Course on Decision Modeling
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Mod08 Lecture37
Maximal Flow Problems (Contd.)

Today we begin with the maximal flow problems once again, if you recall you know last class we just gave some introduction to the maximal flow problems.

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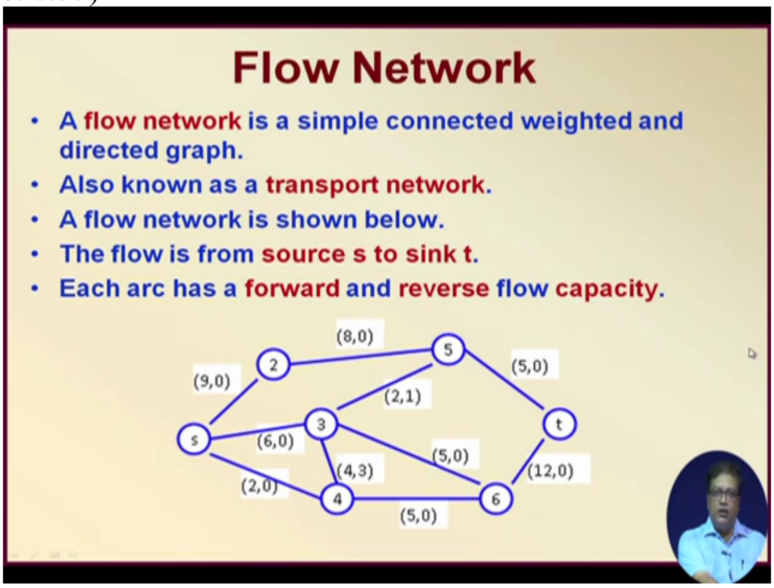

So we know we have seen that maximal flow network, you know it is a simple connected graph and I have also explained in the previous class that usually, it is an weighted graph those in a flow network most of these things are essentially flow capacities. Now when they have in a network, the flow can be both sides, right from suppose a node is I, J, it could be from I to J or could be J to I. So I to J we call it forward flow capacity and J to I is a backward flow capacity. So both forward and backward flow capacities are possible.

So here if you look at that most of the flow capacities are mentioned in terms of forward flow capacities and there is no backward flow, but there could be some exceptions, for example, look at 3 to 5. Here, the forward flow capacity is 2, but there is a backward flow capacity also, which is 1, right. Now these are called transport networks and the flow is from the source S to the sink t and each arc (())(1:46) has a forward and reverse flow capacities.

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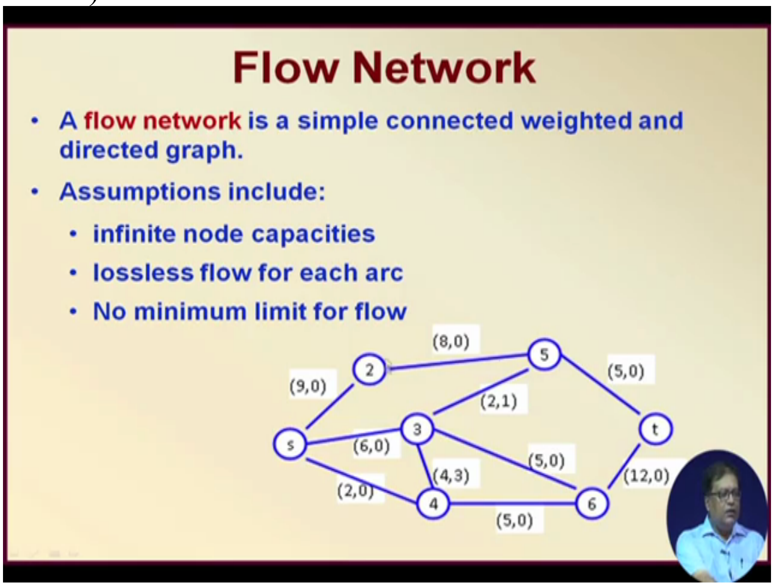

Previously we have also discussed the 3 broad assumptions; obviously there is a 4th assumption also that could be multiple sources and multiple sinks. So what I said that you join the multiple source to a single sink called a super sink and sorry super source multiple source to a super source and multiple sinks to a super sink and then carry on and all these additional edges are moving to have infinite capacity. Other assumptions are these node capacities are infinite that means no restriction in this particular all these nodes. Then lossless flow, there is no loss of flow and no minimum limit for flow. So these assumptions we have already seen.

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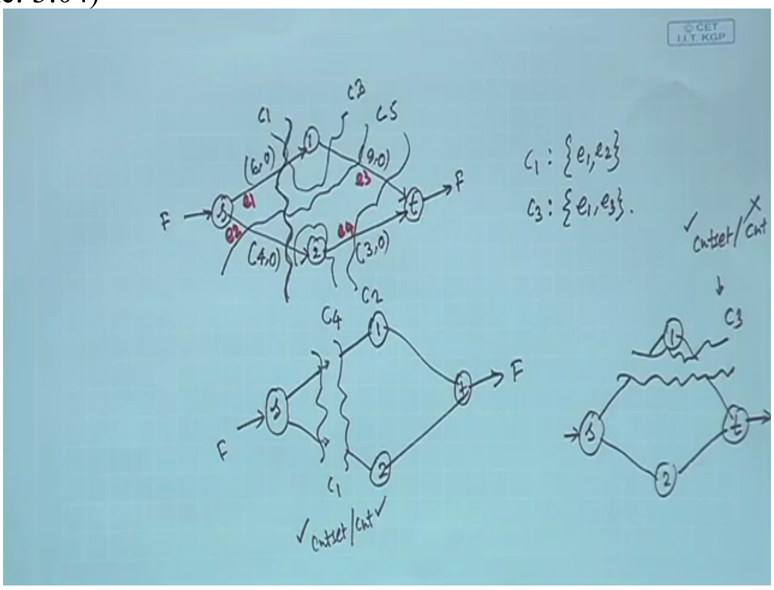

Today we start with very important concept that is called cut and its capacity, see we have already seen the concept called cutset. So let us take an example you know supposing just take a very simple network. Let us say this is source, this is sink and there are 2 more nodes here, node1 and node2, so flow is in this directions, right. So some flow is coming in here and the flow is going out of here.

Now S21 suppose capacity is (6,0)(9,0)(4,0) and (3,0), so what are these? These are capacities of these flows. So this is all unidirectional that means here these are the forward directions, I have shown. Now what could be some cutsets, right look at this is a cutset c1, this is another cutset c2, but you can also think of other cutset like c3 and c4, obviously they are could be like this also c5. So I hope you have , you can see those cutsets, so supposing you know let us use another pen supposing, we name the edges suppose this edge is e1, this is e2, this is e3 and this is e4, then these cutsets c1 is e1,e2 right. Now what is the cutset c3, c3 is e1, e3.

Now let us understand that how it looks after a cutset c1 operates, so you see this is a cutset c1, it has operate how it operates. So you see this is s, this is this portion and this is the other portion and subsequently 2 t to f is this side and f is this side. So you see, this is where we have cut, so you think this way that from s, there is some flow that is taking place, now what will happen if we cut here, right. So it is a cutset called c1, so this is a c1 cutset, so you see the material that goes you will fall, right because there is a cut and since, the flow you know water or whatever, it will fall, it will not be able to reach sink, if it still able to reach sink, then it is not a cutset, right or to be more précised cut in this sense.

Let us state another example called let us take c3, you see what happens in c3, the cut is actually here, see this is c3. Now tell me if some flow comes from s can it reach t, obviously it cannot reach t through node1, because node1 is cutout by the cutset c3, but the flow can still reach from s to t. So in that sense c3 is a cutset, but it is not a cut, it is a cutset but not a cut, c1 is a cutset and it is a cut as well.

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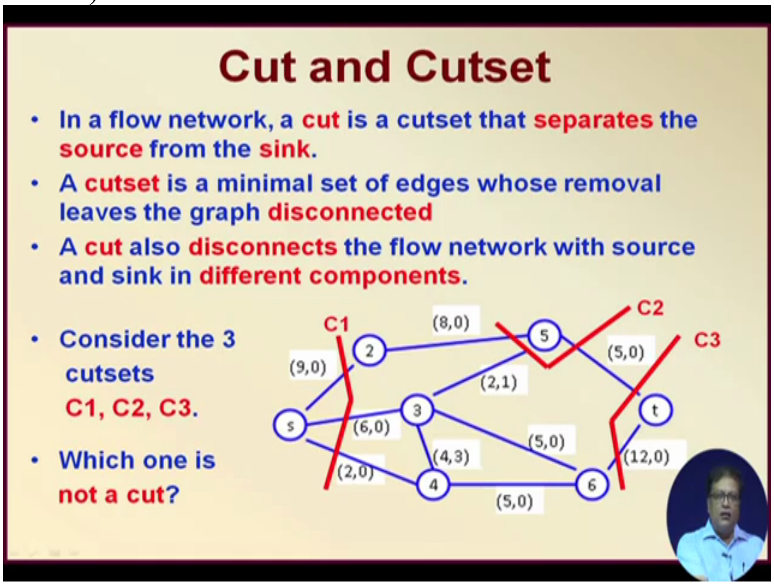

So look at the slide now, you know cut a cutset is a minimal set of edges whose removal leaves the graph disconnected. A cut also disconnects the flow network with source and sink in different components. So in that sense a cut is a cutset, but not all cutsets are cuts, additional requirement for something to become a cut is that after it operates the source and sink should be on the 2 different sides, right. So now tell me is c1 a cut? It is a all these c1, c2, c3 they are all cutsets, but is c1a cut? Answer is yes, it is a cut, because it separates the s from the t, right flow will fall.

Now if these is a cut here flow will not be able to reach t and if as if you know it is cut, so they say it will bleed (())(8:26) alright what about c3? It is a cut or not again it is a cut, because it separates the source from the sink? What about c2? C2 is not, because c2 does not separates source and the sink. So if you say a cutset partitions the network or the graph into 2 different sub sets or sub graphs then source and sink should be on the you know source on one side, sink should be on the other side. If source and sink both remains on the same side then it is not a cut.

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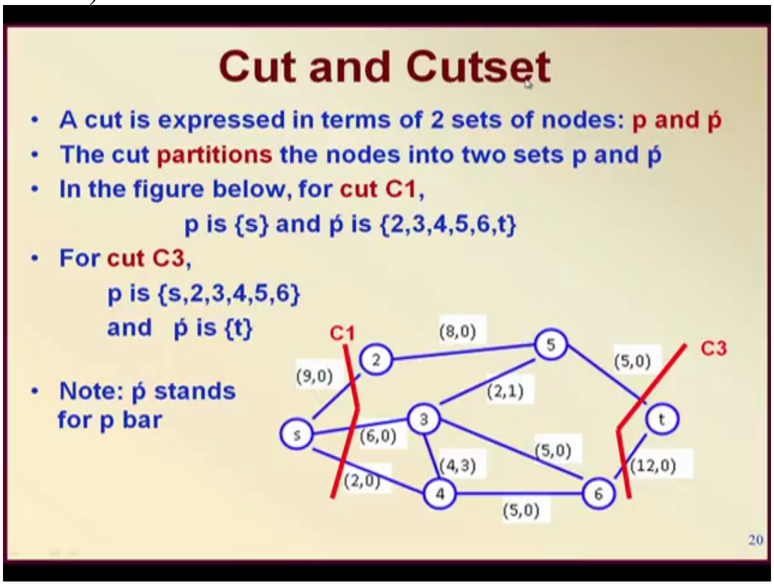

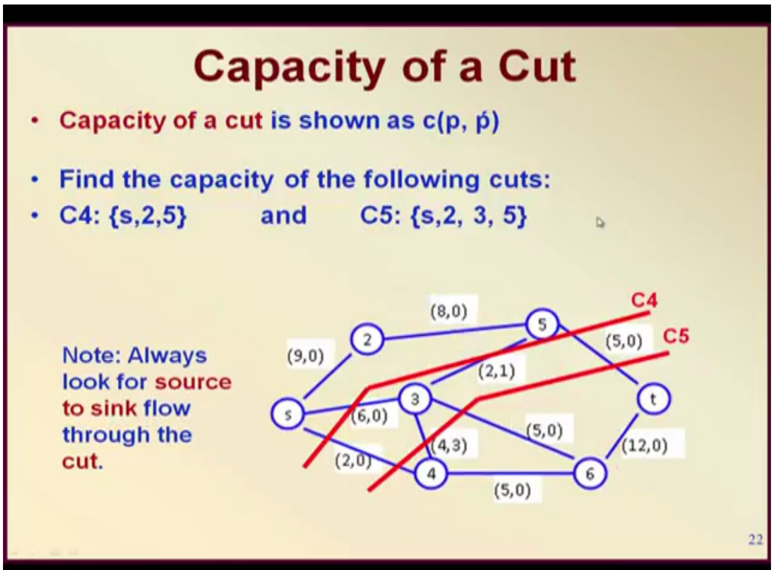

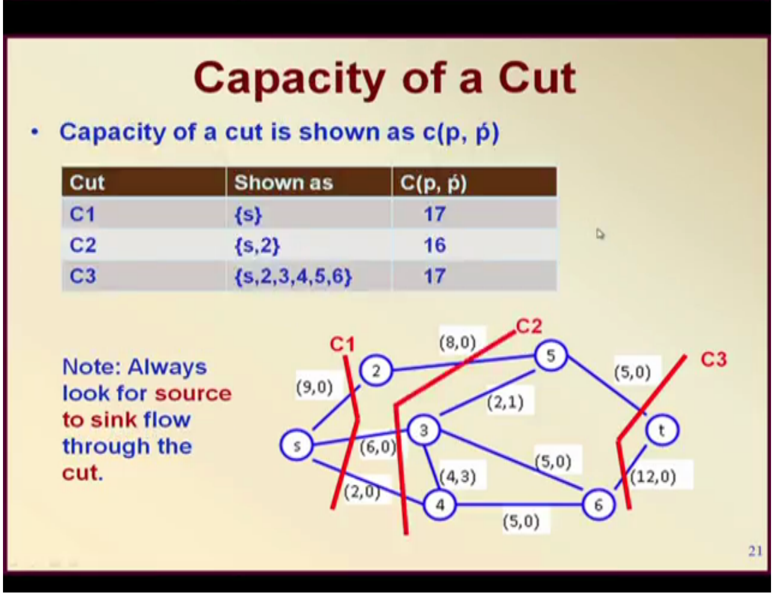

Now comes a very important concept that is the capacity of a cut, right. Now to express the capacity of a cut we need to know that what a cut does the cut partitions the nodes into 2 sets. What are those 2 sets, the p and p bar, right? So look at the cut c1, it partitions the nodes into 2 parts on one side is p, which is s, other side is 2, 3,4 5,6, t, right. So this is p set p is s and set p bar is 2, 3,4,5,6 and t alright. So you know every cut then separates them these nodes into 2 sets, so obviously can you now answer what are the 2 sets for c3, right one side is s,2,3,4,5,6 all these nodes and the other side should be t, right. So that is the partitions that cut does.

Now you can well understand the capacity of a cut is therefore expressed as a function of a both this p and p bar and actually the capacity is written in the form in the form of what is known as the p p bar right and you now the cut set itself can also be shown as p, p bar, so it is like if the cutset is p, p bar then the capacity of the cutset is c of p, p bar, but interestingly we know that the you know p bar is nothing but the rest of the nodes other than p, right. So whatever nodes are there in p the remaining nodes will be p bar. So it is very easy therefore you simply a write the p side of the nodes to express a cutset this is simple simplest way.

So if you simply write s the s is partitioned rest are on the other side. If you write s,2 then you know can you see c2 is really s,2 c2, because s,2 on one side rest of the nodes are on the other side and s,2,3,4,5,6 so that c3. Now what is the capacity of a cut? The capacity of a cut is how much flow will go out of the cut if it actually operates how much flow goes out, right how much flow goes out.

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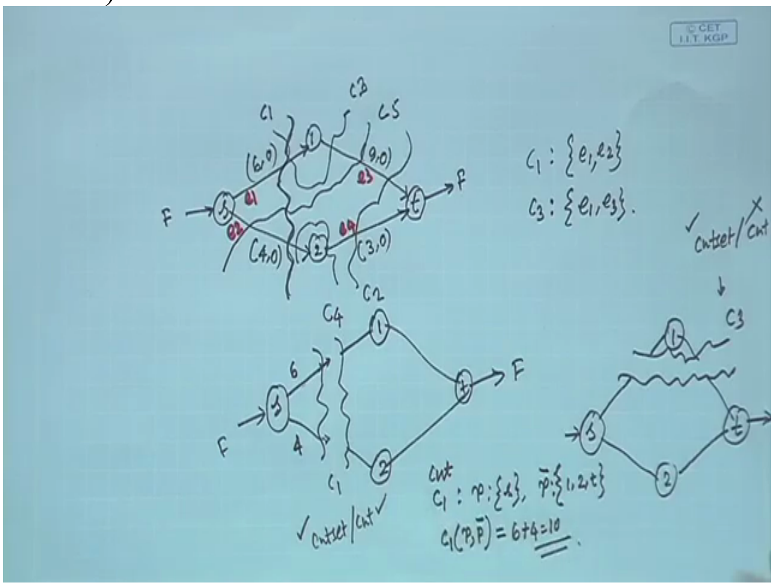

So in these example look at these example, which we are talking about. in this example what is the capacity here the forward flow capacity is 6 and what is the forward flow capacity here that is 4. So what is the capacity of the here p cutset c1 sorry, cut c1 p is s and p bar is the rest that is 1, 2, t alright. So what is the capacity of the cut c1 p, p bar will be 6 plus 4 that is 10. Interestingly one very important point one should remember, a not all the time we may have to take the forward flow particularly we might have to take the reverse flow also depending on the situation.

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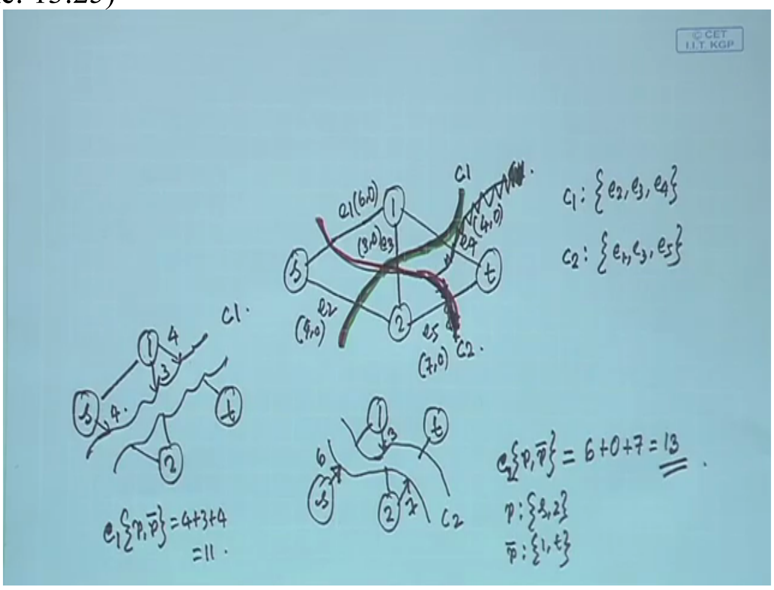

Let us look at one more example supposing we have s, 1, 2, t and stick these example. Let us take this example and compare this with another cutset that is this one. So you see this is c1 and this is c2, right. So let us use different colors so that it is clear, so you see, this one c2 and this one is c1, I hope it is very clear now. So now there are different edges, so this is e1, this is e2, this edge is e3, this is e4, this is e5. So now the two cutsets c1 includes e2, e3, e4 and for c2 e1, e2, e3, e5 again for uh sake of simplicity let us take only forward flow capacities and let us assume that there are no backward flows. So what is the forward flow capacity at e1, let us say 6,0 at e2 4,0 at e3 let us take 3,0, e4 4,0 and e5 7,0.

So you see, when we are separating you know by c2 let us look at the cut c2 e1e3e5, right. So what is happening s and 2 on one side 1and t on the other side to first of all there is a problem you know this is not a cut, so we modify c2, so you see this is not a cut we modify c2 like this, right we modify c2 like this, because s and t should be on the different side, right. So therefore c2 this is c2, so c2 should separate s from t (())(17:07) its fine s and 2 on one side, 1 and t on the other side note (())(17:11) it is fine number problem at all no problem so this is c2 here is let us put a different color, so here so this is c2 no issue s and t on the other side, right.

So now look here what will be the capacity then, right. So here you know this is also cut, so this is the cut c2 right this is the cut c2. So look at this cut c2, suppose you want to find its capacity then you know from s to 1 what is the flow? The s to, this is the cut c2, s to 1 the flow is 6. What is the flow from 1 to 2 the capacity, maximum is 3, but what is the you know 2 to t, the 2 to t the flow could be 7. So what is the c you know p, p bar in this case that p is s 2, so p, p bar, hence write p, p bar for c2 what will be c2 p, p bar where p is s side that is s and 2 and p bar is the 1 and t. what will be its capacity? What will be its capacity?

So look here, how much flow can pass from s to 1, 6 how much flow is 2 to t 7 no problem at all, but what about 1 to 2? The 1 to 2, the maximum flow is 3 only, it is all right 1 to 2 is 3, but in this case the flow is cut from 2 to 1 not from 1 to 2, because 2 is on the s side. So we really do not need to take 1 to 2 capacity, but 2 to 1 capacity which is 0. So it should be therefore, 6 plus 0 plus 7 equal to 13. So I hope you understand that if sometimes but on the other side if you look at s 1 then this is s 1 and this is 2 t, so s to 1 there is no problem, 1 to t it is separated, 1 to 2 it is separated and s to 2 it is also separated.

So this is a separation, you see s and 1 on one side 2 and t on the other side. So this is a cut c1 so this is the cut c2, so when there is a c1 cut then 1 to t, the capacity was 4, so 4i was coming here, 1 to 2 the capacity was 3 and s to 2 capacity was 4. So if c1 p, p bar would be 4 plus 3 plus 4 that is 11, so I hope you understand that how you look at such kind of node capacities.

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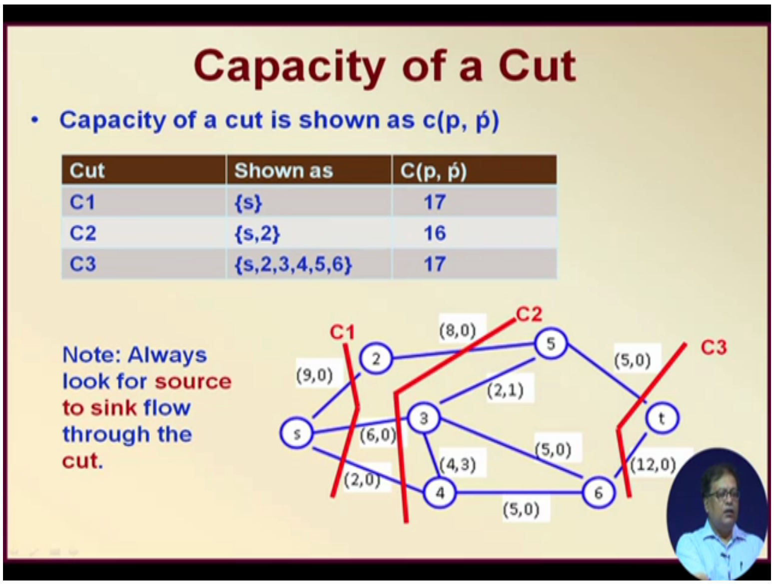

So here look at these example, there are 3 cuts c1, c2 and c3, so what would be their capacities? If you look at c1 then it is very simple s is on one side and rest of the nodes on the other, so it should be 9 plus 6 plus 2, 17 no problem. When it comes to c2 you know it separates this this and this, so here capacity was 8, because you see s and 2 on one side, 6 and t on the other side. So very simply they are all cut in the forward way, so it should be 8, 6 and 2 total is 16 that is the capacity and what is the capacity of c3, because it cuts this 5 that is from 5 to t and 6 to t, it should be 17, right.

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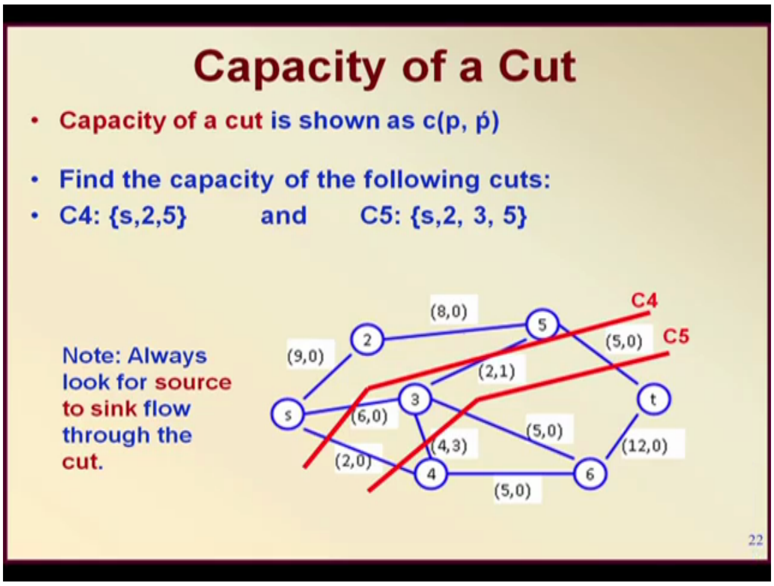

So these are the capacities of the cut for this particular network, right, but interestingly you know this is an interesting example what would be their capacities of the cut c4 and c5 that notice whatever I was trying to tell you is note is given here, always look for source to sink flow through the cut right. So if you look at the source to sink flow through the cut what is the capacity of c5, because look at the 3 4, the 3 4 the capacities the 3 to 4 is capacity is 4, but 4 to 3 capacity is 3.

Similarly, 3 to 5 is 2, but 5 to 3 is 1, so what is the capacity of c5? Look here then this is one portion and the other side is the other portion, so what happens? These portion when you separates then 5 to t the flows are taking place from source side to sink side is 5 to t, 3 to 6, 3 to 4 and s to 4 that is what happens in cut c5. So here the forward flow capacity should be taken. Here also forward flow capacity; here also forward flow and here also forward flow. So total comes be 5 plus 5, 10 plus 4, 14 plus 2,16, so that means the capacity of the cut c5 would be 5 plus 5,10 plus 4, 14 plus 2,16, but what is the capacity of the cut c4? Capacity of the cut c4 if you really look at the cut c4 separates s, 2 and 5 on the other one side and 3, 4,6 and t on the other side. So since 3 on the other side then the flow has to be taken from 5 to 3.

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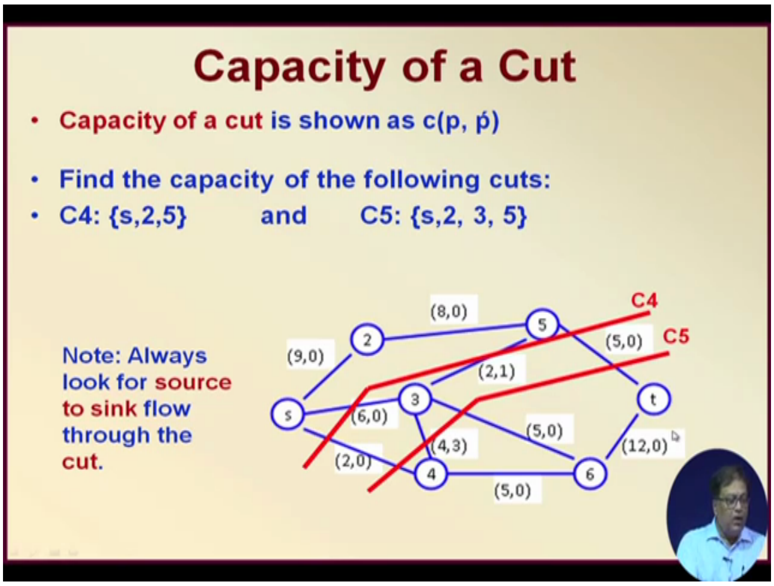

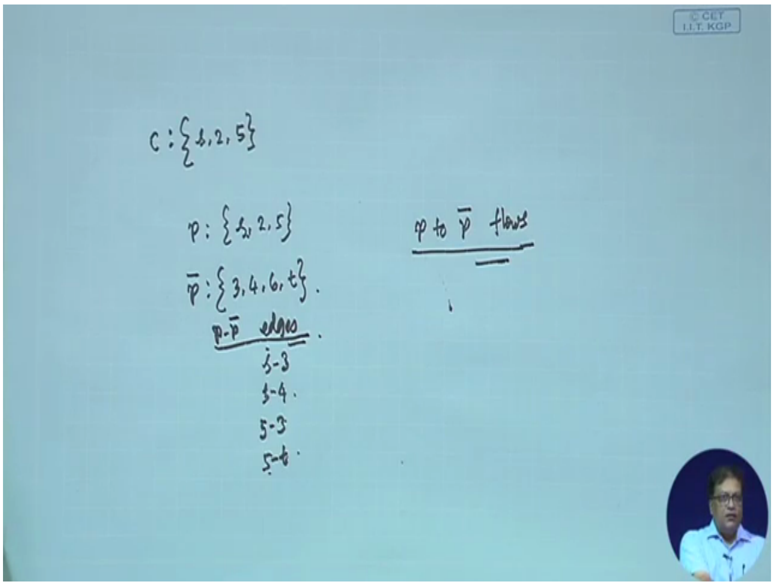

So you see interestingly very you know you can write down in a very interesting manner, you see the cut is s, 2, 5 that is the cut. So basically p is s, 2, 5 and p bar is the rest, so what are the rest? Rest are 3, 4, 6, t look here. So you can then understand if you compare then the you see p to p bar flows do not have to draw the diagram also p to p bar flows. What are the different you know combination? There could be 12 combination, but not all 12 combinations are available you know if you look at here, s to 41, is available s to 3 is also available, but s to 6 not available s to t is also not available. So out of them what are the things there available is s to 3, is available s to 4 is available then 2 nothing is there no no arc is there from 2 to this and from 5 you know 5 to 3 is available and 5 to t is available look here s s, s 5 5, 3, 4, 6, t on the other side. So these are edges you can call them p, p bar edges p to p bar edges.

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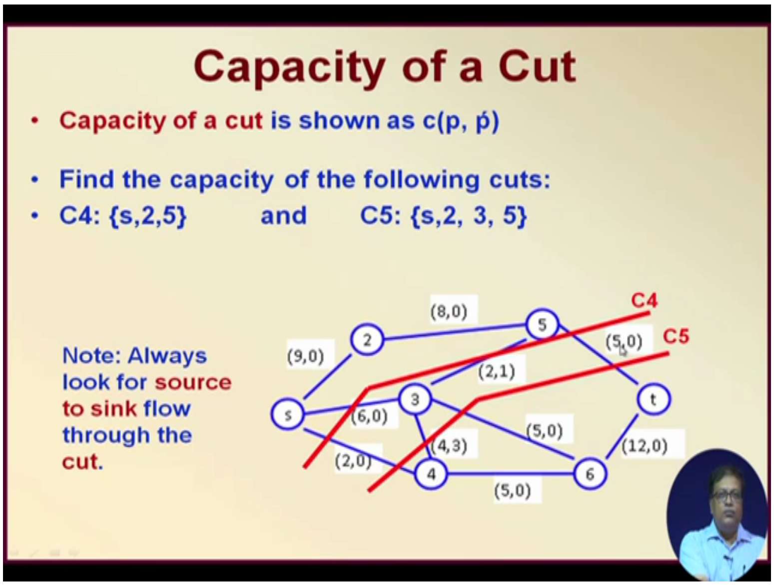

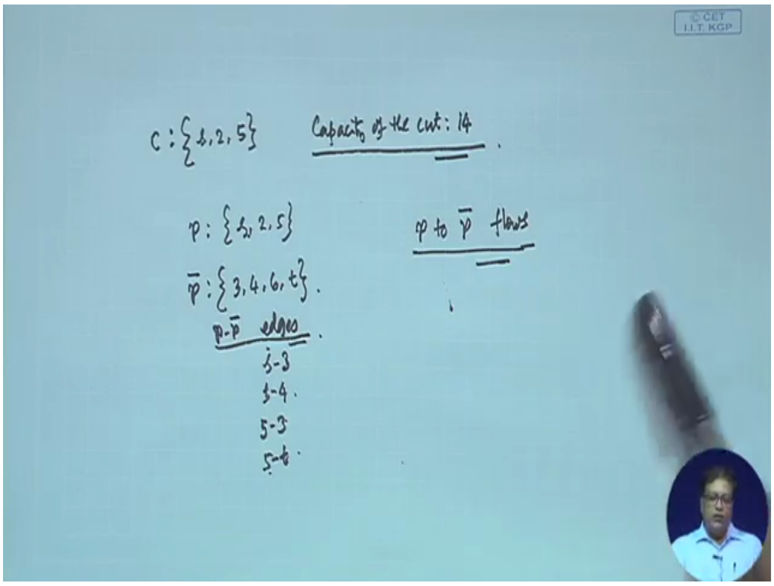

So again look here just look you know this slide you can see that you know s 2 just please look at the slide, so you know s 2 s to 4 and s to 3 these capacities are 6 and 2, 8 but 5 to 3 capacity is 1, right we do not have to take 3 to 5, we have to take 5 to 3, so because we have to count 2 plus 6, 8 plus 16 that is 9 plus 5 that is 14. So the capacity of the cut is really 14 right capacity of the cut is really 14. So that is how you have to compute the capacities of the cut

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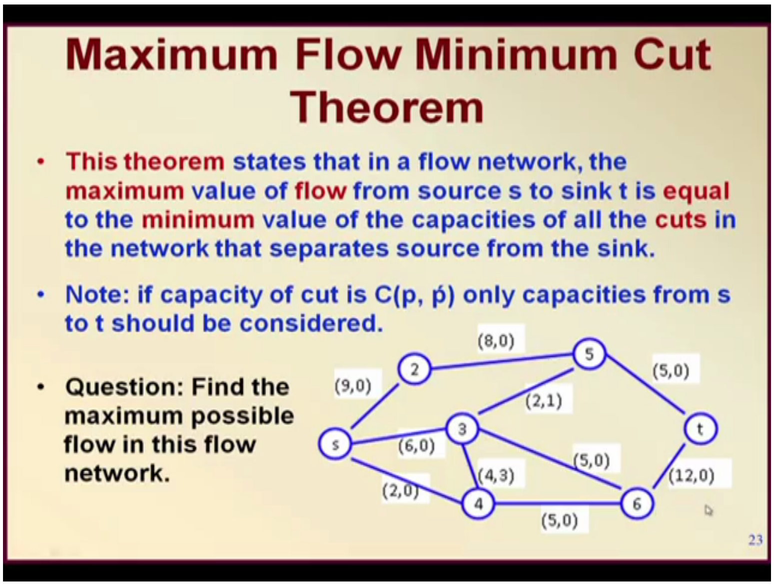

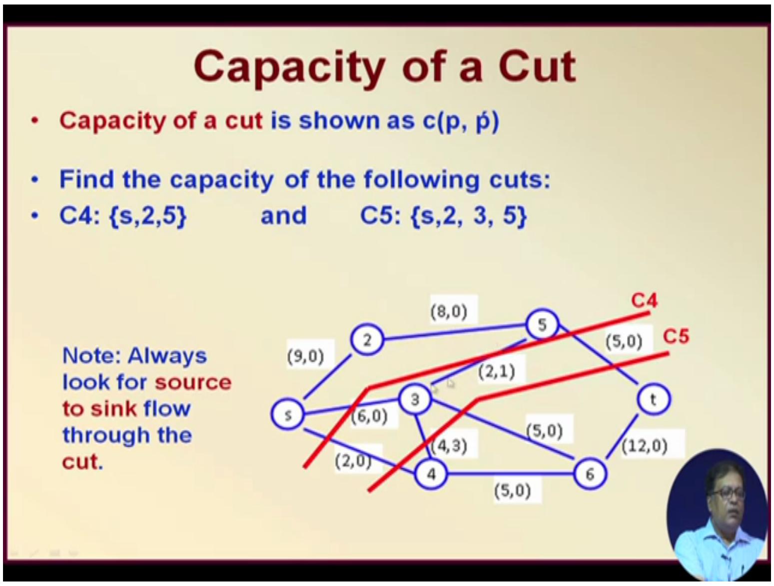

And there is a very important theorem you know that is called the maximum flow minimum cut theorem. This theorem states that in a flow network the maximum value of flow from source to sink is equal to the minimum value of the capacities of all the cuts in the network that separates source from the sink. So if the capacity of cut a c only capacity from s to t should be considered. Obviously that is already known about cut, right some people you know which si not maximum flow minimum cutset, it is the maximum flow minimum cut and cut is separate from cutset.

A cut is nothing but a cutset that separates source from the sink. Is it all right? So if that is so then the maximum flow minimum cut theorem basically says if you look at these networks, what is the maximum flow that is possible from s to t. So out of all the cuts, so basically you have to look at all the cuts and taking all the cuts you have to then understand what is that cut which is giving me the minimum capacity that is the maximum flow possible, it is like the simple btelenic (())(28:05) theory that maximum flow is possible you know a chain is like say as strong as its weakest link that is called theory of constraints, right same thing happens here, you know for a flow the maximum flow is possible only that much which is the minimum possible cut.

So what is the minimum cut here? You know we have seen different cuts. What is a capacity of these cut? They should be 17, this is this cut is 8 plus 6 plus 2, 16. These cut 15, this cut 17, so what is the minimum? You know interestingly go back once we have seen a cut c4 see the cut c4 what is the capacity that we have computed, this side is 5, this side is 1, this side is 6 and this side is 2 that means 5 plus 1,6 plus 6,12 plus 2,14, right. So there is a cut c4 we had already found whose capacity was 14, right.

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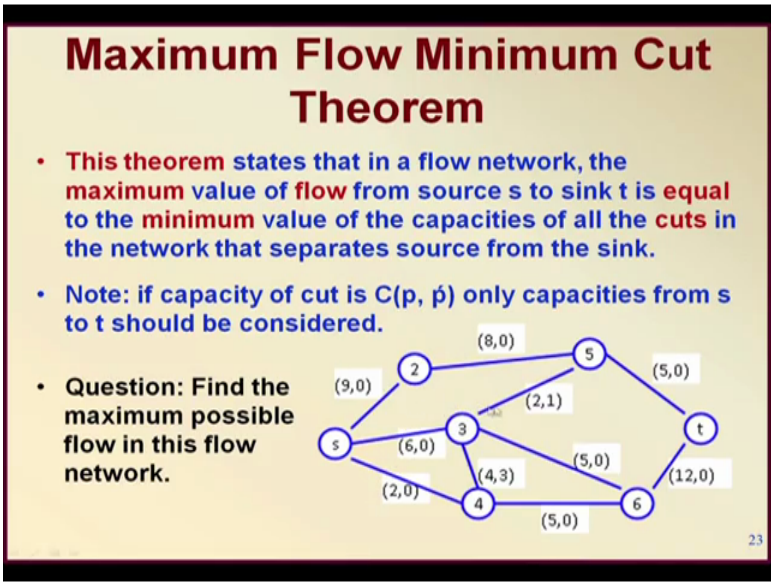

Now look at this graph. Here a flow of 9 can go, flow of 6 can go, flow of 29 can go, so it looks as if you know a flow of 15 can pass through these network, because there is no limit here 17, there is no limit here 14, you now apparently it looks that 15 is possible, but really 15 is not possible, because there is a cut here from here to here to here to here to here, is all right and this cut is capacity is 14.

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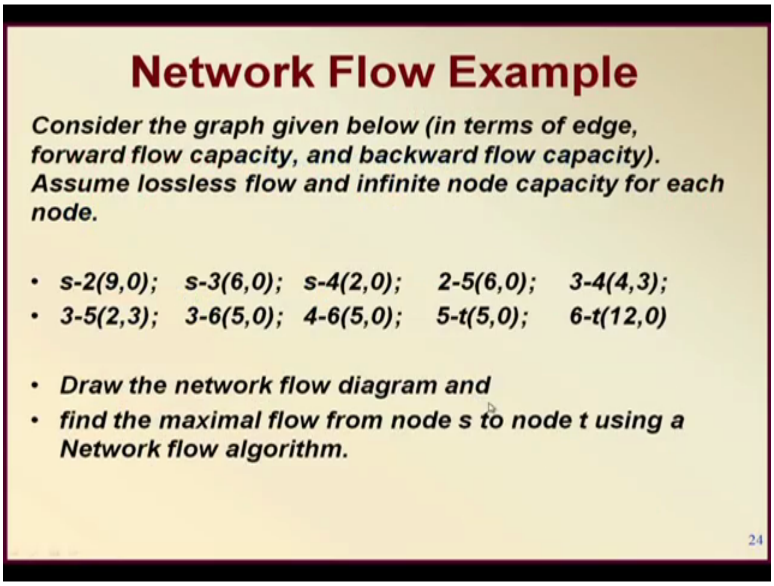

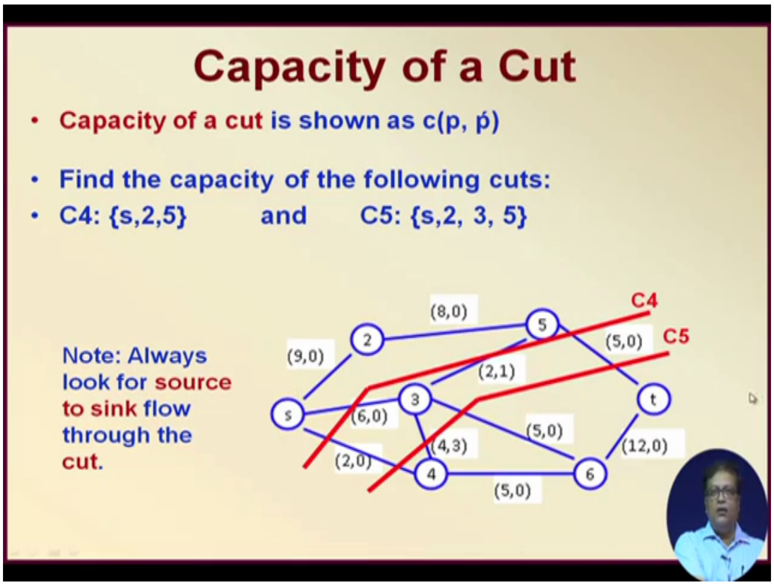

So look at this graph once again there is c4 and its capacity is only 14, so maximum flow that is possible in this particular network is only 14, right. So this point has to be remember that maximum flow that we can have at a given this thing is given by the minimum cut. So thank you very much.