Course on Decision Modeling
By Professor Biswajit Mahanty
Department of Industrial and Systems Engineering
Indian Institute of Technology, Kharagpur
Lecture 33
Module 7
Operations on a Graph, Tree and Spanning Tree

Right, so previous class we have seen some graph theoretical concepts specifically the Euler graph and how an Euler graph is useful in solving Chinese postman problem and an Hamiltonian circuit is the answer for a traveling salesman problem. Now we move ahead and some other important concepts particularly the operations on a graph and tree and spanning tree, so these concepts we shall discuss in this particular class.

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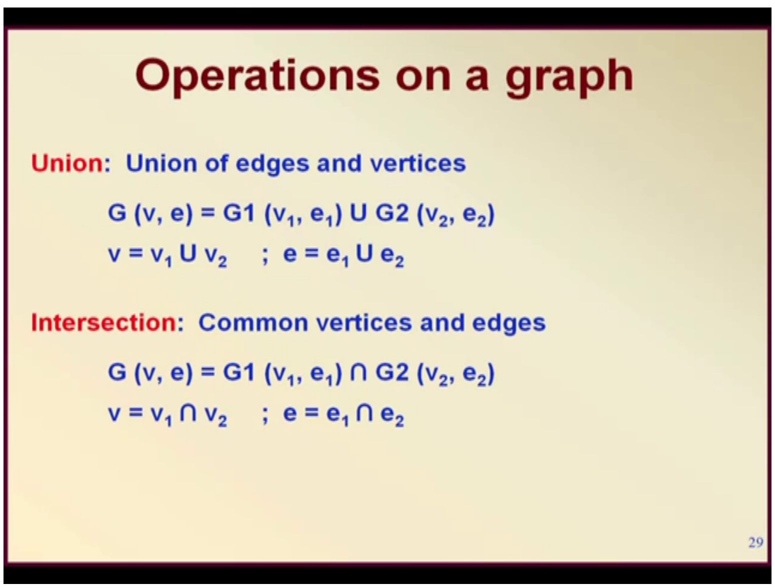

So to begin with the operations when we talk about the graph theoretic thing, there are several operations that we can do on graph. The very first two operations are usual the union and intersection operation, right? So supposing we have 2 graphs and there is maybe some common portion then what is an union?

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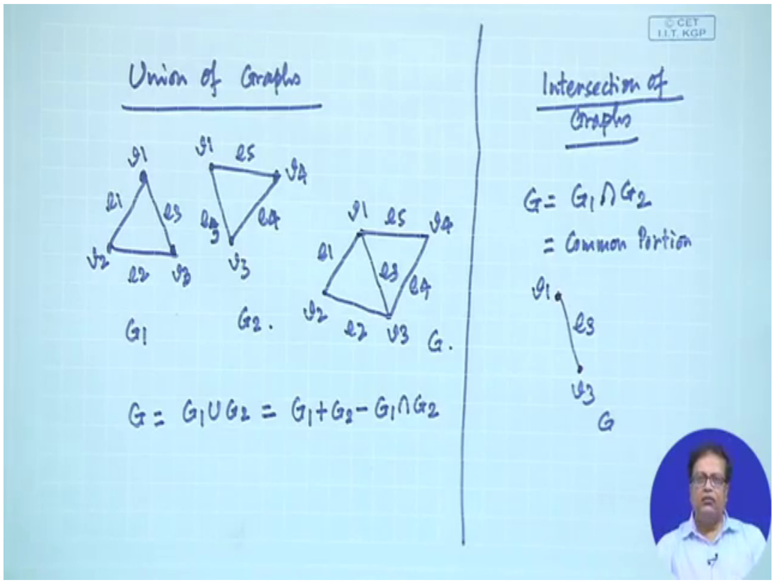

So the union of the graph suppose I have union of graphs let us say we have two graphs here one is this one and another is this one, right?

So these are the 2 graphs, now in these 2 graphs let us see suppose this is a vertex v1, this is vertex v2, this is vertex v3 and here we have v1, v3 and v4 and these are the edges e1, e2, e3, e3 sorry, e3, e4, e5. So if you look at these 2 graphs G1 and G2, right? Then in these 2 graphs there are some common vertices and there are one common edge called e3, right? So if I have the resulting graph G which is an union of G1 and G2 then what we should do. Then we know that we have to add G1 and G2 and minus the intersection portion.

So if you add them you know what will happen there is v1, v3 and e3 they will come twice because the common portion will come from both the graphs, so you have to retain only one set and you have to forget the other set. So what will be the union? The union would be simply let us draw here this is this one, then this one and this one and then this one, so this is our v1, v2, v3, v4, e1, e2, e3, e4 and e5. So this is the resulting graph G by which we have carried out the union operation. Similarly, the intersection of graphs can be also carried out by a process that is G equal to G1 intersection of G2. What is that intersection? This intersection will be nothing but the common portion.

So tell me what is the common portion? The common portion would be the you know your this portion that you know this line only, so this will be G that is v1, v3 and e3. So in this case the G would be like this. So this side is union, this side is intersection. So what happens that if you only look at the common that can be called as the intersection of the graph. So these are simple concepts about union and intersection only thing that this union operator will apply both for vertices and edges and intersection of operator will also operate both for vertices and edges.

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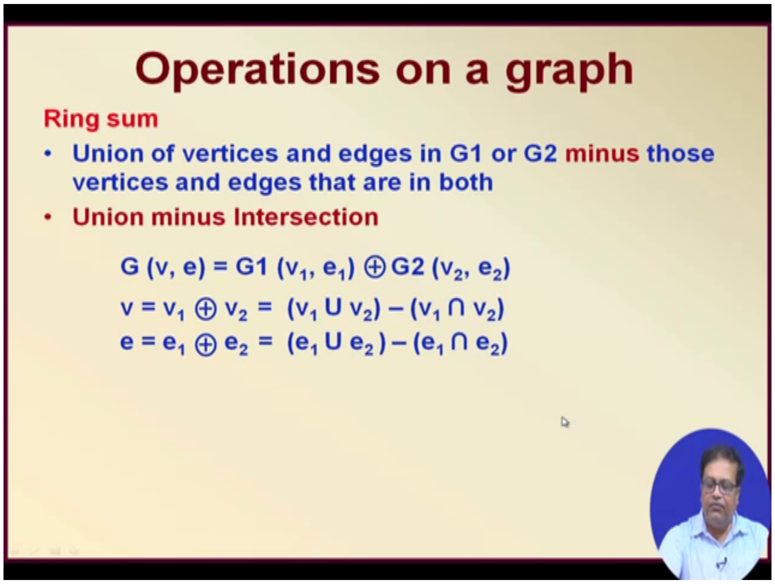

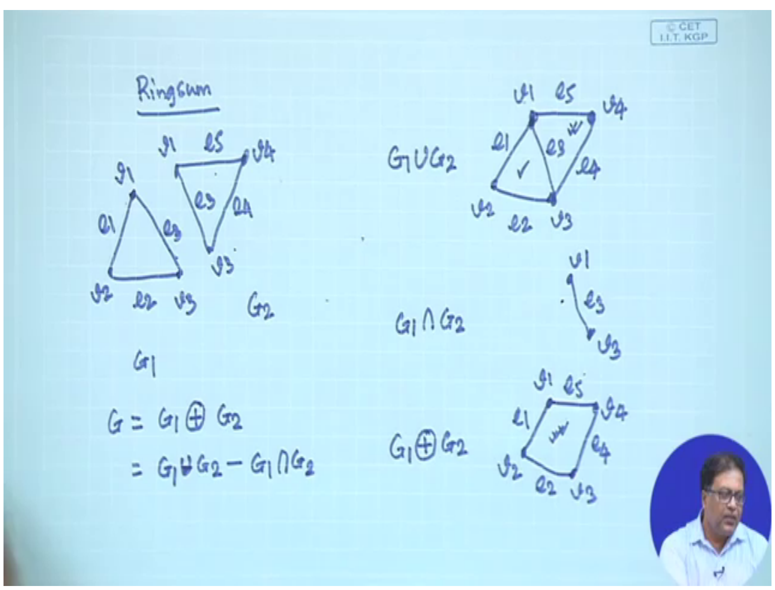

Now comes a very very important graph theoretic operation which can be called Ringsum, very interesting operation that is called Ringsum. So let us look at the same graphs once again, right? So we have taken these 2 graphs so one is your let us put it here, so this is what the graph we had so v1, v2, v3, v1, v3 and v4 this is e1, e2, e3, e3, e5 and e4. So we have these 2 graphs that is v1, v2, v3 and v1, v3, v4 so let us call this as G1 and this one as G2, so these are the 2 graphs G1 and G2.

Now the Ringsum operation is G equal to G1 you see that plus and then circle that is the ringsum symbol, G1 ringsum G2. So you see this is very interesting this is union G1 union G2 minus G1 intersection G2, right? So both the union and intersection operation that we just now learnt both are required. So one portion you make the union and the other portion we do the intersection and take out the intersecting portion from the union.

So what is the union? The union of these 2 graphs is this that we have already seen this is the union, what is the union? v1, v2, v3 and v4 then e1, e2, e3, e4 and e5, right? So this is the union, and what is intersection? Intersection is only this much. So if you take out from the union the intersection portion, then what will remain? What will remain is very interesting is the ringsum, so G1 intersection G2 would be nothing but this one, so that is what we can have you know in this case you know if you look at this intersection portion, then the intersection portion graph would be that v1, v2, v3, v4, right?

So one thing to remember here that the common age is taken out the common is taken out but the vertices should remain, right? The vertices should remain but the common age should go out. So that is what we do in the ringsum operation that the common age is taken out but the vertices are taken. So what is the advantage? The advantage is suppose look here if this is one circuit and this is another circuit, right? Supposing you know we start with this graph let us say we start with this graph and then you are told that you know how many circuits are there in this particular graph, right?

So you can say that this is one circuit and this is another circuit but there is a third circuit also there is a bigger one (())(10:01) these are the 3 circuits. So if you have these 2 circuits and you do a ringsum then you get a third circuit. So that is what is the ringsum operation helps us that if we have you know these 2 circuits and you can find out basically with the help of ringsum you know the third circuit also. So these particular specific you know advantage that the ringsum gives that we shall make use of later as you proceed.

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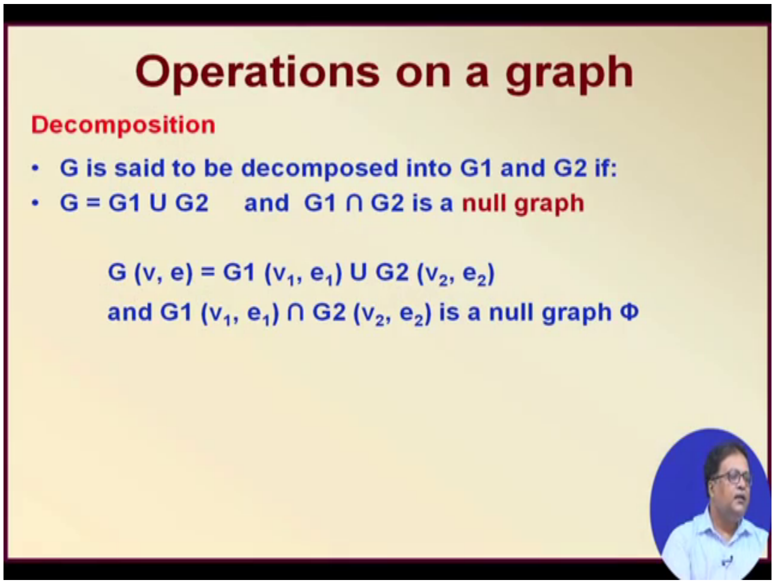

So we move on and then see another concept which is called the Decomposition. The G is said to be decomposed into G1 and G2 if G equal to G1 union G2 and G1 intersection G2 is a null graph, right? So the intersection should be a null graph means nothing. They that means that graph is completely these two has got no intersection that is called a true decomposition, right? So that is what happens that if the intersection is a null graph, then we call that as a decomposition process.

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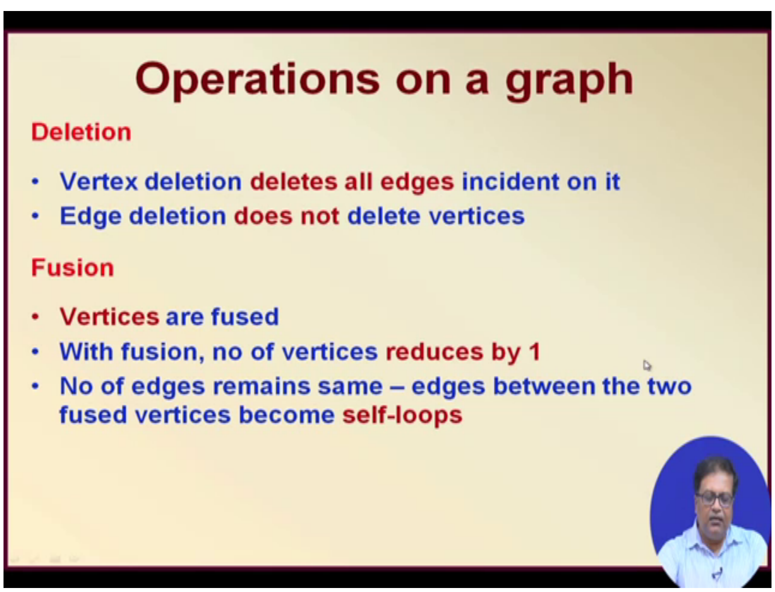

Now look here two more very important concepts, one is called the deletion of a vertex and deletion of an edge. Please remember if you delete an edge the vertices you know that are connecting those two edges I mean sorry, the two vertices that the edge connects they will not be related. So if you delete an edge only the edge is gone vertex is intact, but if you delete a vertex then all edge that is incident on that vertex that is gone, is it alright?

So one should remember this very very carefully about it, say for example some student records, right? Here student is like a vertex and these individual you know different subjects that the students has taken that student to subject connections you may call them as edges, is it alright? So if you delete a particular subject that student has taken, then that student subject link will be deleted only, so that is like an edge deletion. But if you delete the student then all subjects are taken by that particular student are not the subject is deleted that student subject link all the student subject links will be deleted, right?

So must remember this and that is what is done for the deletion operation. A particular type of thing is another one that is called the fusion.

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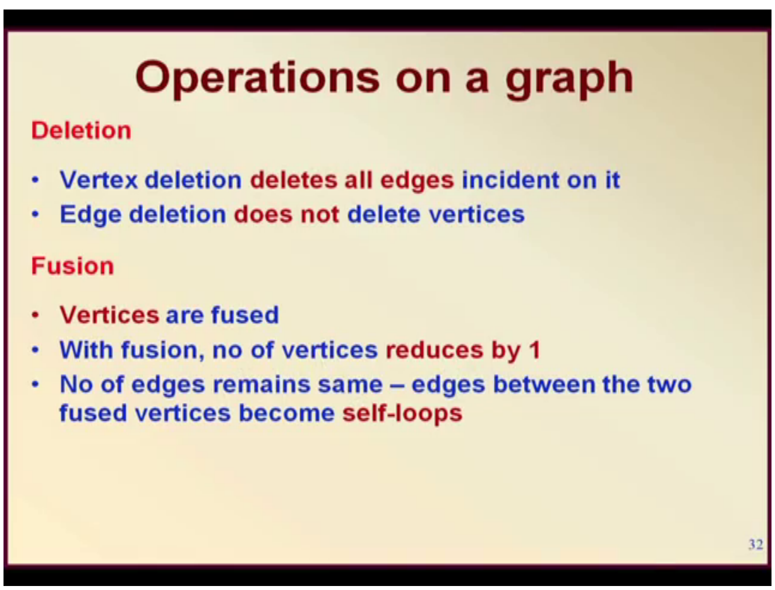

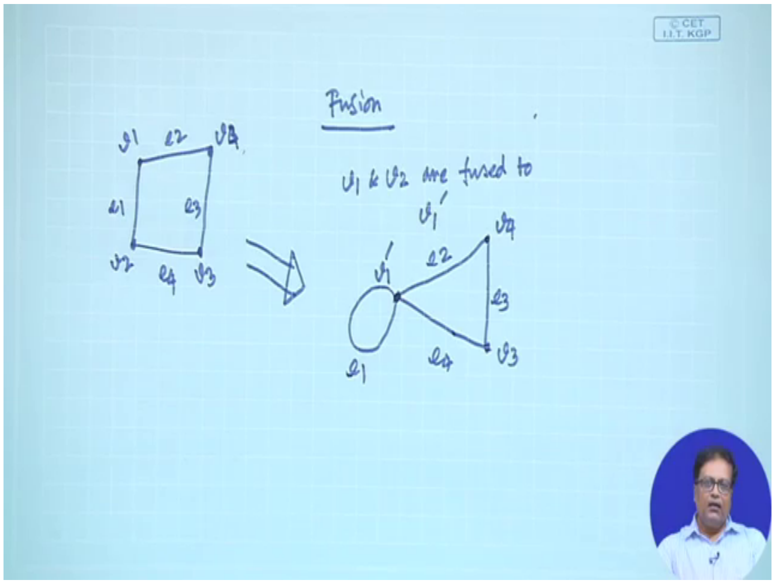

What happens two vertices can be fused, so supposed I draw a graph for simplicity I am not drawing the edges I mean I am not naming the edges, so suppose this is a graph v1, v2 sorry v3 and v4. Now v1 and v2 are fused, right? So v1 and v2 are fused to say v1 dash, so v1 dash is the new vertex. So how this fusion operation would look like.

So from this graph what we shall get is this one, so you see v1 and v2 will be replaced by the new vertex v1 dash and all the edges that where earlier connected to either v1 or v2 will now be connected to this new vertex, so this is v4 and this is v3 so these connections will remain. Now you see this edge will be this and this edge will be this and this one remain anyway as it is, but will happen to this particular edge which was there suppose we call it e1, e2, e3 and e4 so this is e2 now this is e4 and this is e3, what will happen to e1? e1 will now become a self-loop.

So this is called the fusion operation, right? So you can fuse a particular two vertices and as you fuse the resulting graph would be like this, so this is about deletion and fusion operation.

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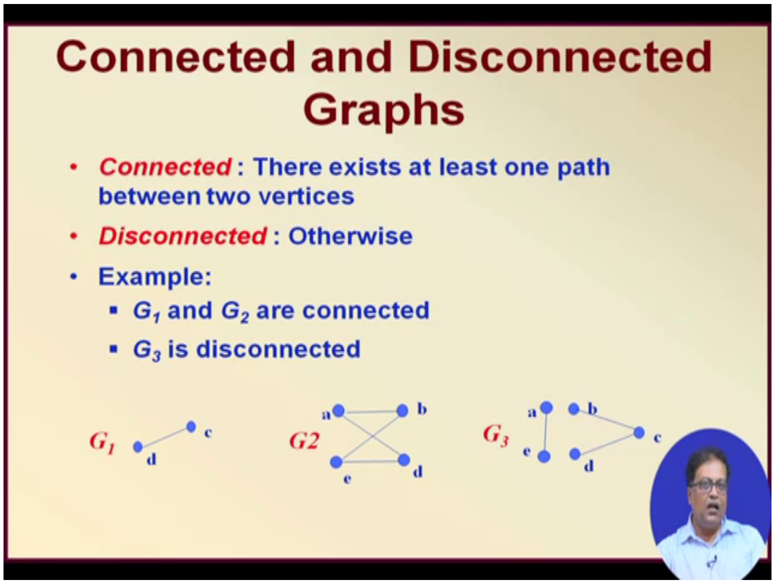

Now connected and disconnected graphs, so in a connected graph there exist at least one path between 2 vertices, right? So if you look at this graph G3 look at this graph G3, can you find a path between a and c? No, answer is no. So if there is at least one path so if there is no path between a and c obviously you have a path between a to e, but you know for every two vertices between any two vertices if you have a path, then you call this as a connected graph, right?

So disconnected graph is a graph where you know such a path is not available. So this is a connected graph, this is a connected graph, this is not because there is no path here between every pair of vertices, so it is a disconnected graph. And in disconnected graph has got connected components, so here you may call this as a component and this as another component. So that is the basic idea about the connected graph and the disconnected graph.

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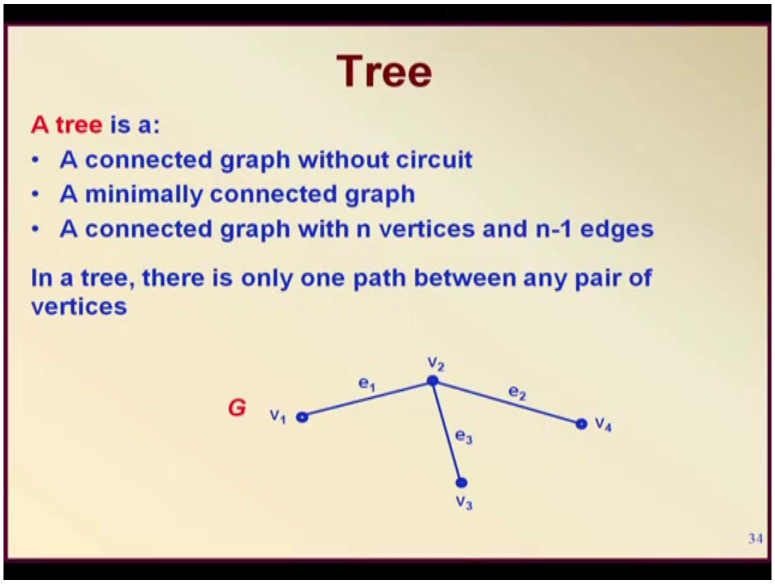

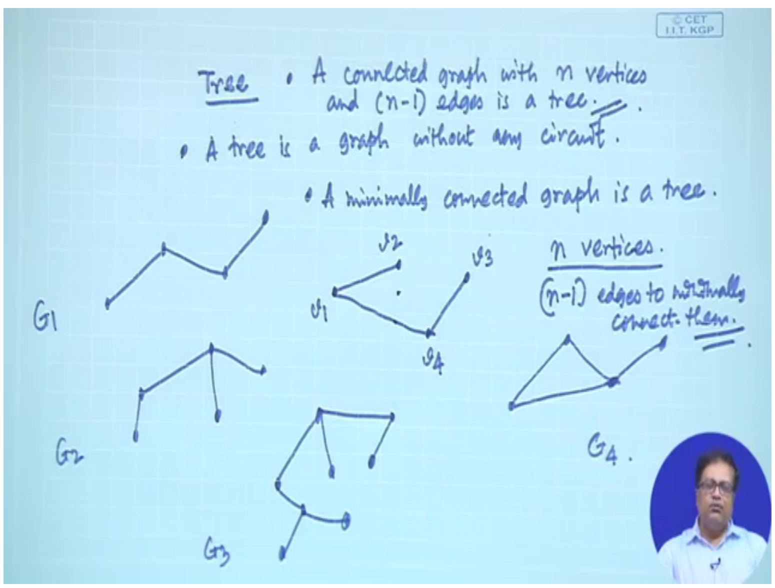

So you know this brings us to a concept called tree, first of all what is a tree? A tree is a graph without any circuit, right? That is the definition of a tree. So tree is a graph without any circuit, so suppose I draw something like this or suppose I draw something like this, right? Or I draw something like this, right? So G1, G2, G3, right? Now we can also draw something like this G4, now tell me out of these 4 things that I have drawn G1, G2, G3 and G4 are all of them trees? Are all of them having circuits? Answer is here in G1 there is no circuit, in G2 also there is no circuit, G3 no circuit and G4 however is not a tree because that is a circuit, right?

So this is how you look at the tree but sometimes we can also define tree in a very simple way, a minimally connected graph is a tree, a minimally connected graph is a tree. So you see suppose we have this v1, v2, v3, v4 4 vertices and we want to minimally connect them, right? So how many vertices? Let us say n vertices, how many edges do I require to minimally connect them? We need n minus 1 edges to minimally connect them so that the resulting graph is a connected graph and not a disconnected graph because you see suppose we connect v1 to v2 so one connection is gone.

Now you connect v1 to v4, so you see two connections are gone. Now the third connection suppose you use between v2 and v4, then you have already utilized your n minus 1 connections, alright? And but then you get a disconnected graph v3 is not connected. So you have to minimally connect that 4 vertices so you have no options by choosing the v2 to v3 or v1 to v3 you can choose any one of them suppose we choose sorry v4 to v3 suppose we choose then we get a tree. So that is a minimally connected graph so a tree is also a minimally connected graph is a tree, right?

In fact you know you can also put this in another way a connected graph with n vertices and n minus 1 edges is a tree. So this is also a another way you can define because as long as you have only n minus 1 edges with n minus 1 vertices it has no option but to become a tree because it becomes minimally connected and because it is minimally connected there cannot be any circuit. So the best definition is always a tree is a graph without any circuit but alternate definitions could be a minimally connected graph is a tree and a connected graph with n vertices and n minus 1 edges is a tree.

Another way you can say in a tree there is only one path between any pair of vertices, look at v1 to v3 is only one path, v1 to v4 only one path. Now the question is that what is the significance of tree?

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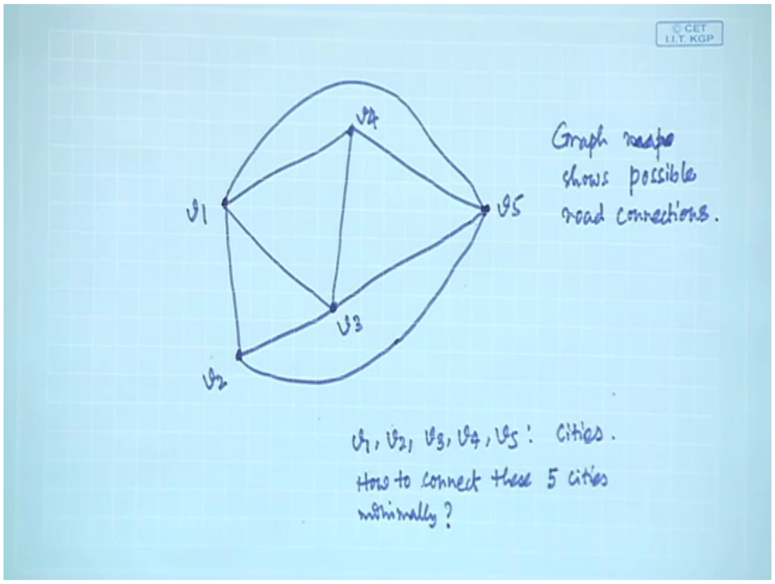

You see assume a particular graph, suppose let us assume this particular graph or maybe if you want to make it slightly more complicated then we can also take this. So let us call it a graph so v1, v2, v3, v4 and v5, see assume these are cities, v1, v2, v3, v4 and v5 are cities, right? And then you want to minimally connect them, right?

How to connect these 5 cities minimally you see this graph represents possible road connections, these roads are not actually there these are the roads that could have been built but our budget is limited and we must give the minimally connectivity so that any person in any of these cities can go to any other city there is a path available. So how to do that? How to minimally connect them? That is an interesting question we will come back to that question little later.

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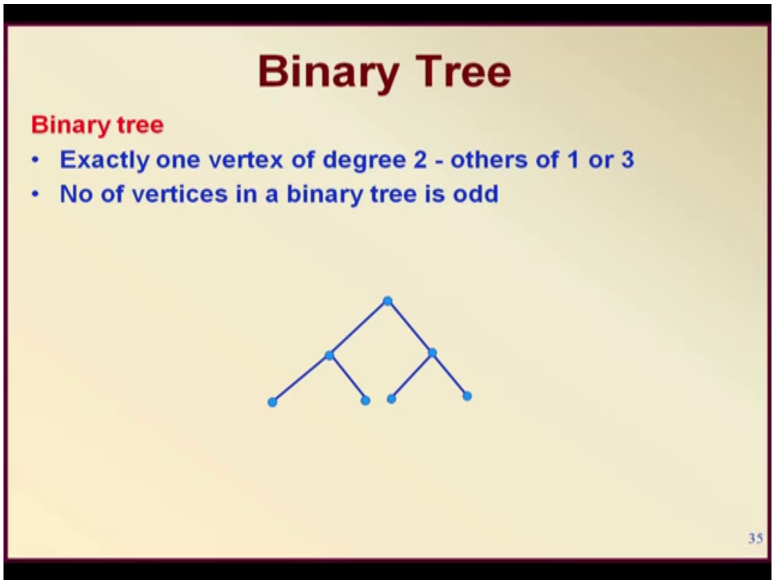

Now look at this there is another type of tree that is called a binary tree. So what is a difference between a binary tree and a other tree, that in a binary tree is a tree definitely but there is exactly one vertex of degree 2 and others are either 1 let us see this vertex which is a pended vertex has got only 1 degree or degree 3, right? And number of vertices in a binary tree is an odd. So actually there are lot of applications of binary tree particularly in the search algorithms and many such situations, but however we are mainly interested in the flow problems so we are not so much concerned about binary tree and we will proceed further and see other type of important applications of tree.

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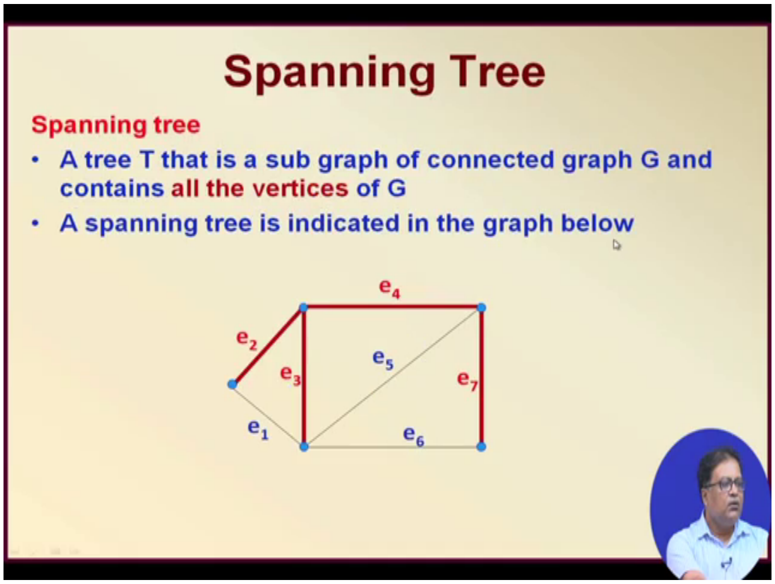

So that is called a what is known as a spanning tree. A spanning tree is a you know is a sub graph of a connected graph G and it contains all the vertices of G you know but it is a tree that is the essential idea. So look here in this particular graph there are you know 1, 2, 3, 4, 5 vertices and you know these edges if you take 4 edges because there are 5 vertices we can take 4 edges.

If you take these 4 edges then that forms a spanning tree, right? So that is the essential idea of a spanning tree. Now come back to the question that I have asked, let us see whether an answer is found to this particular question.

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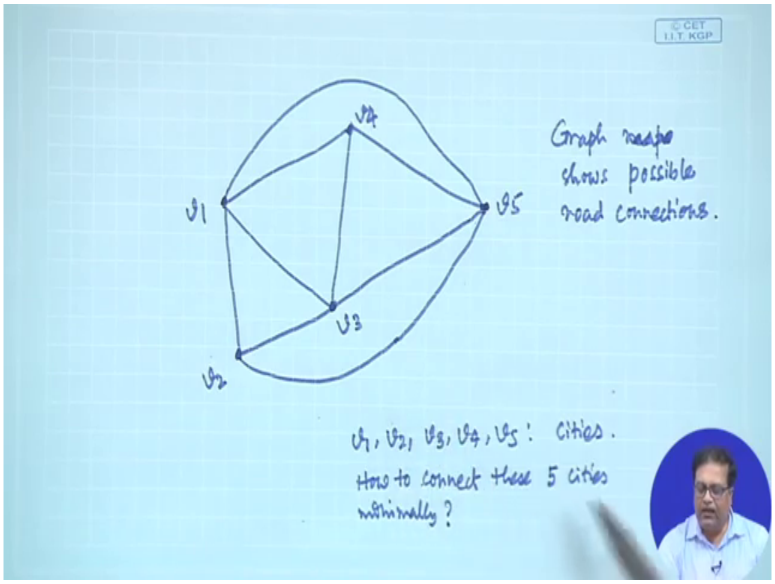

What was my question that the graph shows possible road connections v1, v2, v3, v4, v5 they are cities and the question was how to connect this 5 cities minimally, right? So we need to build the least number of roads. So how many roads will you build out of these 5 cities? You see we discussed that if the number of vertices are n, then to minimally connect them we require n minus 1 edges.

So since there are 5 roads, I mean sorry 5 cities to connect minimally we had connect only 4 roads. Now question is which 4 roads? That is a very important question which 4 roads? Which 4 roads that question we shall deal little later because you know these each road has a length or a particular distance. Now obviously we would like to minimize on the road length also so that is a bigger question but initially let us only look at that what are the possible minimal connections.

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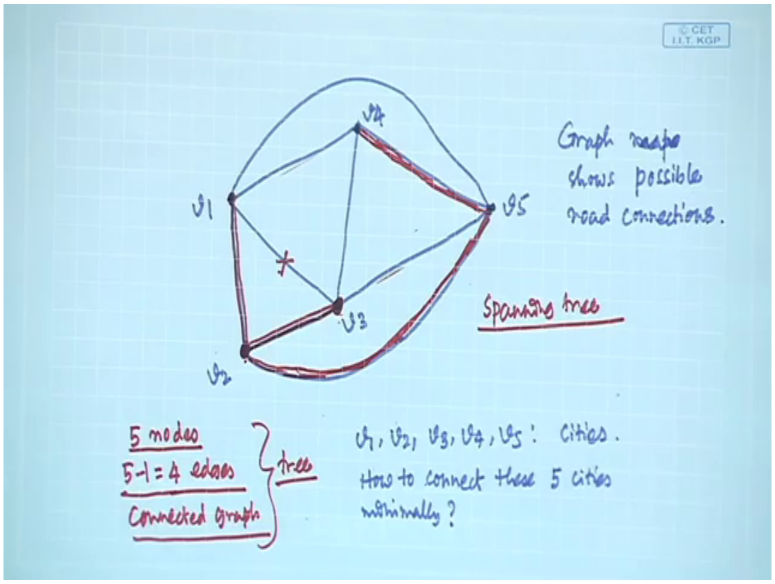

So one such minimal connection would be that choose 4 roads, so which 4 road we should choose? Let us first let us start with say let us connect v1 to v2 so this could be one road, then maybe another 3 roads we have to choose so let us also take another road say v2 to v3 so two roads are chosen, right? So make them double line, right? Two roads are chosen. Then perhaps you take v4 and v5 so this is another road is taken, right? So you see these are the 3 roads chosen now you have only 1 road to connect.

Now obviously look here you can take you know which roads you cannot take, let us look at, can I take this road? Answer is no, it cannot be taken because then it forms circuit and then they are not connected. Can I choose any of the remaining 5 nodes, 5 edges? Answer is yes, we can take this one, we can take this one, we can take this one, we can take this one and we can take this one also, right? So supposing we check this one, okay so this is what we have got what is known as a spanning tree, okay because this is a particular, why? Because it has got there are 5 nodes and we have got 5 minus 1 equal to 4 edges and it is a connected graph.

So a connected graph n nodes or vertices and n minus 1 edges. So n nodes n minus 1 edges connected graph so it is a tree, so it is a tree and is a spanning tree because it spans all the vertices, but is it the only spanning tree? Answer is no, there are other spanning trees also, what are the other spanning trees? If I would have taken this, then another, if I would have taken this, another, if I would have taken this, another, if I would have taken this, another instead of this, so we could have got many other spanning trees.

So first of all why this two, we could have chosen this and this also. So like this you know lot of options a large number of selections of spanning trees are particularly possible. In fact spanning tree is a very very important concept and we shall continue our discussions on spanning trees further, right? So stop here for today, thank you very much.