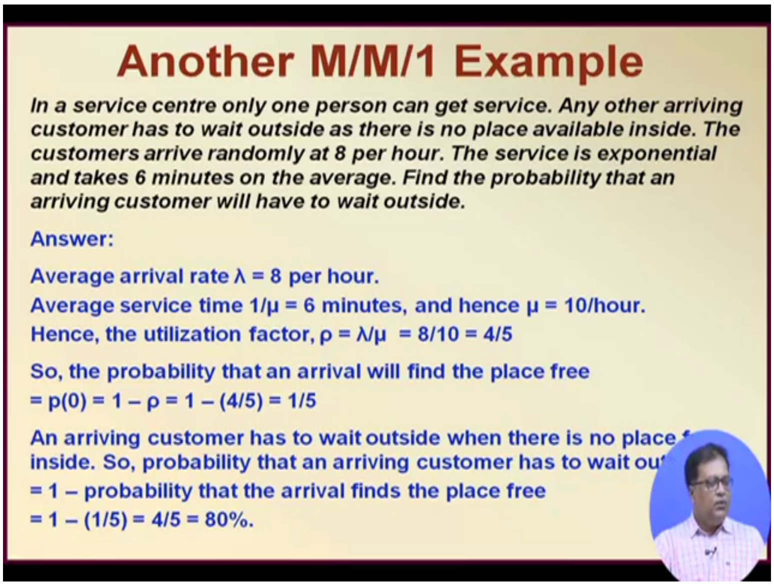

So in this context let us see that problem that we have taken in our previous class that is what we said that in a service Centre only one person can get service, any other arriving customer has to wait outside as there is no place available. The customers arrive randomly at 8 per hour, service is exponential and takes 6 minutes on average, find the probability that an arriving customer will have to wait outside right. So what is basically said, basically said it is here that this is the queue in fact, there is no queue space right.

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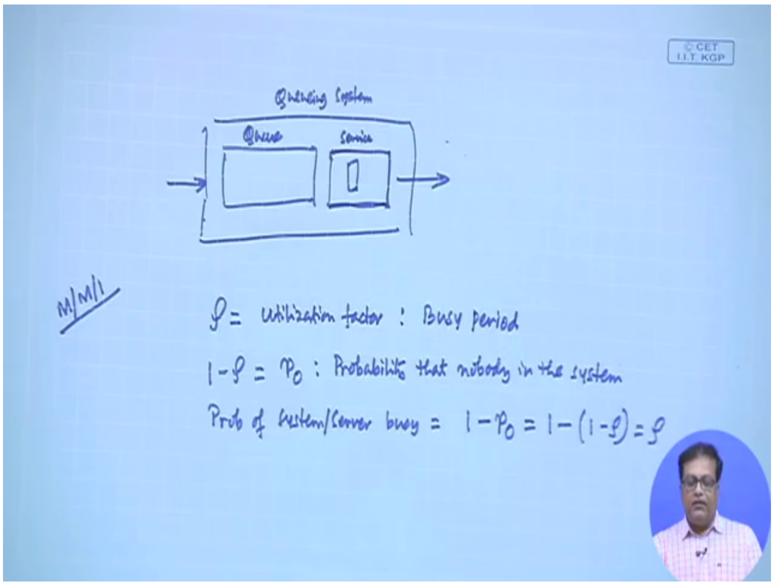

So this is the service but the queue space is outside, it is not that there is no queue space, the system is still the entire thing, this is the queuing system, this is the queuing system and this is the queue, but this queue space is outside that means people are waiting outside and there is a single place where there is no place to wait and this is where the service counter is right and this is where the arrival custom and this is where serviced customer. So what we have to find here? What we have to find here is that an arriving customer will have to wait outside that means what is the probability that the person is in service and it should be 1 – that, is it okay? So is it like a busy period right? It is like a busy period. So let us look at that how this busy period is coming to be equal to right.

So average arrival rate is Lambda is 8 per hour, average service rate is 6 minutes hence Mu is 10 per hour so utilisation factor Rho is Lambda by Mu that is 8 by 10 equal to 4 by 5. So what is the probability that there is nobody in the system that is P 0, that is 1 – Rho and 1 by 5. So if the system is empty, there is nobody in the system so a person who arrives immediately get the place free and get into the service, what is the probability? That probability is nothing but P 0 that is equal to 1 by 5 so there is a 20 % chance that the person will find the place free.

And since there is no place to wait right and an arriving customer has to wait outside when there is no place free inside, so the probability that an arriving customer has to wait outside is equal to 1 – probability that the arrival finds the place free equal to 4 by 5 is 80 percent. And this 4 by 5 is same as the 4 by 5 that is utilisation factor Rho, sometimes it is also called the busy period because what C that once again that Rho is equal to utilisation factor and 1 – Rho is the P 0 that means probability that nobody in the system. Please remember, all the discussions we are doing here they all pertain to MM1 right so that is P 0. So probability that nobody in the system, so what is the probability of system busy or server busy equal to 1 – P 0 equal to 1 – 1 – Rho equals to Rho.

So utilisation factor in case of MM1 queue is also an indicator of busy period, so in the problem that we have done we have got 80 % as the utilisation factor so 80 % is the busy period. And since there is no place to sit, it is also imperative that the people have to wait outside will be 80 percent. But is it all the time this exact same formula is true? That means the probability of the system that is server busy is equal to a person have to wait outside then it may not be so, look at this another problem.

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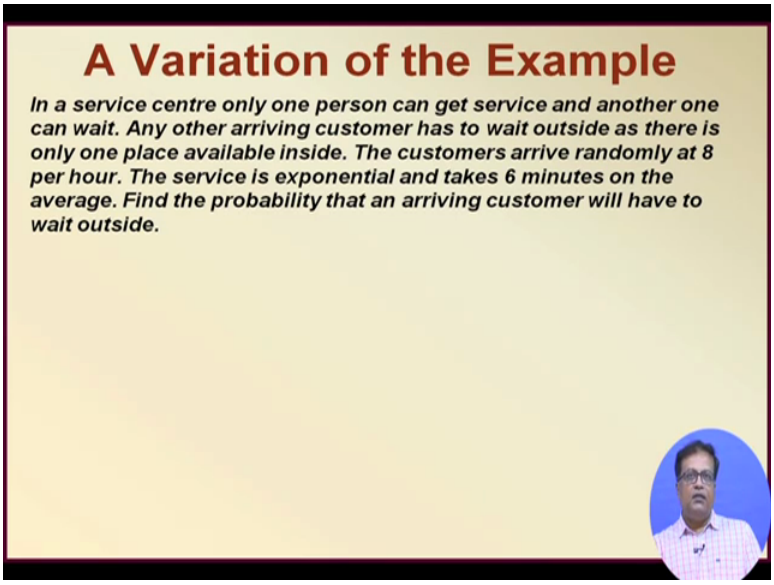

In this problem, one person can get service and another one can wait that means it is like a barbershop with one chair, is it not? Something like that. So any other arriving customer have to wait outside as there is only one place available inside. The customer arrive randomly at 8 per hour, the service is exponential and again 6 minutes on average. Find the probability that an arriving customer will have to wait outside. Look here in this case it is not simply P 0 you know that should be excluded. Suppose even if there is 1 customer in the system right and then he is busy, another arriving customer can still get a chair right, so what should be the probability what should be the probability that the system is free right, then there is nobody in the system or there is one person in the system.

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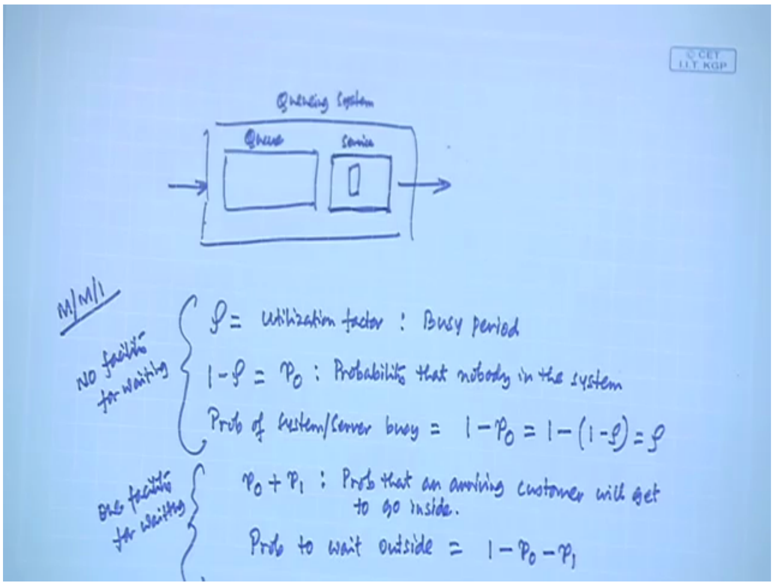

If there is nobody in the system you go directly to the service, if there is one person in the system he must be in the service so go and get the chair. Only if there are more than 2 people I mean only if there are 2 persons in the system then you have to wait outside or 2 or more is it all right? What is that probability? That probability for the second case right, so the first case what we have done that is no facility for waiting and the second case, one facility for waiting that is the second case. So in the second case what is happening, P 0 probability that nobody in the system, P1 is the probability that one person in the system this is the probability that an arriving customer will get to go inside. So what is the probability to wait outside is 1 – P 0 – P 1 right, so that should be the probability to wait outside.

So here it is given that since the Rho is 4 by 5 we have already seen, probability that arrival will find the place free is P 0 + P 1 because either you can get the service or you can wait inside and that is equal to 1 by 5 + 4 by 25 equals to 9 by 25 right. So an arriving customer has to wait outside when there is no place free inside, so probability that an arriving customer has to wait outside becomes 1 – 9 by 25 equal to 16 by 25 or 64 percent. So you see what has happened, the probability of you know waiting outside has reduced from 80 % to 64 % with just giving one facility right. Let us take another problem but this problem is one of uniform distribution right.

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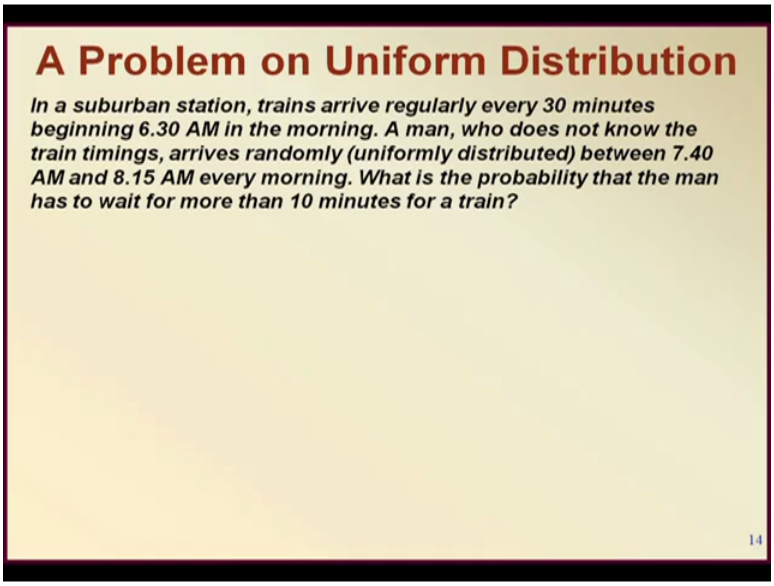

In case of uniform distribution what really happens is that you know the things are not random, in a random facility what is usually happening is that the probability is according to a certain distribution right that is Poisson distribution or exponential distribution if you take the inter-arrival time. But in this case is a much simpler probability problem where reasonably assume an uniform distribution, let us look at the problem. In a suburban Station trains arrive regularly every 30 minutes beginning 6:30 AM in the morning. A man who does not know the train timings arrive randomly uniformly distributed between 7:40 AM and 8:15 AM every morning right, so what is the probability the man has to wait for more than 10 minutes for a train? Is it all right?

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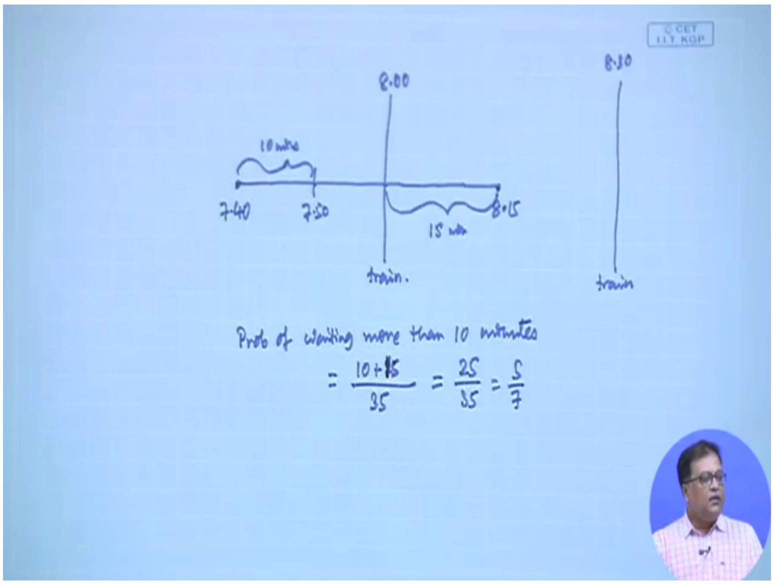

So here you know it is a very simple analysis but really we have to understand this very clearly. So it is like 7:40 the person can come between 7:40 and 8:15, so this is 7:40, this is 8:15, so if I if I draw continuum then this is how the thing is. Somewhere here is 8 o clock and this is where the train is arriving. See the person can come anytime between 7:40 to 8:15, suppose he comes at 7:40, he has to wait for more than 10 minutes, suppose he come at 7:50 so let us say somewhere here, so this is 7:50. If he comes at 7:51 you know little more than 7:50 then he has to wait less than 10 minutes, is it all right? But if he comes after 8 that means train is gone, the next train is that 8:30 right this is where this is the next train is the another train that is at 8:30 because every half an hour there is a train.

So you know he has to wait because he comes by 8:15 so he has to wait all this time anyway. Now, the moment he is gone you know the train, obviously he has to wait that means this is the time and this is the time, these are the time you know when if he comes then he has to wait more than 10 minutes. How much time is this? This is 10 minutes, this is 15 minutes, but if he comes by 7:50 + then obviously he has to wait less than 10 minutes, so probability of waiting more than 10 minutes equal equals to 10 + 15 divided by 35 equals to 25 by 35 equals to 5 by 7 right.

So probability of waiting more than 10 minutes would come out to be 25 by 35 equals to 5 by 7. So look at this, the man has to wait more than 10 minutes that is P 0 X between 0 to 10 and x between 20 and 35. So X is a uniform random variable between 0 to 30, so desired probability will be 5 by 7, (())(14:00) so that is the problem on uniform distribution. Now a very important question that we had discussed so many times but you know still I thought that this point has to be focused one more time.

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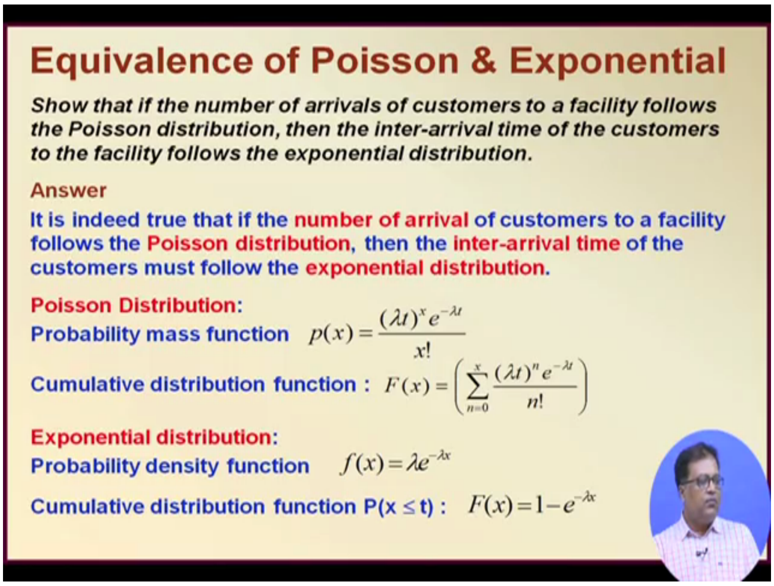

Show that if the number of arrivals of customers to a facility follows the Poisson distribution, then the inter-arrival time of the customers to the facility follows the exponential distribution right, now how can we show this? Maybe we can take an example. So what happens, we find out a result from Poisson distribution point of view and again we find the same result from exponential distribution point of view and see that both the results are similar right, so let us see some of the distribution values. So it is indeed true that number of arrivals of customers to a facility follows the Poisson distribution then the inter-arrival time between the customers will follow the exponential distribution.

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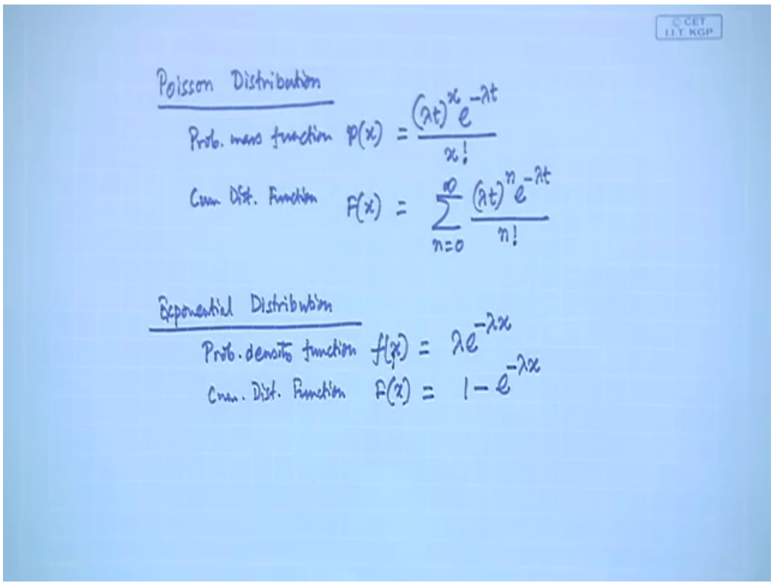

So Poisson distribution which is an discrete distribution, the probability mass function equal to P x equals to Lambda t to the power x e to the power – Lambda t by factorial x right. And if you really sum it then you get the CDF or cumulative distribution function CDF, F x will be sum of n equals to 0 to infinity then Lambda t to the power n e to the power – Lambda t by factorial n. On the other side for exponential distribution, probability density function F x equals to Lambda e to the power – Lambda x right, and cumulative distribution function becomes F x equals to 1 – e to the power – Lambda x right, so these are the results which are going to be useful to us for this understanding, now let us take a particular example right.

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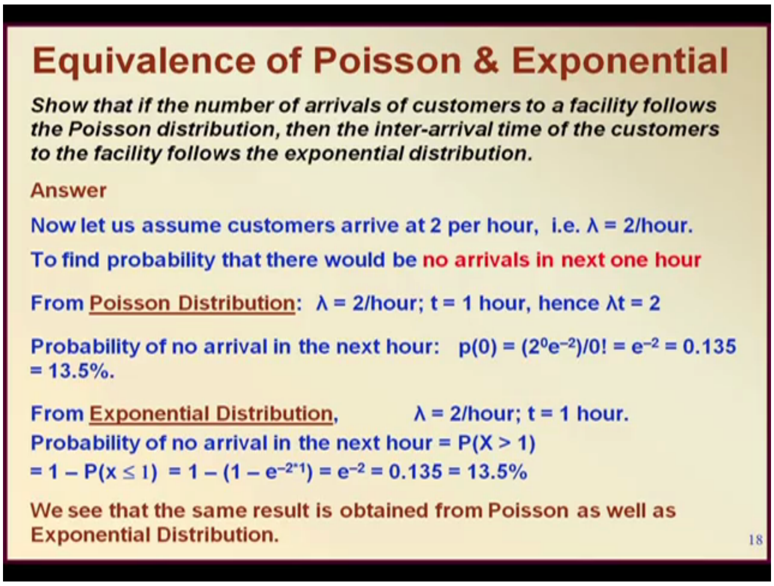

So if we take a particular example that is first of all what we need to do let us take an example just say Lambda equals to 2 per hour and let us take a time of one-hour duration, so what is Lambda t? Lambda t will be 2 into 1 equal to 2. Now what is the probability, suppose we are really asking what is the probability of no arrival within one hour, Probability of low arrival within one hour so P done. What is P done we need to calculate from both sides right, or sometimes we may simply call it as P 0 also that nobody arrives. So what is P 0?

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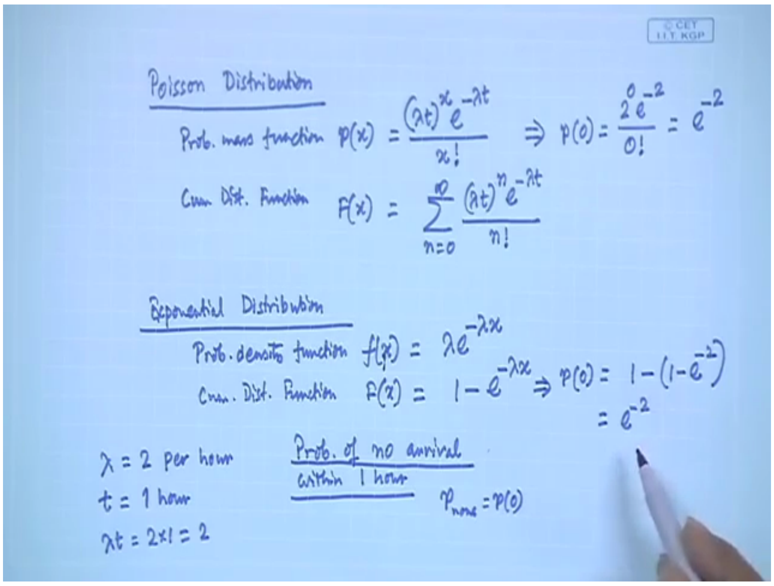

Now look at this probability mass function is Lambda t go to the power this thing so let us put here probability of x equals to 0 and Lambda t equals to 2. So from this equation we get P 0 equal to what is Lambda t, Lambda t equal to 2, x equal to 0, e to the power – 2 by factorial 0. 2 to the power this is becomes 1 so this becomes e to the power – 2 right so this is what we get from the your Poisson distribution. Similarly suppose we use this one, please also remember that this F x is nothing but P x less than equal to 1 so that means what will be the probability that nobody will arrive right, so we need to find out probability of no arrival from here P 0 that will become 1 – 1 – e to the power – 2 right that also becomes e to the power – 2 right.

So must remember that when we have to find out the probability of no arrival then this is the cumulative density function that means the entire arrival time is between 0 to 1 right that is what if you want to find out that you have to find out this formula I mean within the given time and if we want to find out that none arrives within this given time that will become 1 – of this function so that will be 1 – 1 – of e – 2 equal to e – 2, so both the figures are similar and look at the slide similar thing is also described there that for Poisson distribution we find probability of no arrival P 0 equals to e 2 the power – 2 is 13.5 % and from exponential distribution it is P x greater than 1 which is 1 – P x less than equal to 1 right. That means this is the thing that we shall get actually the particular function the cumulative density function so this we can do and we can get e to the power – 2 equals to 0.135 is 13.5 % right, so this is what we can get.

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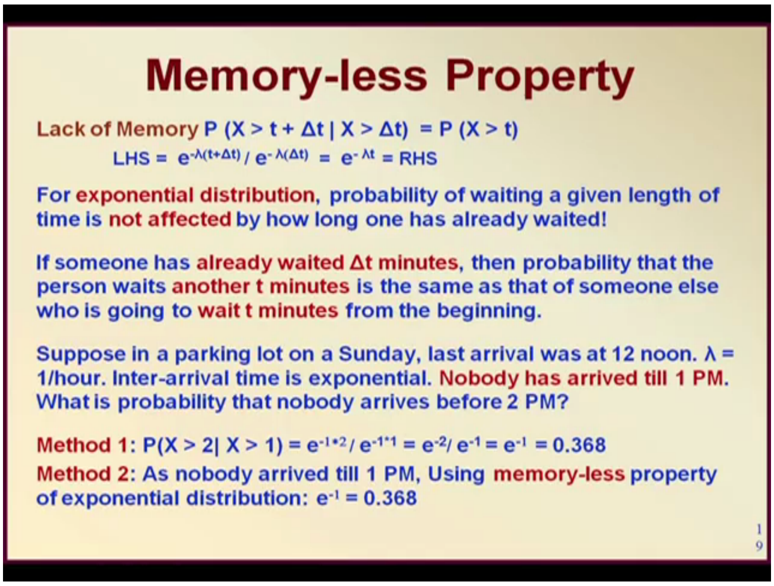

Now another very very important the property which we have discussed earlier also which is known as the memory less property and why it is called memory less let us understand this once again. Say suppose let us look at an example first before we look at those equation, if some suppose on a parking lot on a Sunday, the last arrival was at 12 noon right 12 noon. Now so Lambda is 1 per hour, inter-arrival time is exponential, nobody has arrived till 1 PM, what is the probability that nobody arrives between 2 PM?

You see this problem is like there is the last arrival was at 12 noon, up to 1 PM nobody has come, what is the probability that you know nobody arrives before 2 PM. So you tend to think that if you start thinking from 12 then what is the probability that nobody arrives in 2 PM given that nobody arrived by 1 o'clock, is it all right? So maybe you tend to calculate that this particular thing that is the conditional probability, probability of X greater than 2 given that X is greater than 1, is it alright? But already we have seen that probability of something so here itself let us know we can do this side.

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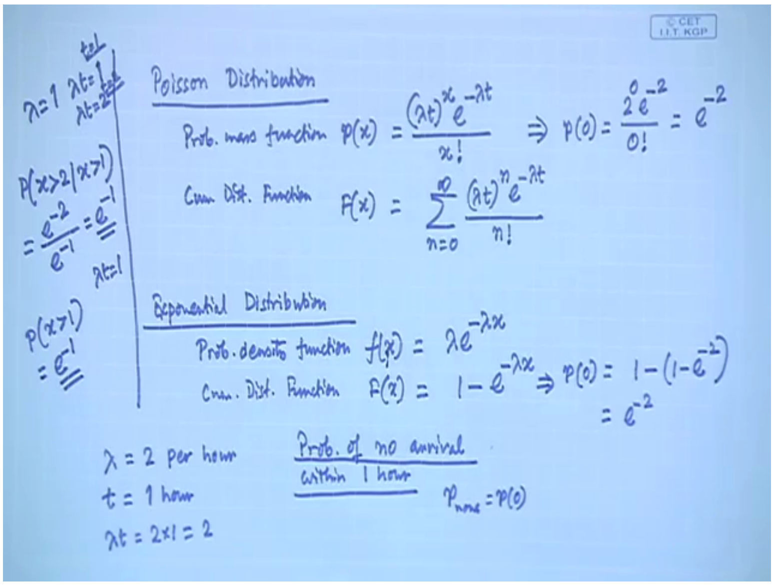

So supposing what is the probability of X greater than 2 given that X is greater than 1, is it all right? So it is like a conditional probability. So what is the probability of X greater than 2? This is nothing but e to the power – 2, so already we have seen what is the probability of X greater than 1 and there it was Lambda is 2, here Lambda is equal to 1 right so Lambda t is also 1, is t equal to 1, Lambda t equals to 2 if t equals to 2, this is t equals to 2 right. So this will be therefore e to the power – 2 by e to the power – 1, so it will now come to nothing but e to the power – 1. But if you would have directly calculated that you know it is like nobody has arrived till 1 PM so simply start time from there and not at 12 PM right. So 1 PM is the start of time, what is the probability that next arrival will be by 2 PM?

It is nothing but probability of x greater than 1 right, nobody arrives by 2 PM so it is nothing but e to the power – 1 because Lambda t equals to 1 so this is this value is same as this value, this is all right. So this is exactly what happens that you know if someone has already waited Delta t minutes then probability that the person waits another t minutes is the same as that of someone else who is going to wait t minutes from the beginning? Supposing you are playing game of dice right and you will win if you get a 6. You have assumed that it is a perfect Markovian process and it is an unbiased dice and casting it has absolutely no issue. So supposing you have already played 5 times and you have not got a 6, what is your probability of getting a 6 six times?

You may be thinking no it is almost certain, but in reality it is exactly same as someone who is casting the dice for the first time, this is all right there would be no difference at all right.

So if you look at this memory, lack of memory P X greater than t + Delta t given that X is greater than Delta t, it is a conditional probability, it is nothing but P X greater than t is all right. So LHS is e to the power – Lambda t + Delta t by e to the power – Lambda Delta t, it comes out to be e to the power – Lambda t right, so this is precisely what is known as the memory less property. So if we already know that nobody has come by 1 PM, we can easily calculate those things from there and we need not really worry about the other things.

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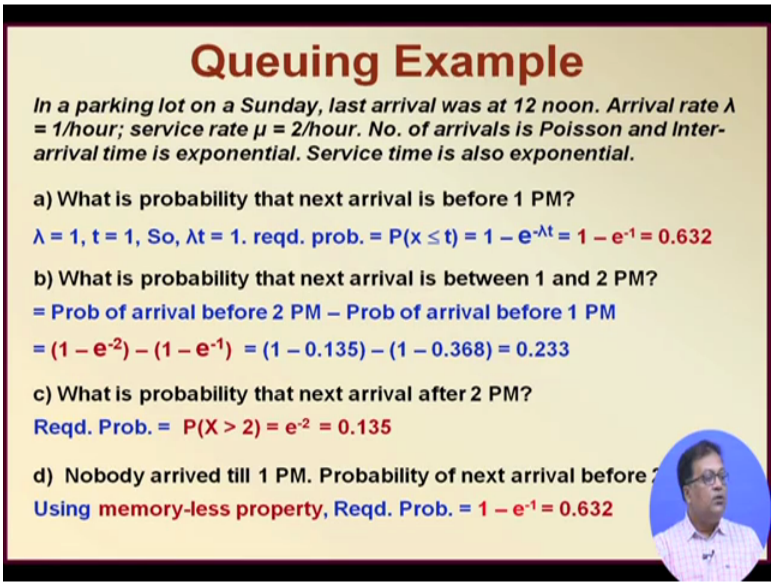

So here is an example, in a parking lot on a Sunday, last arrival was that well-known. Arrival rate was 1 per hour, service rate was Mu equals to 2 per hour, number of arrivals is Poisson and inter-arrival time is exponential, so service time is also exponential. So what is the probability that next arrival is before 1 PM right? What is the probability that next arrival is between 1 and 2 PM? What is the probability that next arrival is after 2 PM? And last one, if nobody arrived till 1 PM, what is the probability of next arrival before 2 PM?

In all these questions what we really have to understand is you know that in the first case what is the probability that next arrival is before 1 PM right? So in this case Lambda equals to 1, t is also 1 right, so it should be Lambda t equals to 1 so how much it should be? The probability will be, what is the probability that the arrival will be beyond 1 PM? It will be e to the power – 1, it will be the reverse of that. So let us look at this that Lambda equals to 1, t equals to 1 so Lambda t equals to 1 and therefore, required probability will be 1 – e to the power – Lambda t is 1 – e to the power – 1.

In the second case, what is the probability that next arrival is between 1 and 2 PM? So what is the probability of arrival before 2 PM – probability of arrival before 1 PM, So you know that already we have seen for 1 PM 1 – e 1, so before 2, 1 – e to the power – 2 so the first one – the second one. What is the probability that next arrival after 2 PM it will be e to the power – 2 already we have seen it and nobody arrived till 1 PM, Probability that next arrival is before 2 PM, again we can use memory less property and it should be one – e to the power – 1.

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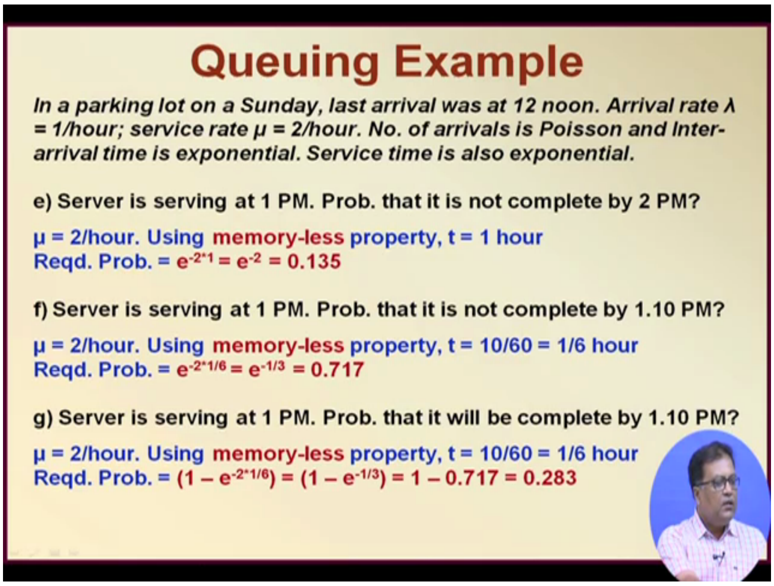

Let us look at some of the service side also, server is serving at 1 PM, probability that it is not complete by 2 PM, not complete by 1:10 PM right and not complete by the last one you know we wrote wrongly that is okay the same figure is taken but let us ignore the third part, let us look at the first 2, so what is the probability that server is serving at 1 PM, probability that it is not complete by 2PM right. So Mu equals to 2 hours using memory less property t equals to 1 hour so required probability is e to the power – 2 star 1 is e to the power – 2.

And second case the t is 1/6th over so it should be e to the power – 1 by 3 right. So these are the some of the problems that we can really work out and in the next class we shall continue to see certain other systems other than MM1 right so thank you very much.