Course on Decision Modeling
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Lecture 36
Module 8
Maximal Flow Problems

In our previous class we have seen the cutsets as a concept and those concept are very important particularly in a set of problems which may be known as maximal flow problems. So we will take up maximal flow problems and we will continue for next few classes particularly also looking at probably the maximal flow minimum cost problems, but before really getting into those type of problems there are certain important concepts which we should understand first with regard to the fundamental matrices.

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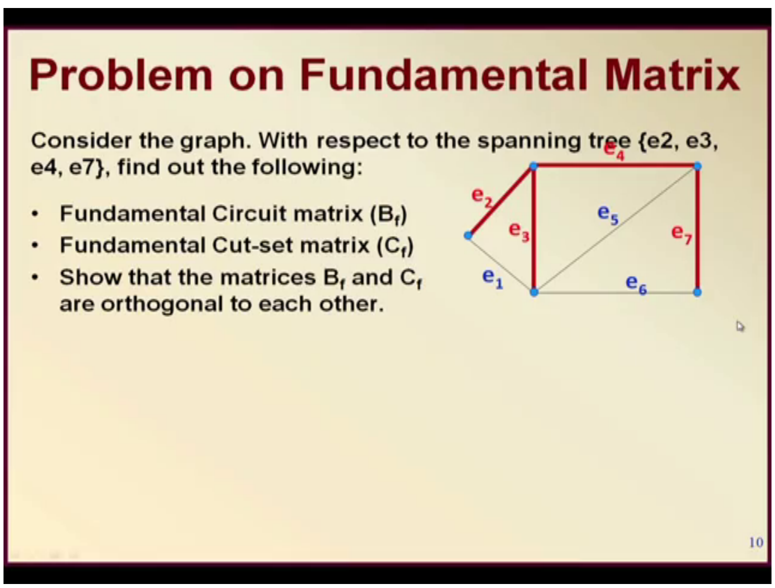

Now, there are certain fundamental metrics the two of them are so important they may be called fundamental circuit matrix and the fundamental cutset matrix. Now you know if you recall why no we are identifying the cutsets and earlier also the circuits in a particular graph we have understood that this spanning tree is very important concept and because a particular kind of problems like a kind of linear programming problems or similar such problems the solution is basically to be found from one of the spanning trees, right? Which optimizes a particular you know problem solution.

So therefore from this spanning tree point of view if we really have to find out all the cutsets or all the circuits you know from a particular graph simply by exploring we can see that sometimes it may not be possible to really identify all the cutsets. Much more important is how to find those maybe the fundamental ones from here we can actually generate all the remaining cutsets or circuits as required, right? So let us look into this particular thing first and then we move over to the maximal flow problems.

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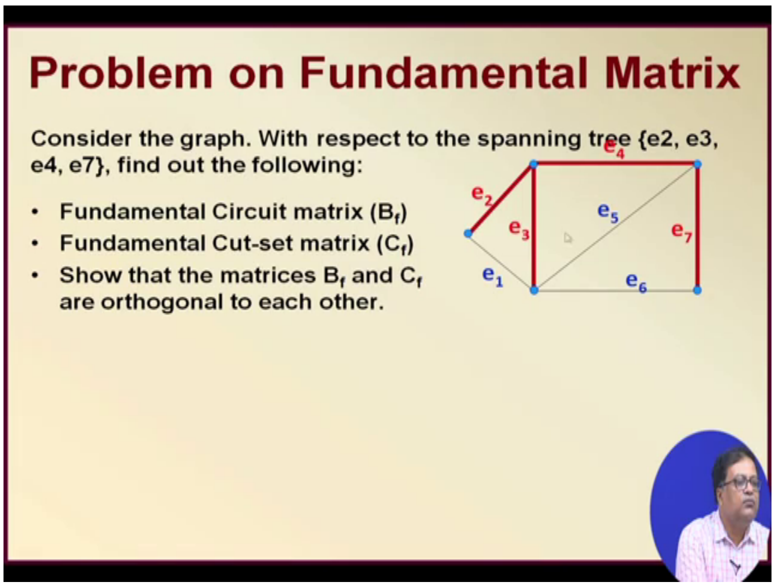

Now consider a graph with respect to the spanning tree that is e2, e3, e4 and e7 and find out the fundamental circuit matrix fundamental cutset matrix and show that the matrices are orthogonal to each other. So this is the question that we shall take up now, look here please always remember this fundamental circuit matrix or fundamental cutset matrix they are always defined with respect to a particular spanning tree. So if you have another spanning tree of the graph, then the fundamental circuit matrices will be different, fundamental cutset matrices will be different, is it alright?

So these fundamental metrics a always defined with respect to a given spanning tree. So if anyone asks you find the fundamental circuits, immediately your question should be what is a spanning tree we are talking about? So we respect to that particular spanning tree only will be able to find out the fundamental circuit or the fundamental cutset matrix.

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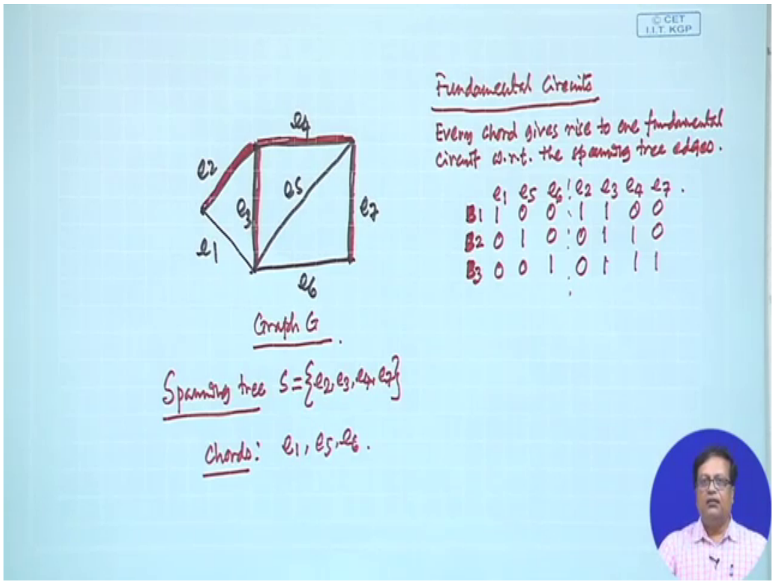

So let us look at let us draw this graph here that is there is a triangle here and there is a square here and then there is a connection, so this one is e5, this one is e4, e7, then e6, then e1, e2 and e3, right?

So this is our graph and a particular spanning tree is given to us that is e2 obviously there is nothing about this particular one, one can always choose another graph another spanning tree of the graph and continue the discussions but (())(4:41) let us go ahead with this particular one, right? So this is the spanning tree so this is the graph G and the spanning tree S equal to e2, e3, e4 and e7. So this is the spanning tree that we have taken and for a graph G, and then what we need to do first? We have to find out the fundamental circuits we can have a matrix also that is the circuits on one side and the edges on one sides and we can put 0 if the edge is not taken 1 if the edges is taken.

So we can make the fundamental circuit we can show in the form of a matrix and that matrix is called the fundamental circuit matrix. So you see what we have to do here you know every chord gives rise to one fundamental circuit with respect to the spanning tree edges, this point you must remember, right? What you have to remember that you see here these are the spanning tree edges, so what are the chords, what is the chords? The chords are e1, e5, e6 there are 3 chords.

So how many fundamental circuits will be there? If there are 3 chords because every chord will give rise to one fundamental circuit matrix and if there are 3 chords then there should be 3 fundamental circuits, right? This point you must remember. So if you remember that, then you see every chord will give rise to because once you have taken one chord you cannot take another chord, so if you take e1 look here e1, e2, e3 that will make a fundamental circuit, alright? So we can write here c1, we can write here c2, we can write here c3 although we will not write c1, c2, c3 the fundamental circuits are actually shown not in terms of c1, c2, c3 but on b1, b2, b3, why? Why not c? Because c is reserved for the cutset matrix, so we write b1, b2, b3.

So b1, b2, b3 are my 3 fundamental circuits coming out of the 3 chords and here we will not just write e1, e2, e3, e4, e5, e6, e7, right? We will use an ingenious way of writing this, what we shall do? We shall write e1, e5 and e6 first, why e1, e5, e6? Because they have the chords and then we write e2, e3, e4, e7. Now look this you can think of a partition here as well. Now e1, e5, e6 every chord will give rise to 1 fundamental circuit so we make a simple thing we just make a unity matrix here. So if you make an unity matrix, then you know you get those 3 because e1 is the first one, e5 for the second one and e6 for the third one.

If I take e1 what will be some spanning tree edges that will make a fundamental circuit e5, e2 and e3 so we put one there and zeros here. What about b2 when you take e5 then e3 and e4, what about the third one if I take this one look here e3, e4, e7, right? So e3, e4, e7 so look here we have got our fundamental circuit matrix.

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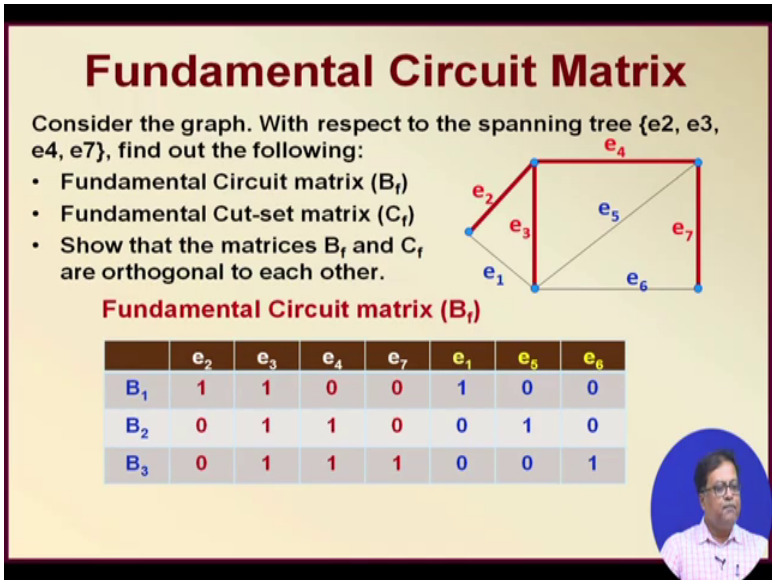

So just now for your information just look at this particular thing what we have this is drawn in a different way that is this side it is shown e1, e5, e6 so 1, 1, 1, 0, 0 that is the unity matrix from the chord and this is where the spanning tree edges, alright?

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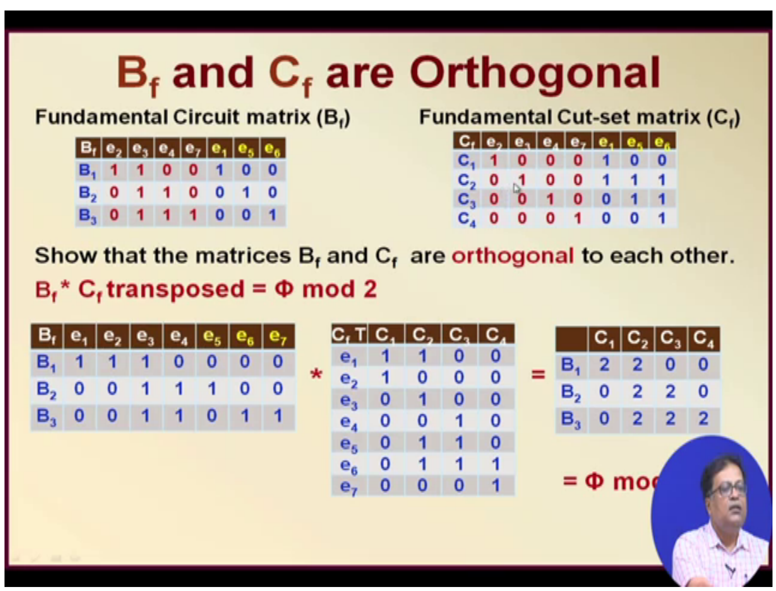

Now what we can do? We can also rearrange this one. So you see this is our fundamental circuit matrix here, and what we have done? We have put the fundamental circuit matrix rearranged. So this our matrix the matrix is rearranged how it is rearranged just look only this part that this is the fundamental circuit matrix Bf and after rearranging we put e1 1, 0, 0, e2 which was 1, 0, 0, 1, 0, 0, e3 1, 1, 1 e3 1, 1, 1 and we can put. So that becomes our fundamental circuit matrix, right?

So this is our fundamental circuit matrix so you understood that how to make a fundamental circuit matrix. The question is that if this is our fundamental circuit matrix how do I get other circuits?

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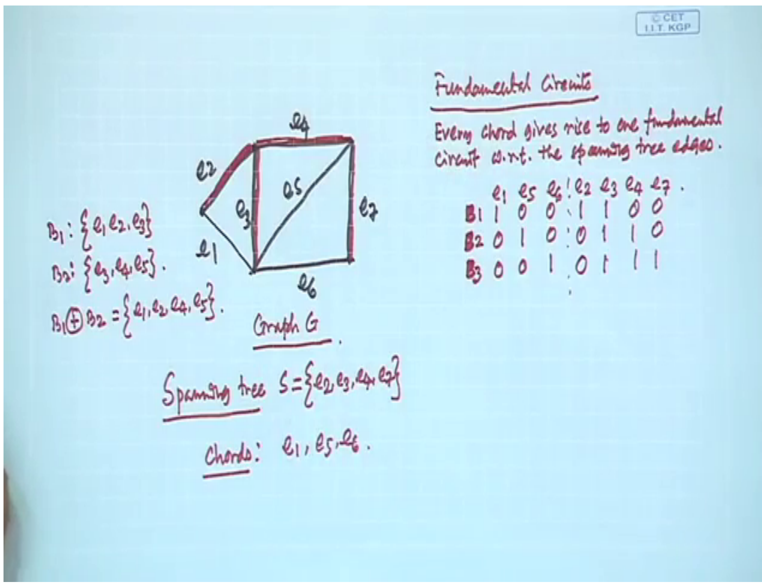

How do I get other circuits? So just imagine B1 is given by e1, e2, e3, and what is given by your B2? B2 is given by e5, e3, e4, e3, e4, e5 so you see simply do a ringsum, what is the ringsum of B1 and B2? Remember ringsum, in a ringsum what happens the it is a union minus the intersection. So what is a union all of them, what is the intersection? e3, so e3 should be removed. So the ringsum will give rise to e1, e2, e4, e5 can you identify e1, e2, e1, e2, e4, e5 just see can you see that circuit, right?

So like this you can keep on doing this ringsum operations to identify other circuits, is it alright? So now look here that we have only 3 fundamental circuits and we have to we have work with them only there is no need to really find all the circuits all the time, is it alright? The fundamental circuit matrix is sufficient to generate all the other circuits by doing ringsum between them. So we can do ringsum between B1 and B2, B1 and B3, B2 and B3, B1 and B2 and B3, is it alright? So we can keep doing all these ringsum operations and we can obtain all the circuits of this particular graph, right?

So this is the advantage of the fundamental circuit matrix once we generate the fundamental circuit matrix obviously remember with respect to the spanning tree only. Another spanning tree, another fundamental set of fundamental circuits, right? So once the spanning tree is fixed you can get the fundamental circuits and you can generate all the circuits by ringsum operations, alright?

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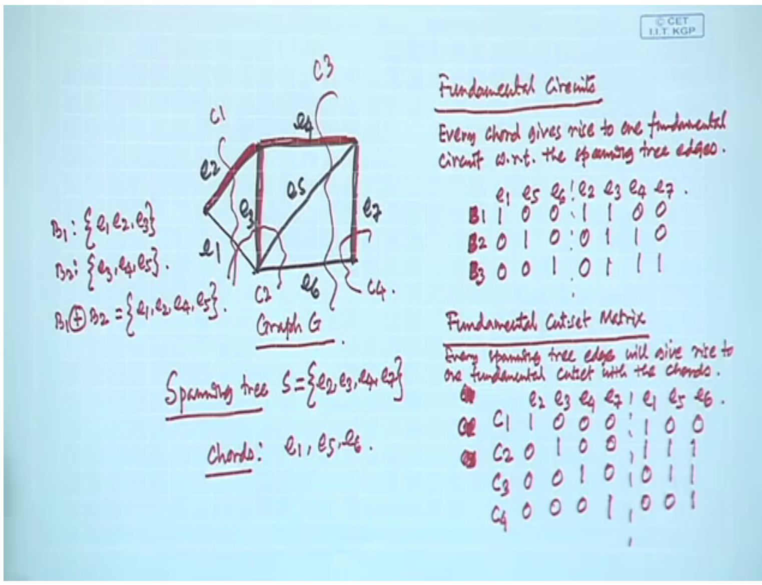

Now after that let us look at the fundamental cutset matrix, again what we do here fundamental custet matrix.

So C1, C2, C3, how many will be there? How many spanning tree edges are there, right? Like in fundamental circuits we have every chord gives rise to one fundamental custet here every spanning tree edge will give rise to one fundamental cutset with the chords, alright? So every spanning tree edges will give rise to 1 fundamental cutset with the chords, so how many cutsets can you think of fundamental cutsets? How many spanning tree edges are there? 4, so you can think of 4 fundamental cutsets.

So these are the 4 and let us write these e2, e3 this order is not very important because finally again we make it sorted and e1, e2, e3 so you can write this side first and these are the 3 chords e1, e5, e6 so again take the unity matrix with regard to say earlier we took unity matrix with regard to chords now we are taking unity matrix with regard to the spanning tree edges, right?

So, right? Now look here if you take e2, then this is our cutset that means you will take e1 so this is fundamental cutset 1.

Now if you take e3 only e3, right? Then you can take only chords mind you, so which chords will you take, you know only this because you cannot take another spanning tree edge so all the 3 chords you can take and that will your c2. So you have taken e1, e5, e6 all of them and c3 is about e4 so this is e4 so this is c3, so c3 would be e4 and then e5 and e6, alright? And what about c4? It is with e7 so just this one so this is c4, right? So the c4 would contain e6 and e7, e7 is already there so we have e6.

So you see we have now generated the fundamental cutset matrix. So we have the fundamental circuit matrix and we have the fundamental cutset matrix. So here you can look that we have made in the slide already these answer and that is the fundamental cutset matrix, right?

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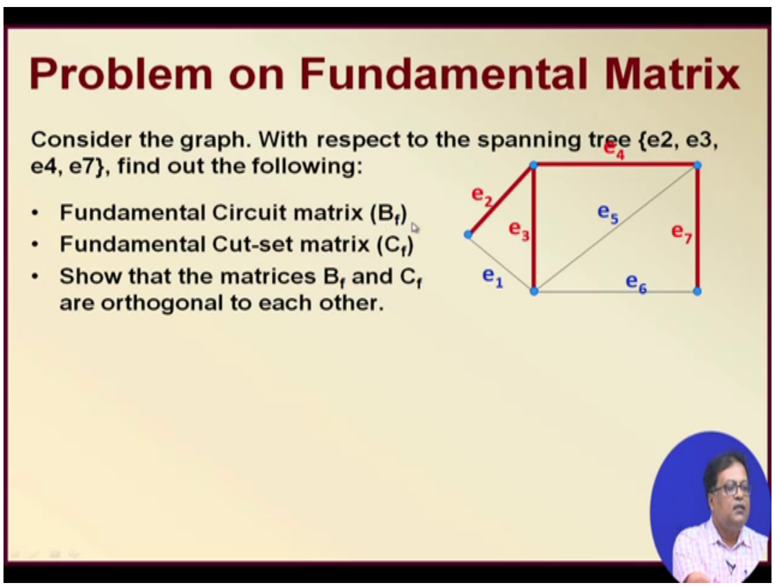

Now an interesting question if you look at the fundamental cutset matrix and the fundamental circuit matrix Bf and Cf they are orthogonal to each other.

You know you see we will not go very deep into this but at least this much we will tell if we you know multiply Bf with the transpose of Cf, then we get a null matrix. Obviously not really null because these are all binary matrices so two should be taken as 0 only, so this you can call as 5 mod 2 matrices that means the entries will be either zeros or twos, is it alright? So the advantage is that if we have one matrix it should be possible to generate the other matrix by some operation and lot of applications are there if I have a switching circuit you know if I have the circuit matrix it should be possible to generate the connections not from cutset matrix but another relation where we have the incidence matrix we can generate.

But any how we will not go into those things we will simply look at that how they are orthogonal.

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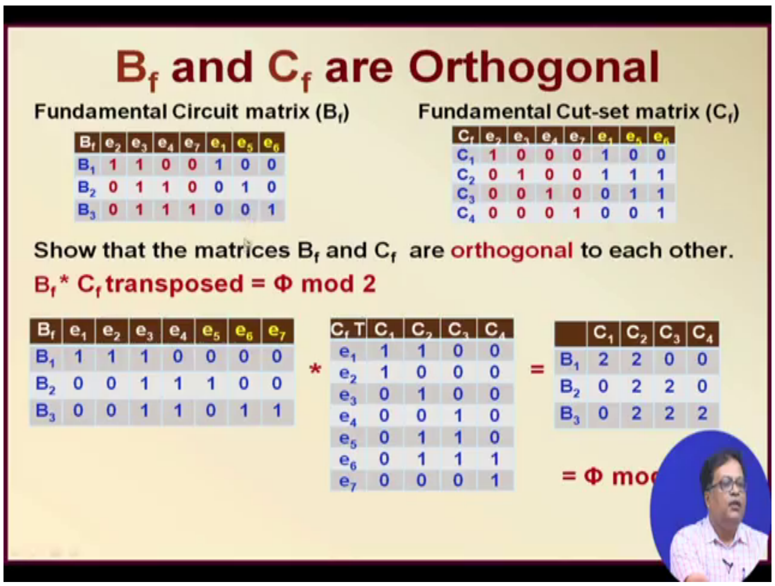

So now let us look at that here this is the fundamental circuit matrix and we have put it in this manner e1, e2, e3, e4. Similarly this is the fundamental cutset matrix and that is also we have arranged in a transpose manner because if you have to do the matrix multiplication you know this side and this side should be exactly same, so if this is 3 by 7 this should be 7 by something, so this is 3 by 7 this is 7 by 4 and the result will be 3 by 4.

So this is the result that you get just look here if you multiply Bf with Cf transpose then we get a null matrix obviously in the modulation 2 domain, right? That is the entries are zeros or twos, right? So that is how we say that these two matrices orthogonal to each other. The advantage of orthogonality is that within certain limitations and assumptions if we have one matrix should be possible to find the other one, alright?

Anyhow this is the essential idea of the fundamental circuits, now let us see how we can use them in maximal flow problems.

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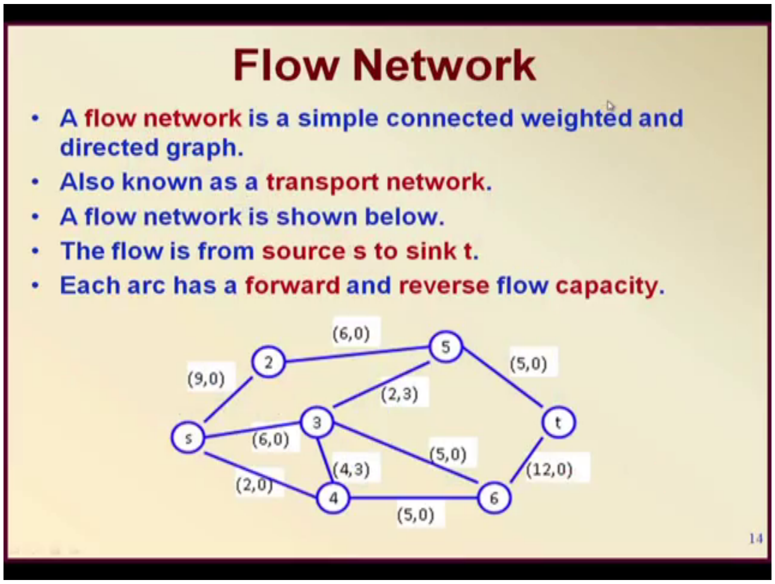

So first and foremost thing when we take up the maximal flow problem the first thing that we need the idea of what is known as a flow network, what is a flow network? A flow network is a simple connected weighted and directed graph, alright? So what are the things first of all it is a simple graph that means there are no parallel edges and there are no what is called self-loops that is why it is a simple graph, I think you know that simple graph is a graph which does not have any parallel edges or self-loops so it is a simple graph.

Second it should be connected, right? That means connected means it should not be having components and it should that means from every node to every other node there should be what is known as a path available like here s to t look here there are there are paths, s to 6 there are paths. So between every pair of nodes there are paths so it is a connected graph. Thirdly you know it is a weighted graph, what is meant by weighted graph? That means each edge is you know having a weight assigned to 8, what could be that weight? The weight is that each edge or each arc is having a forward and reverse flow capacity that is how it is became a directed graph because if you go s to 2 then the capacity is 9 and if you come the reverse direction that is 2 to s capacity is 0.

That means here there are some arcs like 3 to 5 look at 3 to 5 so a flow of 2 is possible between 3 to 5 but a flow of 3 is possible from 5 to 3, is it alright? So that is what is the weight all about and that is how the direction comes in, is it alright? So this is a flow network, what is happening in the flow network a flow is coming at the s that is called the source and the flow goes out of the sink that is t.

So a flow is coming at s and the flow is going out at t, so what is the maximum possible flow that can come in s that is the question, right? So we have a flow network and the question is how much maximum flow is possible in this particular network, is it alright? And second question is that what should be the pattern of that flow, right? What should be the pattern of the flow. Now do we want to know all these? You see there are very important questions that are there in all types of network these network could be a transport network, road network communication network, telecommunication network, is it alright?

There could be different types of networks at different domains and in all of these domain a common question is that how much flow is flowing through a particular path because there are demands and those demands are to be satisfied, right? So what could be one what are the things that we have? We have the capacities, we can change the capacities you see if we change one capacity the entire flow network may change, so you see just look here if you change probably a 5 to t capacity 5 to 0 you make it say 7 and 0, probably the entire flow pattern will change and that would might affect this another link 4 to 6 also.

So it is not like now that to 5 to t let us increase the capacity automatically it would mean that its affect would be maybe in another link. So person who is looking at the total network you cannot really think of you know just one particular link and then increase capacity or decrease capacity because the person knows the its effects could be on the other part of the network and that is where the design comes in, right? So it is not simply find the flow that is available but also what is the impact of the change of capacities and how that is actually going to affect the entire network.

So such questions the answers to such questions are extremely important and you know for all types of networks such kind of analysis is very much required. So let us see how some of these things are useful and important.

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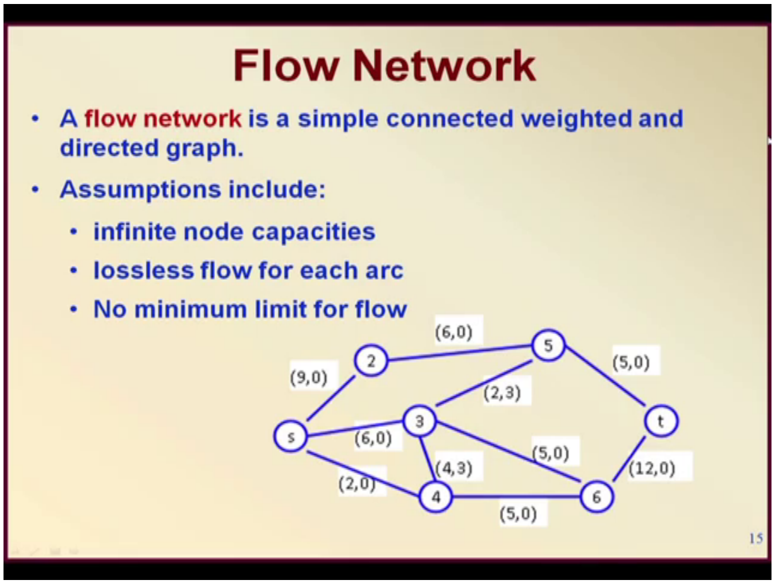

Now there are some very very basic assumptions some of the assumptions are given here there are infinite node capacities, lossless flow for each arc and no minimum limit for flow. So 3 such assumptions we have given in fact there is a fourth assumption also there is a single source and there is a single sink.

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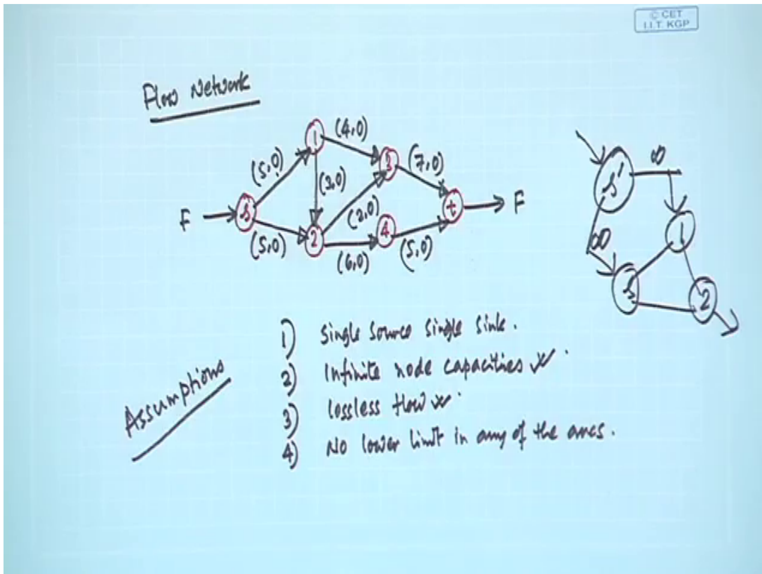

So let us take a particular example, supposing we have a flow like this is source and this is sink so let us take a particular network let us say we have some very simple network let us take 1 and 2 then 3 and 4 that is all, nothing else. So if take this very simple network and let us say these are the flow direction supposing and 2 to 3 is possible, 2 to 4 is possible, 4 to t is possible and 3 to t is possible, alright? And 1 to 2 is also possible, just take only this much network and let us say we have (5,0), (5,0), (4,0), (3,0), (2,0), (6,0), (7,0) and (5,0) just imagine and say a flow is F that is going out and F that is coming out. So let us consider this flow network, right? So what are those assumptions? Single source single sink, second assumption infinite node capacities, number 3 lossless flow, fourth no lower limit in any of the arcs.

So let us say this assumptions what are there implications, so first one single source single sink, here if you really look at that here if you have a single source and a single sink, what happens if there are multiple sources? Imagine a network where we have s, 1, 2 this kind of network and this is a source, this is also source and this is a sink, what to do then? What we can do we can make the super source s dash by connecting all the sources and assume a single flow is coming with infinite capacities to these connections, right?

So this is an infinite capacity this is a infinite capacity so you know you can transform a single multiple source multiple sink problem to a single source single sink problem by connecting all the sources to a super source and all the sinks to a super sink, alright? That is possible. The second one the infinite node capacities, look here the flow of 5 is coming to 1 and flow of 4 is going out of 1 or it can go here 3. Now question is that if the node 1 cannot handle the entire flow of 5, what will happen then? What would happen it would restrict the amount of flow that is possible, is it alright?

One hand you have these flow of 5 suppose the junction capacity is only 4, then only flow of 4 can come there not the entire 5, right? So initially we assume that all the nodes are having infinite capacity and they are not going to give any constraint to our flow problem. The third assumptions says that flow is lossless that means if a flow was 5 is coming from here the entire 5 is reaching these node we know sometimes there could be a loss during the transit the assumption here is no such loss is available.

The fourth one is that no lower limit in any of the arcs sometime what happens that see supposing the lower limit here is put as 0 the reverse flow is 0, right? Reverse flow capacity not just lower limit the reverse flow capacity is 0 but suppose some time what happens in the water pipeline networks particularly in the cold areas there is a minimum flow that has to be maintained because if you do not do that water may actually become ice and when water becomes ice its volume increases, pressure develops and the pipe may burst, right? So minimum flow must be assigned.

So these are our basic assumptions with which we shall consider the flow network problems and all the knowledge that we have got about the graph theory, the cutsets, the spanning tree and etcetera we shall see how all of these are useful and important and those we will discuss in the next few classes, right? So thank you very much.