Course on Decision Modeling
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Lecture 32
Module 7
Introduction (Continued)

Right, so last class we have started our discussions on graph theory particularly the network models and as I have said in the previous class already that in this particular discussion on networks we are basically discussing the graph theoretical concepts specifically for the network flow problems and the shortest path problems, right? Basically the kind of problems that would be really required to solve this kind of network related problems only.

And graph theory is a vast subject and this particular theory is also used for lot of other applications specifically the computer science related applications. So if you take a very quick recap what I said on the previous day that graph is a set of edges and vertices and there are certain things about a graph that you know the vertices 2 vertices are adjacent if there is an edge between them and this edge is incident on those particular vertices and number of edges incident on a particular vertex is called the degree of that particular vertex.

So some of these concepts we have discussed in our previous class let us continue from there.

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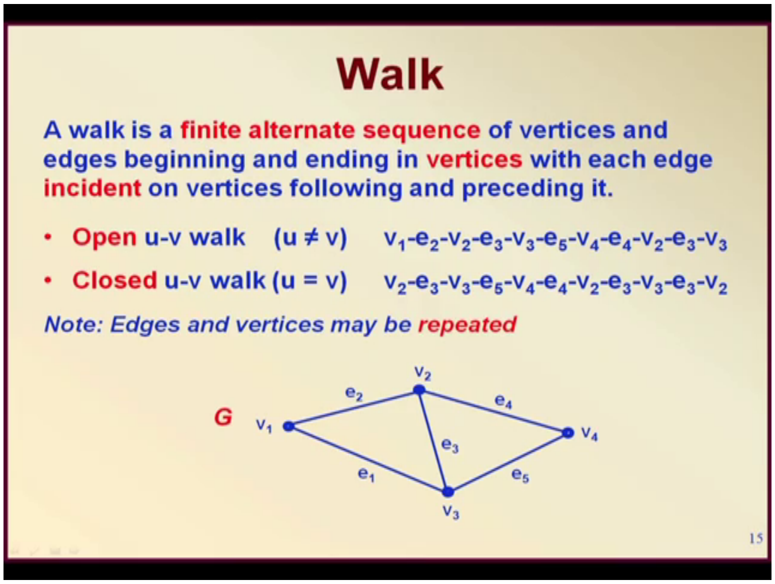

So today we shall see a certain kind of concepts the first one is a walk. Now what is a walk? A walk is a finite alternate sequence of vertices and edges beginning and ending in vertices with each edge incident on vertices following an preceding it. Now I do not know what you really understood from this particular definition but let us really try to see what it is by looking at a particular graph.

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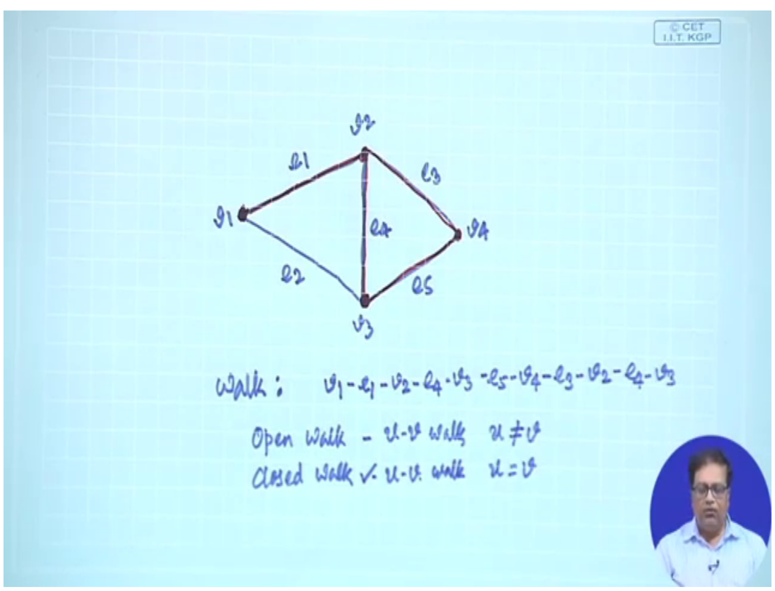

So suppose you know we have a particular graph like this let us take a very simple graph, so suppose we have a graph like this, right? So suppose this is v1, this is v2, this is v3, this is v4 there are 4 vertices and let us say there are 4, 5 edges, right? So very simple graph with 4 vertices and 5 edges. So walk will be let us say we start with v1 then you know it should be an alternate sequence of vertices and edges staring from vertices.

So we started at v1, then let us say we take e1 we go like this so we can take another pen so from v1 we go to v2, right? So that is the first is a first leg, right? Then after that we might come to another vertex v3, then we may go to another vertex v4 so v2 to e4 to v3 to e5 to v4, right? Now you may question that you know if from v4 can we go to v1? Answer is no, we cannot go, why? Because there is no edge available between v4 and v1, alright. So from v4 suppose we again come back to v2, so v4 e3 to v2 and then we may take that path once again so we take that path one more time second time just look we have gone second time, so v2 to e4 to v3.

So this is an walk, so what is an walk? Walk is an sequence of vertices and edges starting from vertices such that each edge considered is incident on the vertex, right? And while doing so we can repeat both vertices and edges, right? So that is the concept, so it is the most general but then again there are 2 kinds of work there is open walk and closed walk. So the kind of walk that we have taken just now is it open walk or closed walk, see if the starting vertex and the ending vertex is a same, then it is a closed walk.

So a simple thing we say that an open walk we call it an u-v walk with u not equal to v, right? And a closed walk is an u-v walk where u is equal to v. So that is the difference between the open and the closed walk.

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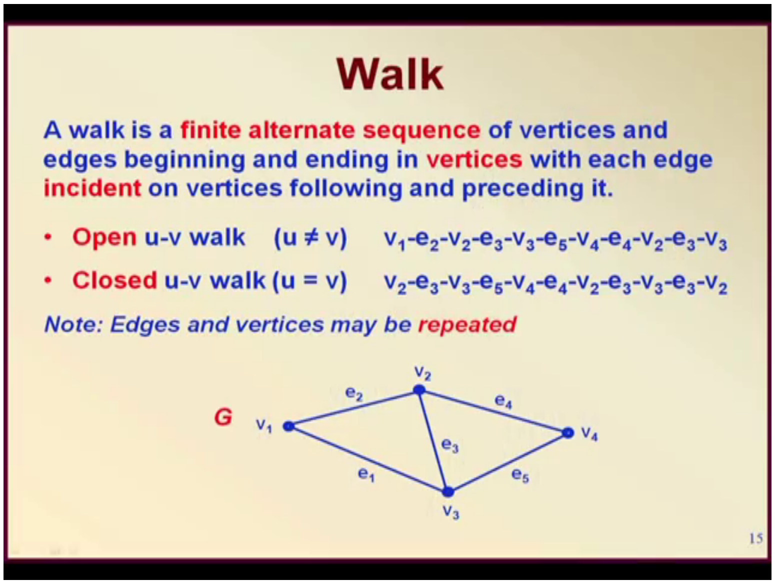

So same thing you just now understand this that open walk in this particular graph v1, e2, v2, e3 then v3, then e5, v4, then e4, v2 and then again e3, v3, is it alright? So close walk starting at v2 and ending at v2. So v2, e3, v3, e5, v4, e4, v2, e3, v3 and then e3 again e3, v2. So staring at v2 ending v2, right? So this is called an walk.

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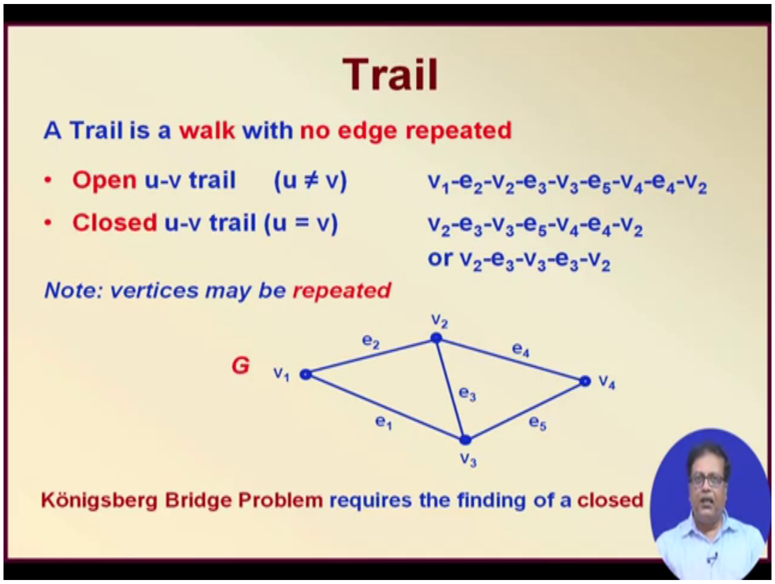

Now the next term what is important in this context is called a trail. So what is a difference between an walk and a trail? The trail is also an work but the all trails are not works, what is the difference that here no edges can be repeated, right?

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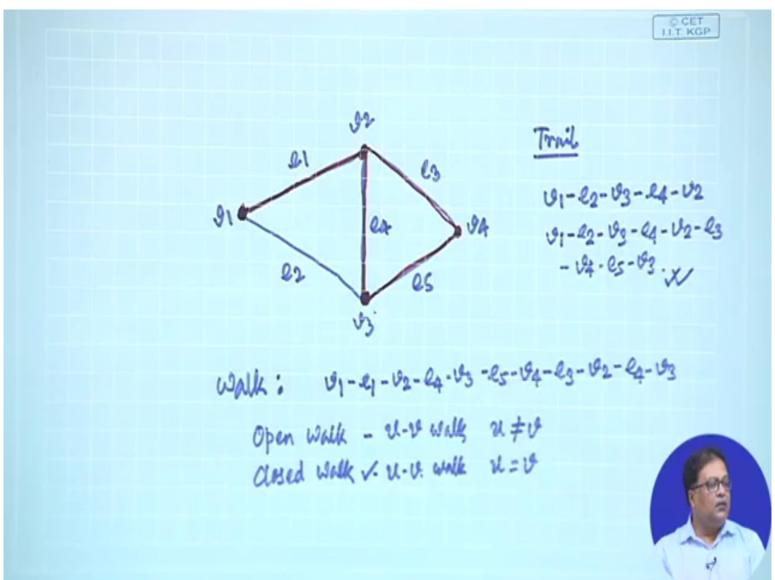

Edges can be cannot be repeated, but vertices can be repeated, is it alright? So in this case let us look at some trails, so trail 1 trail could be that v1, e2, v3, e4, v2, right? So v1, e2, v3, e4, v2 that is a trail, right? Supposing if we have v1, e2, e3, right? v1, e2, v3 then e4, v2, e3 and then continue v4, e5, v3. So you see here the v3 is repeated, right? The same we have gone like this, we have gone like this and we have finally ended here. So this one is a trail definitely because no edges is repeated but it is not a what is known as a closed trail is a open trial.

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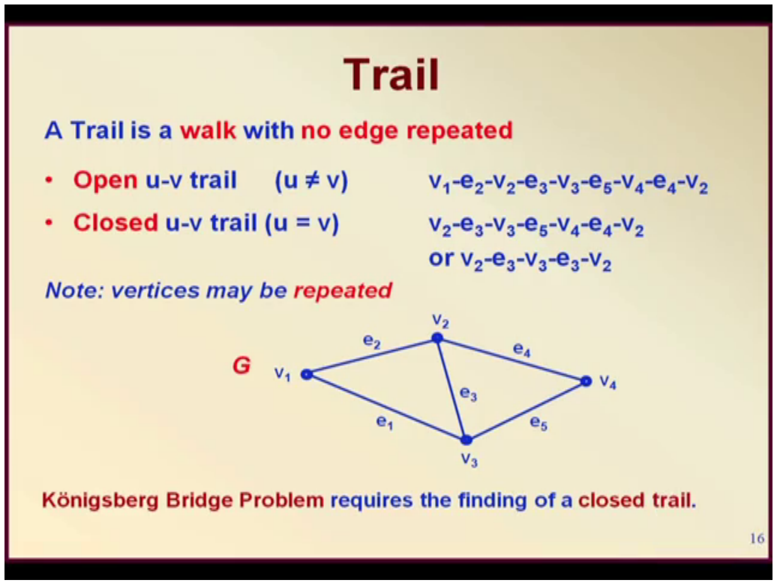

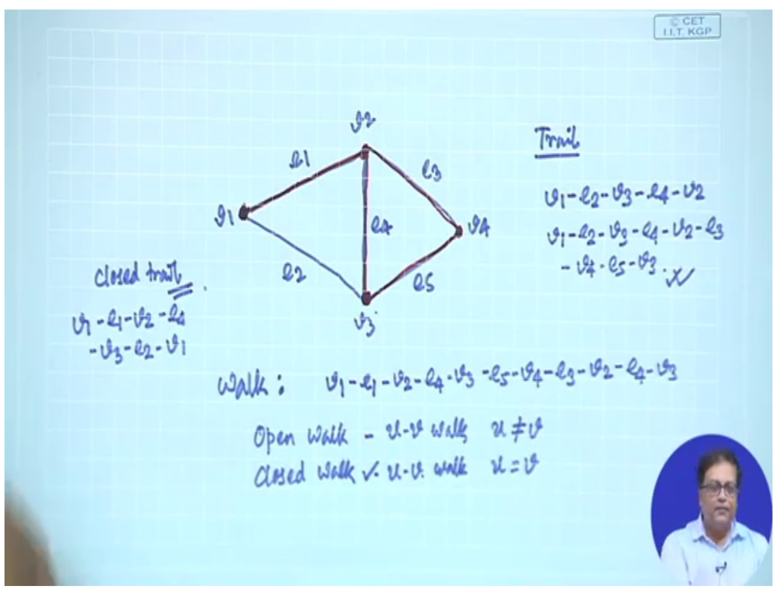

What happens in a closed trail you know in a closed trail we have you know a particular if you look at a closed trail just look at the closed trail the closed trail would be v1, e1, v2, e4, then v3, then e2 and v1 so this is a closed trail, right? Is a closed trail, right? So that is what that no edges should be repeated. Now after that let us look at the next one that is called a path. A path is basically is a trail with no vertex repeated, right? So we have the work we have vertices and edges both are repeated, a trail where edges are not repeated but vertices can be repeated and finally the path where neither vertices nor edges can be repeated, right?

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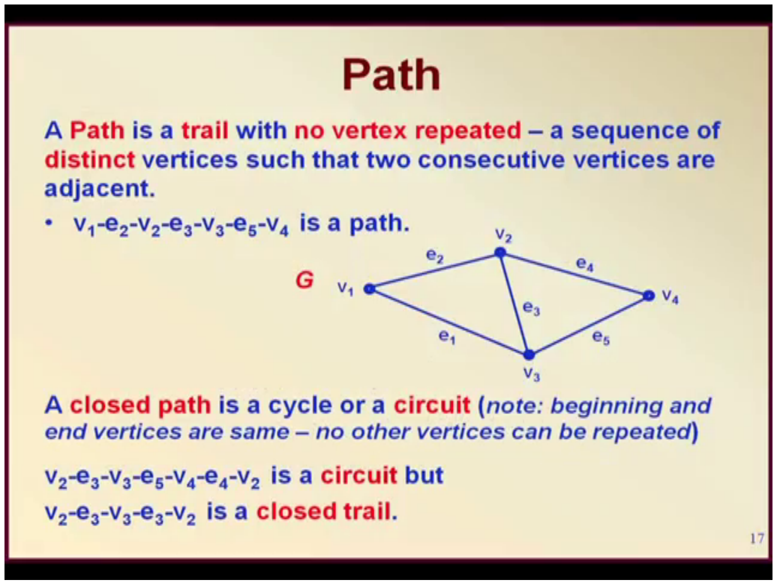

So here in this diagram so look at this that v1, e2, v2, e2, v2, e3, v3, e5, v4 is a path, right? And closed path is called a circuit. So what is a closed path? So v1, e2, v2, e3, v3, e1, v1 so in this case you know see what is happening the beginning and ending vertices are same, right? Beginning and ending vertices same but no other vertices can be repeated. So some question maybe raise that you know how come it is a path because this beginning vertex is repeated but you know we call it a circuit so that is a closed path.

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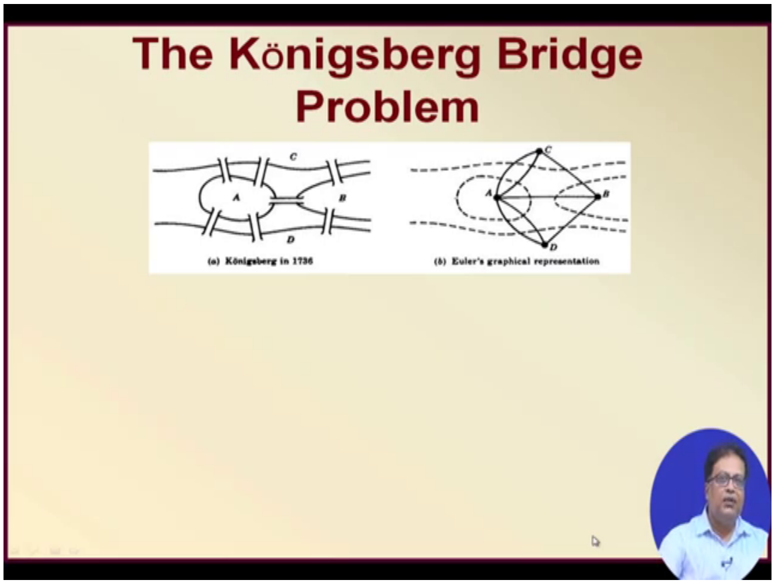

So more about the circuits and other things later and let us know continue and see a particular problem that we are calling the Konigsberg Bridge problem.

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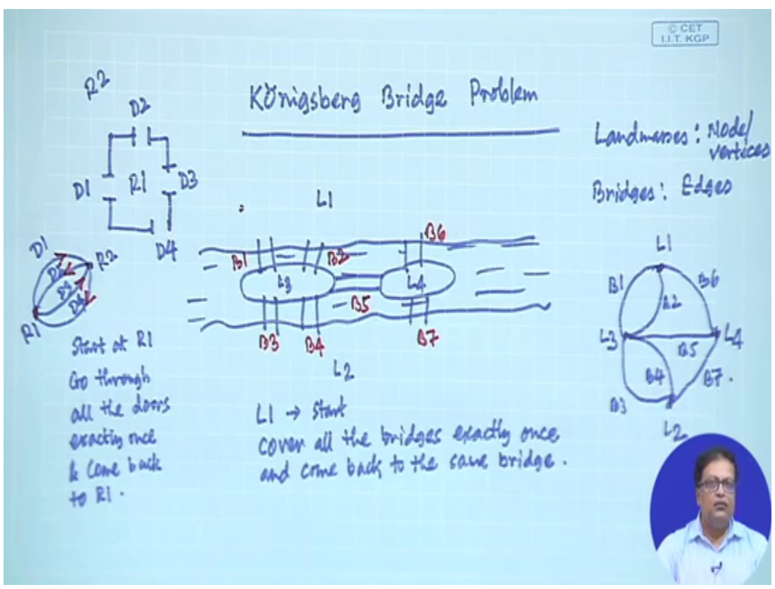

So you see this is a very interesting problem the problem as you can see let us draw this the Konigsberg Bridge problem in this case supposing there are 2 landmasses in a river, right? So this is a river and in this river there are 2 landmasses and there are 2 landmasses 2 more landmasses on the both sides of the river. So we have got therefore 4 landmasses, what are these? This one is L1, let us say this is L2, this is L3 and this is L4.

Now this 4 landmasses are connected by 7 bridges, so what are those 7 bridges? This is 1 bridge, this is another bridge, this is another bridge, this is another bridge, this is the fifth one, sixth one and seventh one, alright? So these are the 7 bridges let us name those bridges you know this is a B1, this is B2, this is B3, B4, B5, B6 and B7 so there are 7 bridges they are available in this particular you know river and you know connecting the 4 landmasses. So what is to be done what we have to do is start from any of the land masses let us L1 start at L1, what we have to do cover all the bridges exactly once, right? Cover all the bridges exactly once and come back to the same bridge, right? Cover all the bridges exactly once and come back to the same bridge. So you see supposing we start from this particular bridge come to L3 again come to L1. Now you know you can come to a landmass more than once, right? But the bridges should be covered exactly once, question is it possible or not possible, right? That is the question.

What is the first task to do is can we convert this to a network problem so that could be our first task let us convert this to a network problem. So what which one should be known and which one should be our vertices, right? Or node or vertices and which one should be arcs or edges. So quick thinking if you do you will find this landmasses will be like landmasses will be like node or vertices and the bridges are like edges, right? So this is understood, once this is understood then we can convert this to a network, so let us take the 4 landmasses L1, L2, L3 and L4 so these are the 4 landmasses.

Now L1 and L3 connected by 2 bridges you look it is like a parallel edge. Similarly, 2 bridges here, single bridge here and a single bridge here so you see these are the 7 bridges. So B1, B2, B3, B4, B5, B6, B7 so this is the graph theoretic representation of the same problem, the question is that starting from a particular landmass can we move through all the bridges exactly once and come to the same landmass. Let us give an alternate problem to the same thing, suppose I have a room and this room has got 4 doors, these are the 4 doors, right? So call it D1, D2, D3 and D4 these are the 4 doors.

Now question is that say this is room 1 and this outside is part of the bigger room call it R2, right? So it is like is a big room within that this small room is there and this room has got 4 doors. Now starting start at R1 go through all the doors exactly once and comeback to R1, so this is the problem that can we start at room 1 go through all the doors exactly once and come back to R1 is it practically possible, right?

So in graph theoretic language how it looking like, so what should be the graph for this particular problem, right? So again look here how many vertices there are 2 vertices, R1 and R2 so we can put R1 here and R2 here, and how many doors? There are 4 doors, so you can connect those 4 doors in this manner, right? So D1, D2, D3 and D4. Now tell me is it possible or not possible? See, very interesting and very easy also all you have to do use any of the door from R1 go to R2 take another door come back to R1, take another door go to R2, take another door come back to R1.

So it is a very simple problem and the solution is easily available that yes the answer is yes it is possible, why? Because you see from R1 have 1 door and to go out and another door to come back, similarly, D3 and D4. So essentially what it boils down to that as long as I you have those doors in pairs or you have edges in pairs you can find the solution to such problems, right?

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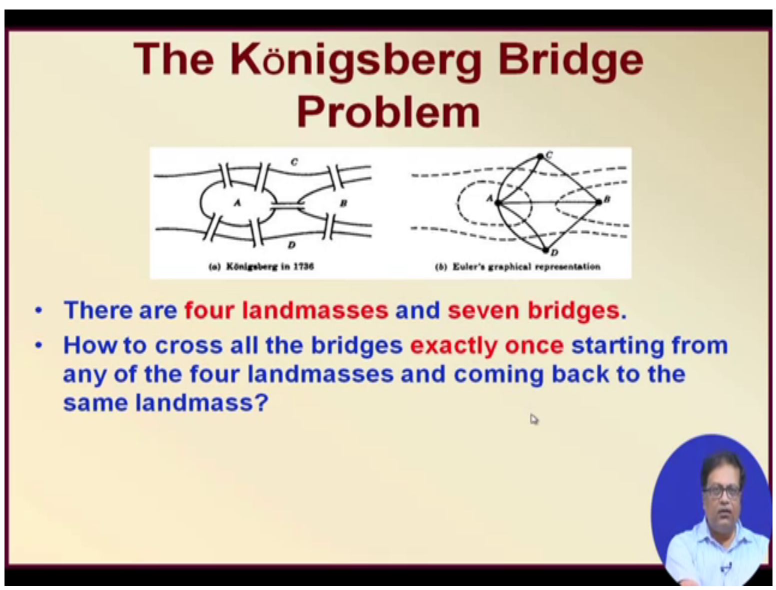

Now let us see that whether it is you know leading to some theory. So there are 4 landmasses 7 bridges that we know and how to cross all the bridges exactly once starting from any of the 4 landmasses and coming back to the same landmass.

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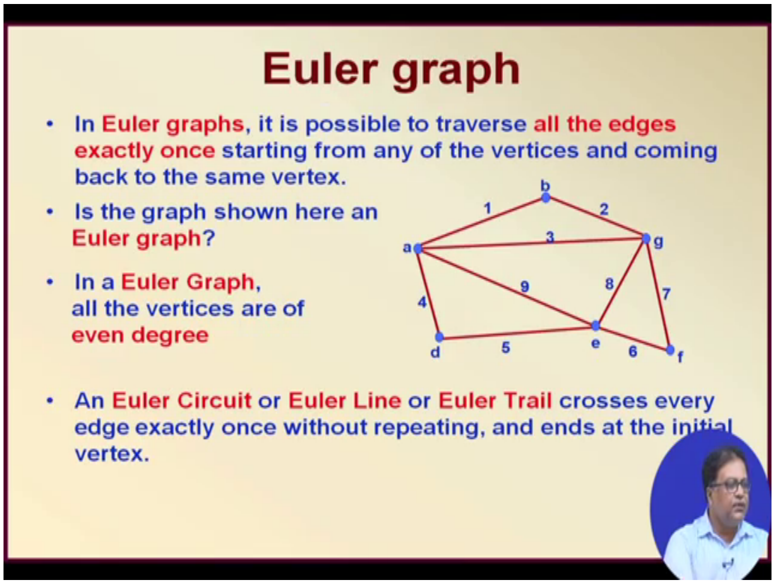

So answer of this question we shall see later but let us look at some concept you know that is called a concept of Euler graph. The idea of Euler graph is that in Euler graphs it is possible to traverse all the edges exactly once starting from any of the vertices and coming back to the same vertex, right?

So can you just now we have gone through the concept of work trail and path. So what does it mean in that language? You see work is a concept where it say set of vertices and edges where vertices and edges both can be repeated. Trail is a concept where vertices can be repeated but edges cannot be repeated. And path is a concept where neither vertex nor the edges can be repeated, is it alright?

So therefore an Euler graph is you know really matching with which concept, here the vertex can be repeated but edges cannot be repeated, right? Additionally we are telling you have to go through all the edges so it is like a concept of the closed trail, right?

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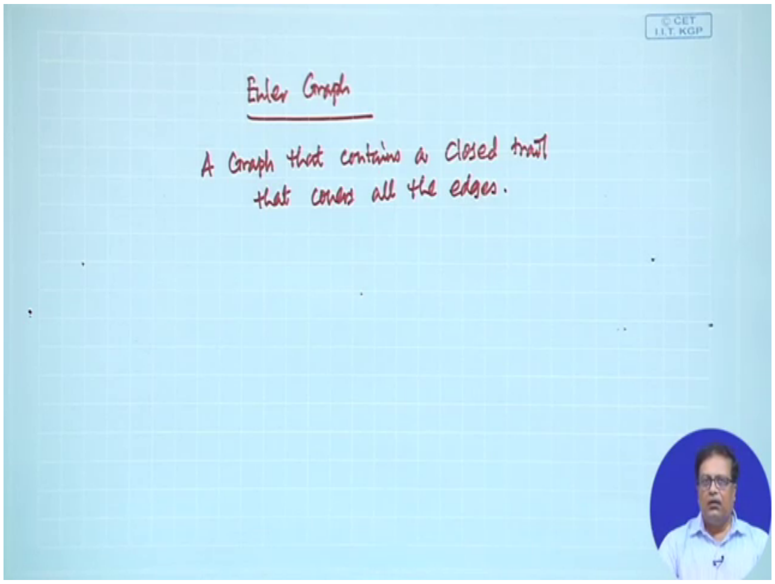

So the essential idea of a Euler graph find a close trail Euler graph is a concept a graph that contains the closed trail that covers all the edges, right?

So if you can find the close trail that covers all the edges then we can call that is an Euler graph. So in this particular graph look at you know we start with A go to B, go to G then go to A and then go to D and then go to E, then F, then G, then E once again and come back to A, right? So 1, 2, 3, 4, 5, 6, 7, 8, 9, right? So it is an Euler graph because we have a closed trail and these is sometimes called an Euler circuit or and Euler line or an Euler trail, right? This crosses every edge exactly once without repeating and ends at the initial vertex, right?

So this Euler trail in the Euler graph all the vertices are of even degree, that is very interesting, why even degree? Think of that door problem once again if you have to come back to the same room you must have one door for going out another door for coming back, (())(22:39). So that means door should be in pairs, so in this case if you go out from A from from 1 edge you must have another edge to come back. If you have come back to A but still 2 more edges are remaining so again you have to go out in 1 and come back in another.

Same thing should hold for all the vertices or in simple words all the vertices should be of even degree, is it. That is the requirement of an Euler graph that all the vertices should be of even degree, right?

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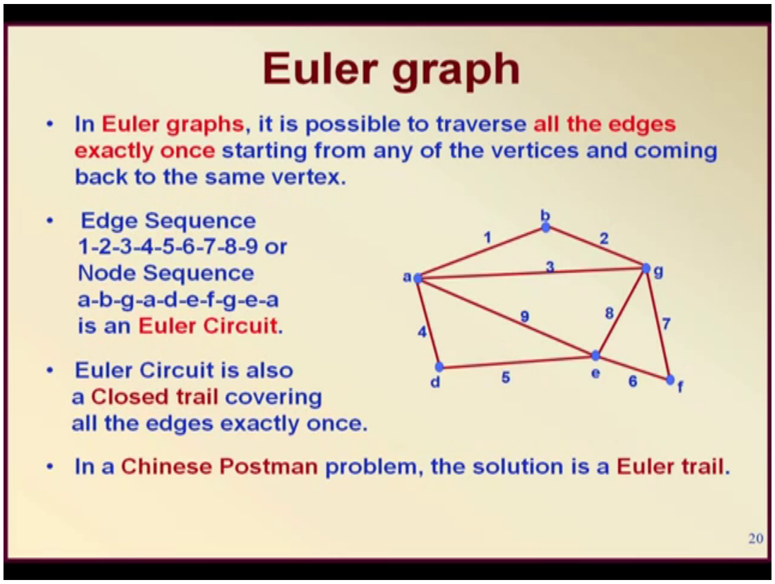

So that is what we find here that we it must be a closed trail all the edges should be exactly once. You know a type of problem that is called the Chinese Postman problem where the basic idea is to find out an Euler trail. So what is a Chines Postman problem?

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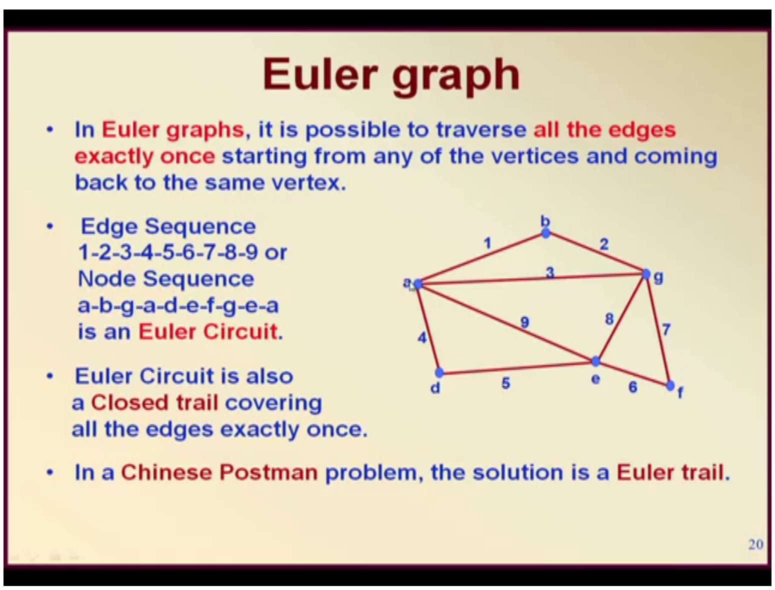

The look at this graph supposing these are the road network in a particular locality, now in this locality you know on each of the road there are lots and lots of houses each of the road.

So assume there are 9 roads and each roads has got lots of houses, the postman is currently at point A, what the postman would like to do? The postman would like to you know deliver letters and packages to all the houses, but you know since the postman has to deliver letters to all the houses it is therefore good for the postman that you know what the postman would like is to travel the minimum. Assume the postman is going from A to B and again comes back to A, then what happens road 2 is not covered so again the postman then has to go through perhaps one and you know then cover road 2 and then maybe take some path and come back. So that way the post man may have traveled 1 3 times, first go and come back and again take 1 and go to door 2. So this way the total travel of the postman is rather high.

So if an Euler trail is available then the Chinese Postman would be able to really deliver all his or her letters to all the houses by covering the minimum distance. So there must be lot of equivalent you know problem which are acing to this problem in all this problem essential idea is can we find then Euler trail.

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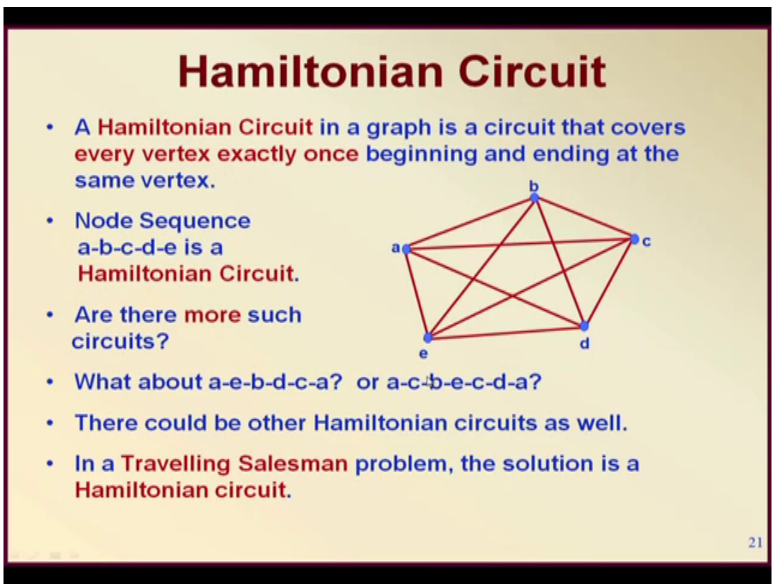

Now while discussing Euler trail a closely related problem is what is known as you know finding a Hamiltonian circuit. The Hamiltonian circuit is you know it covers every vertex exactly once and while beginning and ending at the same vertex.

So what is the difference? The difference is earlier we are covering all the edges now we have to cover all the vertices exactly once. So maybe you can go from A to B, then C, then D, then E and then come back to A, right? So assume these are all the possible path sequences so A, B, C, D, E is a Hamiltonian circuit. Now are there more circuits in fact you may find there are 100 of such circuits, right? That you can also take AC, BE, DA. What is very interesting to note that all these Hamiltonian circuits they are not of the same length, right? Because you see if you go from A to B look in that second path that you know AC there is 1 path that goes through you know A, B, C, D, E. So A to B is included.

In the second path when you take A, E, B, D, C, A or A, C, B, E, C, D, A AB path is not there another set of 5 paths are there. So what is the difference? The difference is that all this path lengths will not be same, is it alright? So the Hamiltonian circuits that we shall have they all will have different kind of if there are weights suppose if there is a road network and there are distances, so distance is like a weight.

So these distances when you add together you may find that total distance is different in different paths. So question that comes is that which path should one consider in order to travel the minimum such a problem is called a traveling sales men problem, right? So essentially the solution is nothing but a Hamiltonian circuit the idea is the salesman has to go from 1 city to all the cities exactly once what should be the path the salesman should follow so as to travel the minimum, right?

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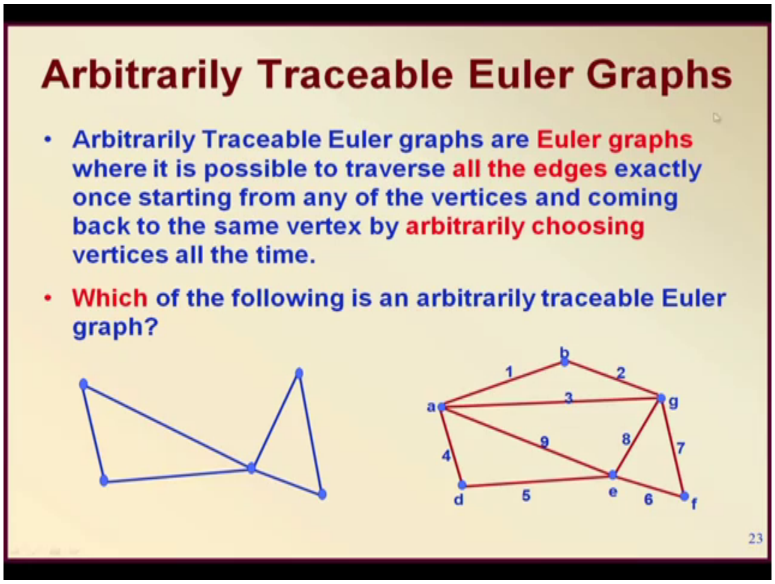

Related concept could be you know the quickly let me show then we will come back here is called an arbitrarily traceable Euler graph. So this is also an Euler graph the only difference is that at every vertex you reach you have the choice to choose any of the paths that you like, right? So here you know if you come here, then suppose you know at this point if you decide to go back this way then you do not cover the all this things, so is it arbitrarily traceable?

So these are interesting question, here suppose you go from A, B, G and then you decide to take another path, right? So it maybe that a particular graph has an Euler graph like here see every degree is even. So there is an Euler path but question is that are, sorry not Euler path the Euler trail but in this case is there is it arbitrarily traceable? Right? Arbitrarily traceable suppose here you decide to go back again you come back but you do not cover this so it is not.

So same thing here also so if you can do that such a graph is called an Euler graph, right?

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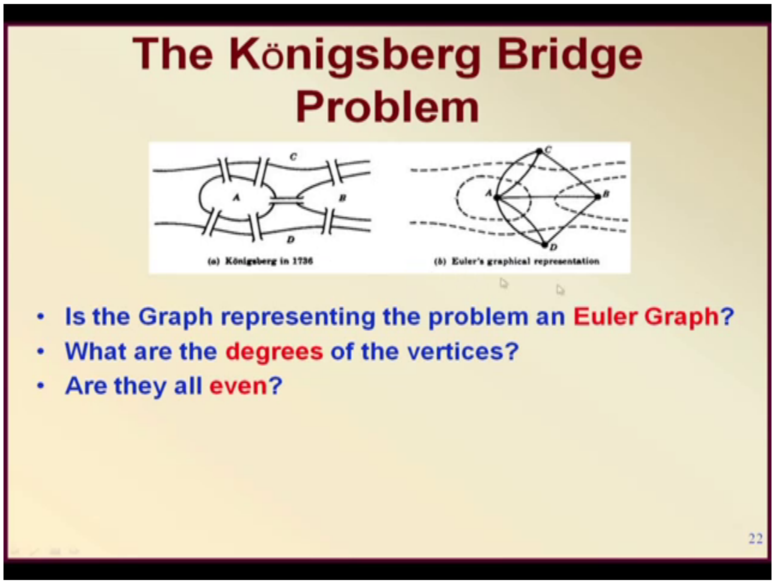

So before I end let us try to see what should be the solution to the Konigsberg Bridge problem. So this is the problem and this is the graphical representation look at the degrees, what is the degree at A? It is 1, 2, 3, 4, 5, what is at D? 3, at B? 3 and C also 3. So since the degrees are not even, right? Will there be an Euler graph? Answer is no this graph is not an Euler graph and there is no Euler trail. Since no Euler trail is there this solution is not possible, so alright?

So we discussed some of the very important graph theoretic concepts today and I stop here. We will continue our discussions but now this time we will see some operations on graph and then the most important concepts of tree and spanning tree, thank you very much.