Course on Decision Modeling
Prof. Biswajit Mahanty
Department of Industrial and Systems Engineering
Indian Institute of Technology Kharagpur
Mod08 Lecture38
Maximal Flow Problems (Contd.)

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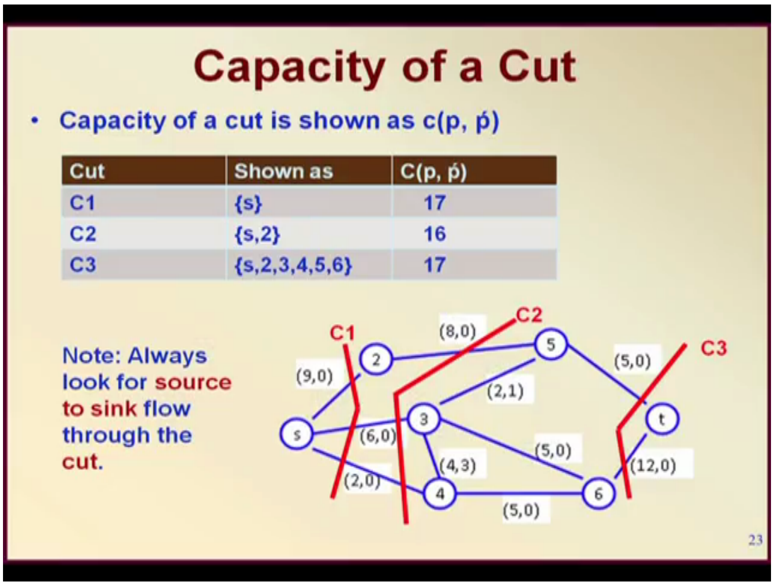

Right so previous class we have turn the maximal flow problems one side of this that is the capacity of a cut. So you very quickly if we just do a review of this then cut is a cutset, which separates the source from the sink. So if you look at the slide, in this particular examples c1 is a cut c2 is also a cut c3 is a cut, but anything that does not separate the source from the sink is not a cut and we have also understood the concept of capacity of cut.

the capacity of a cut is equal to that if a particular cut separates the network into 2 sets of we know nodes say p and p bar all the edges which are existing between p and p bar, capacity of those edges, so some of them will be you know like (1,4) (2,4) (3,4), but some could be something like 5,3 where 3 to 5 forward flow capacity and 3 to 5 backward flow capacity should be taken, because it is coming 5 to 3 and not 3 to 5. So one has to be very careful which capacity are we taking in the computation of the cut. So that is what is required to compute the capacity of a cut.

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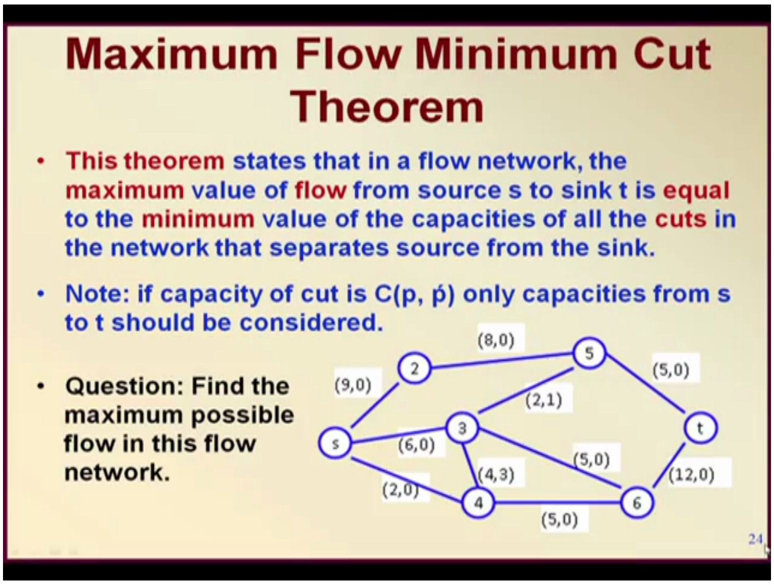

And then we have learned that there is a theorem called maximum flow minimum cut theorem which is based on the theory of constraints and the essential idea is that maximum flow possible in a network is equal to the capacity of the minimum possible cut in that network. This we have already seen.

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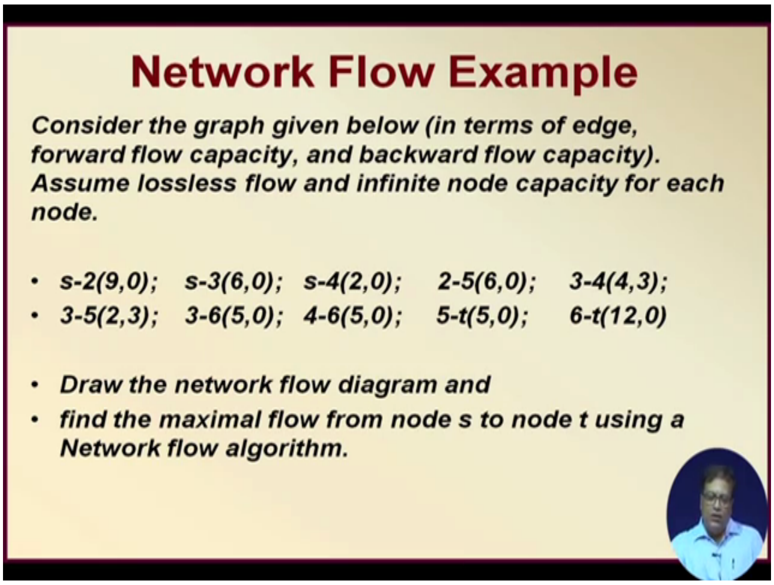

Now let us go ahead and do a problem from scratch. So consider the graph given below in terms of edge forward flow capacity and backward flow capacity assume lossless flow and infinite node capacity for each node. So draw the network flow diagram and find the maximal flow from node s to node t using the network flow algorithm.

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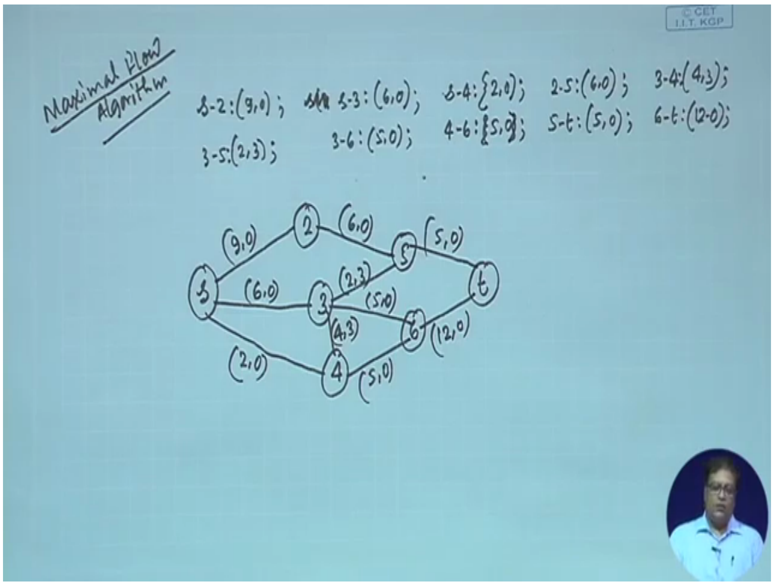

So let me note down the problem and then let us draw it. So s to 2 is (9,0) s to 3 sorry, s to 3 is (6,0), s to 4 is (2,0), 2 to 5 is (6,0), 3 to 4 is (4,3), then 3 to 5 is (2,3), 3 to 6 is (5,0), 4 to 6 also (5,0), 4 to 6 is (5,0), 5 to t is (5,0) and 6 to t is (12,0), maximal flow algorithm. So this is a problem that s to 2 (9,0), s to 3(6,0), s to 4 (2,0), 2 to 5 (6,0), 3 to 4 (4,3), 3 to 5 (2,3), 3 to 6 (5,0), 4 to 6 (5,0), 5 to t (5,0) and 6 to t (12,0). So first the task is to construct the network diagram. So you see we have s-t and 2-3-4-5-6, so let us draw s here, 2 here, 3 here, 4 here and 5 here, 6 here and t here. Then s to 2 we know it is (9,0), s to 3 (6,0), s to 4 (2,0), 2 to 5 (6,0), 3 to 4 (4,3) then 3 to 5 (2,3), 3 to 6 is (5,0), 4 to 6 is also (5,0), 5 to t (5,0), 6 to t (12,0). So see network is constructed, right.

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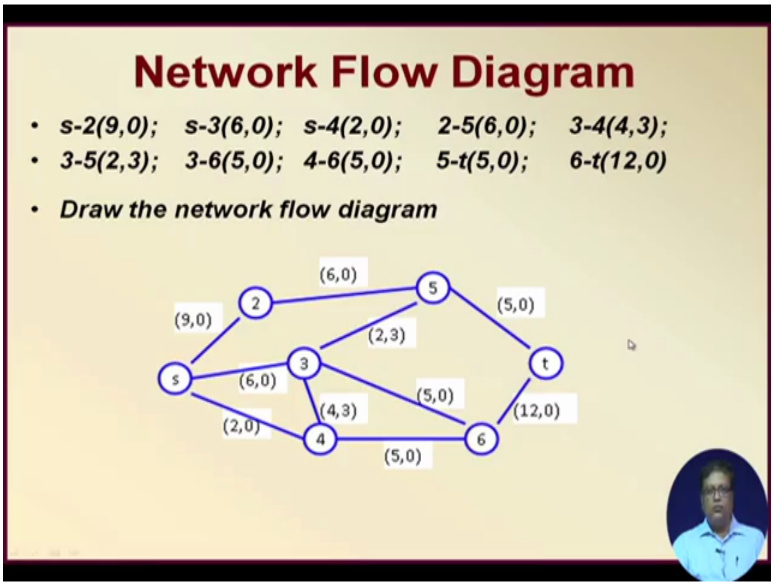

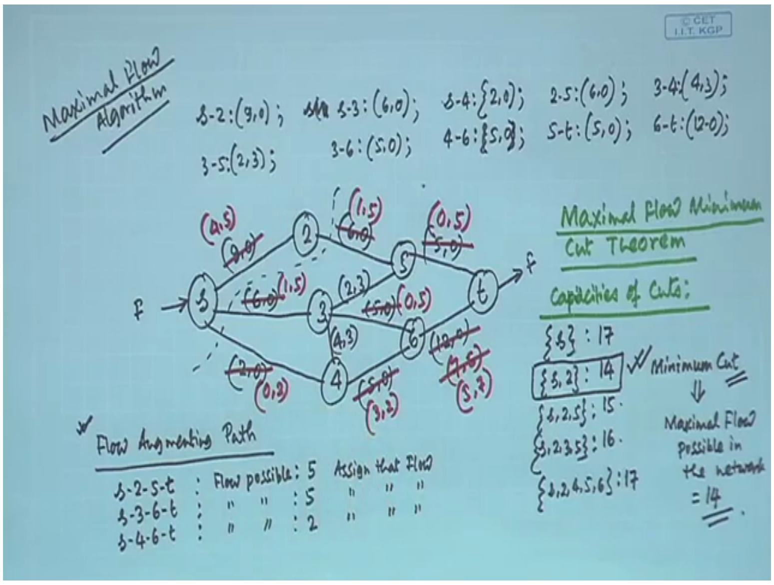

So to check this network let us look at this diagram you know, so more or less we have drawn the same diagram, so that is the diagram that is drawn. Now question is if this is the diagram that is the network diagram, the flow coming at source and flow is going out at sink. So can we apply what is known as maximal flow minimum cut theorem can I apply? So if you want to apply then you have to know the different cuts and their capacities. So what is the capacities of cuts? What are the different capacities of cuts, right? So what is the capacity of the cut s? s is a cut what is that cut just look here, it will cut s on one side and rest on the other side.

So can you see if I have a cut here, it should be 9, 6, 15 and 17 the capacity should be 17 all right. So you see you have to keep exploring, so what is the capacity of s,2 s and 2 on one side and rest on the other side. So can you see that it is 14. Why 14 because if I draw a cut here, let us draw this cut right then in this cut the total flow is 14, right. So this is the cut s-2 this is the s-2 cut right I have drawn very light, just to do not make the diagram to hazy, right. So this is 14 then other cuts or many other cuts are possible, but one has to be careful I mean there is no need to really clatter (())(8:42) all of them, because let us draw this one what its capacity? Just look s, 2,5 what is s,2,5 it will be 5 then 2,7,6,13,15 is also 15, right.

Similarly what is s,2,3,5 this will be this one 5,5,10 plus this side is 14,16, so you please complete you know I am not doing all of them, I stop here by just taking the one more that is s,2,3,4,5,6 that is this one that is 17, right but then there are many other cuts you can be really generate , but then we have to cover many other things. So if you look at only these many cuts obviously that could be more, but we find that minimum is 14 and since I know let me also tell you that this is a minimum not because from only these many, but since I know that is why I am writing.

So this one is the minimum cut, right really to understand that or know that this is the minimum cut you have to work out all possible cuts and then see what is the minimum. So this is the minimum cut that means what is we infer (())(10:23) from here that maximal flow possible in the network equal to 14 that is because we have the minimal cut, but then there should be an algorithm by which we can easily find out this without really going through you know like it is like maximal flow minimum cut theorem is giving me a quick result, quickly we can know that is the maximum flow possible, but one need to know also that what are the flow patterns? How the flow is actually taking place that is the second particularly of our question, right.

So in order to do that we have to really follow a maximal flow algorithm. To the maximal flow algorithm the first task before us is to find out a flow augmenting path. Now what is a flow augmenting path? A flow augmenting path is you know a path from s to t right where a flow is possible, say for example, if I go from s to 2 and then 2 to 5 and then 5 to t, can we get forward flow capacities in all of them? If we get then that is a flow augmenting path. So you see s to 2 to 5 to t, so a flow augmenting path is a path from the source to the sink if we go then if a flow is possible look here the forward flow capacity is 9 a forward flow is possible, it is 6 the forward flow is possible, it is 5 a forward flow is possible, so we can actually assign a forward flow in this path if we can do that then we call it a flow augmenting path all right. So s-2-5-t is a flow augmenting path, so first identify a flow augmenting path, right and then see how much flow is possible through this right.

So flow what is the flow that is possible what is a maximum possible flow in this particular flow augmenting path, you see 9,6,5 it should be the minimum of all the flow capacities there available that means only a flow of 5 is possible. So what we do (we) you assign that flow right assign that flow. So if you assign that flow basically what we need to do is we need to revise the capacities. So we have assigned a flow so this will now become (4,5), because we have assigned a flow of 5 and this will become (1,5) and this will become (0,5), so you see no more flow is possible here, say flow of 5 is assigned. Now find another flow augmenting path, so can you see that we can have s-3-6-t how much flow is possible through this look here (6,0) (5,0) (12,0), so another flow of 5 is possible, so again assign that flow. So if you assign this flow 5 again then we find this will revise to (1,5), this will revise to (0,5) and this will revise to (7,5), right. So this flow is assigned.

Now another flow augmenting path let us find s-4-6-t. How much flow is possible 4-6-t can you see, because it is (2, 0) the flow of 2 is possible 5, 7,5 , so only 2 flow is possible. So again flow possible is 2 assign that flow. So again this (2, 0) will become (0,2), (5,0) will become (3,2) and this will become (5,7). What are we doing? What we are doing since a flow of 2 is assigned you know we reduce the forward flow capacity and increase the reverse flow capacity, because at this point you know you can reverse some of these flow, because a flow of 2 is assigned, you can take out some particularly of it. Originally the flow capacity was zero, but now it is 2, because a flow of 2 is going through this. So you can reverse it, in that sense the reverse flow capacity is now 2, because a flow of 2 is assigned through these arc.

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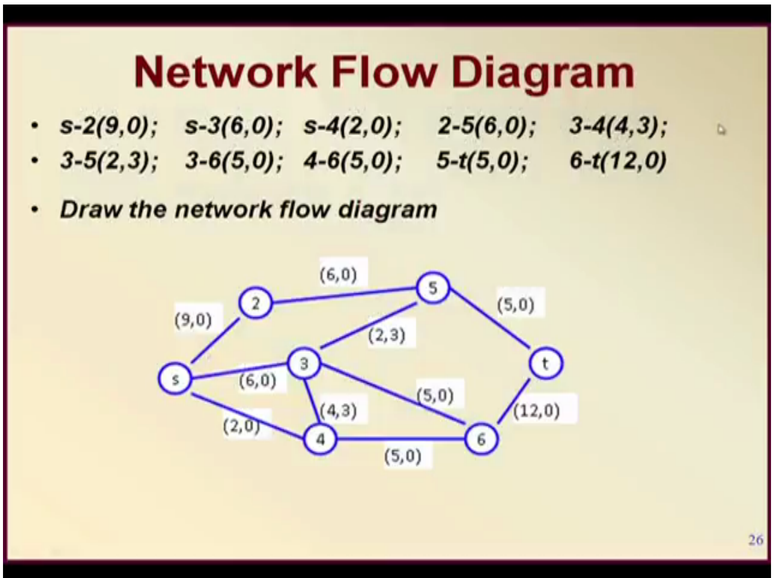

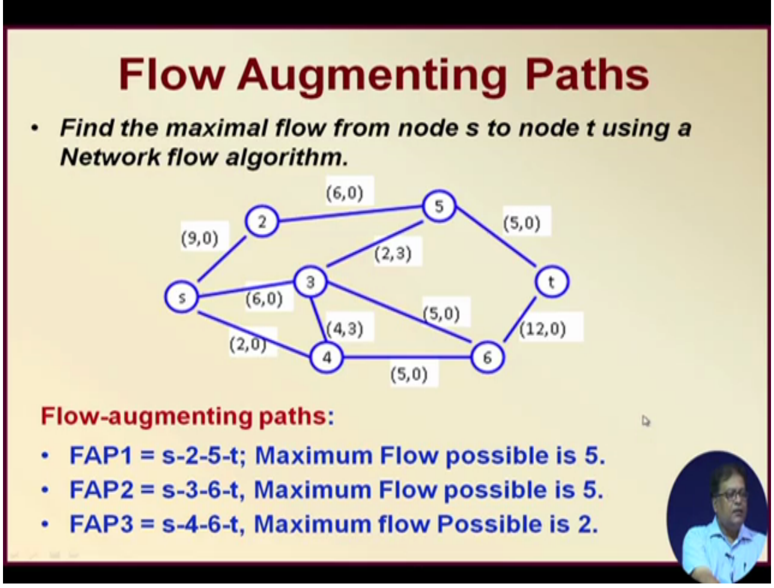

So look at the slide exactly that is what is done here that flow augmenting paths are identified and these are the 3 flow augmenting paths that I have already drawn for you s-2-5-t, s-3-6-t and s-4-6-t and these are the maximum possible flows all right.

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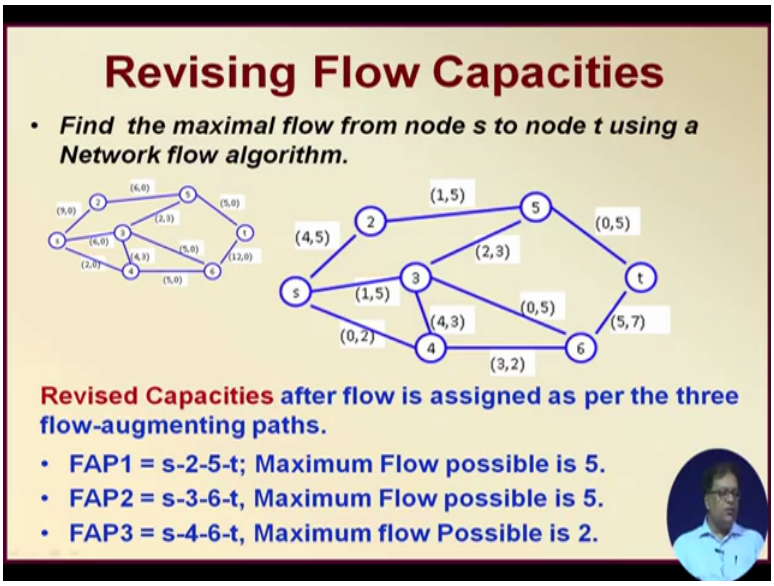

And after these flows are assigned, the flow capacities are revised in these manner, right. So earlier it was (9,0) now it is (4,5) earlier it was (6,0) now it is (1,5) (5,0)(0,5) and this path (1,5)(,0,5)(5,7) from (6,0)(5,0)(12,0) and (2,0)(5,0), so (0,2)(3,2) and this is because 2 are there one from this side another from this side. So total flow is you know is coming to (5,7), because from (12,0) the flow of 7 is assigned. So this is what we do we find out the revised capacities after flow is assigned as per the 3 flow augmenting paths. Is it all right? So that is what we have done that we have really assigned all the 3 through the 3 flow augmenting paths, but is there any more flow augmenting paths possible that we have to understand and identify now.

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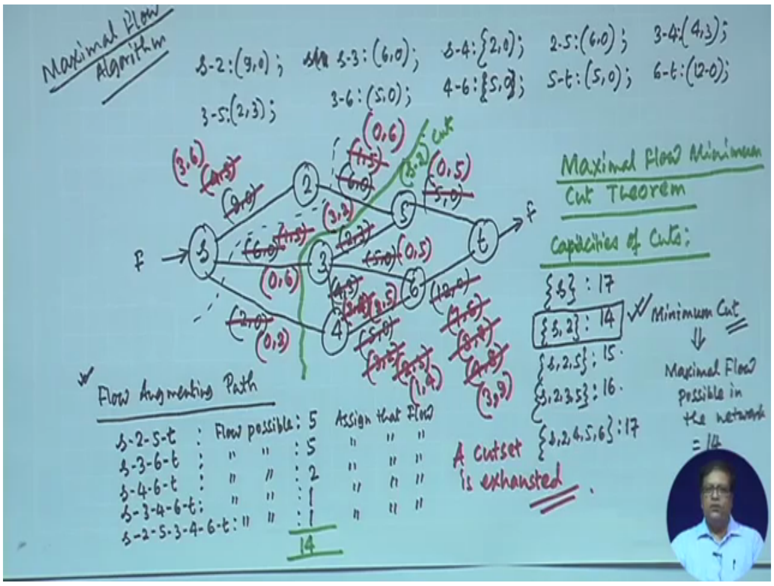

Now let us look at whether some more flows are possible, you see flow of 1 is possible through here, right. Now where from will it go? It will go through this and this and this. So can you see another flow augmenting path is possible that is s-3-4-6-t, so you see (1,5), (4,3), (3,2), (5,7). So in the forward direction maximum possible flow is 1 assigned that flow? So if we assign that flow then we revise the flow capacities. So from (1,5) it will now become (0,6) then (3,4), (4,3) will now become sorry, (3,4) and (4,6) will now become (2,3) and (5,7) will now become (4,8) all right. So this is also assigned.

Now after these assignment is there any more flow that is possible, you see these 2 are not possible, because they are become 0,0 no more forward flow capacity remains, but here something can go. Here also another one can go here and look here, we can come back like this. So s to 2, 2 to 5, 5 to 3, 3 to 4, 4 to 6 and 6 to t, right. So another flow augmenting path is possible that we might call s-2 so hopefully you can see s-2-5-3-4-6-t and how much flow is possible 4here,(4,5)(1,5)(2,3)(3,4)(2,3)(4,8) is it all right? So 1 is minimum. So here flow possible is 1 assign that flow.

Now let us assign these flow and see what happens (4,5) will now become (3,6), (1,5) will now become (0,6), right then (2,3) how much (2,3) will become? Look here the flow is actually being assigned on the reverse side that is 5 to 3 not on the forward side. So (2,3) will become (3,2), because reverse flow capacity will reduce forward flow capacity will go up and (3,4) will now become (2,5), (2,3) will become (1,4) and (4,8) will become (3,9), right. So this is what is maximal flow algorithm identify the flow augmenting paths and assign that flow and change the flow capacities accordingly as you assign the flows all right.

So after we have done all these thing now very interestingly can you see something very interesting, look at these s and 2 cutsets, can we draw this cutsets now, see this is the cutset. So what are these cutsets? This is (s,2) cut or cut basically. So this is the (s,2) cut right. So you see this particular cutset I mean it is a cutset also is a any capacity remaining this is become 0, this is also become 0, this is also become 0. So no so what we call in the language that a cutset is exhausted, no more capacity is left, the cutset is exhausted as a cutset is exhausted no more flow is possible. So what maximum flow minimum cut theorem has told us that total flow possible would be 14 and the cutset was (s, 2).

So after using the maximal flow algorithm, we have seen, a cutset has been exhausted and that cutset has given us exactly a flow of 14 only, because 5 plus 5 plus 2 plus 1 plus 1 is 14, right. t A total flow of 14 is that is possible and the algorithm essentially works by really identifying the flow augmenting paths from the source to the sink and assigning that flow as a we are assigning the flow revise the capacities, as you revise the capacities, then you know we are able to find out whether a cutset is exhausted or not, moment a cutset exhausted we have no more flow possible and we have to stop, right.

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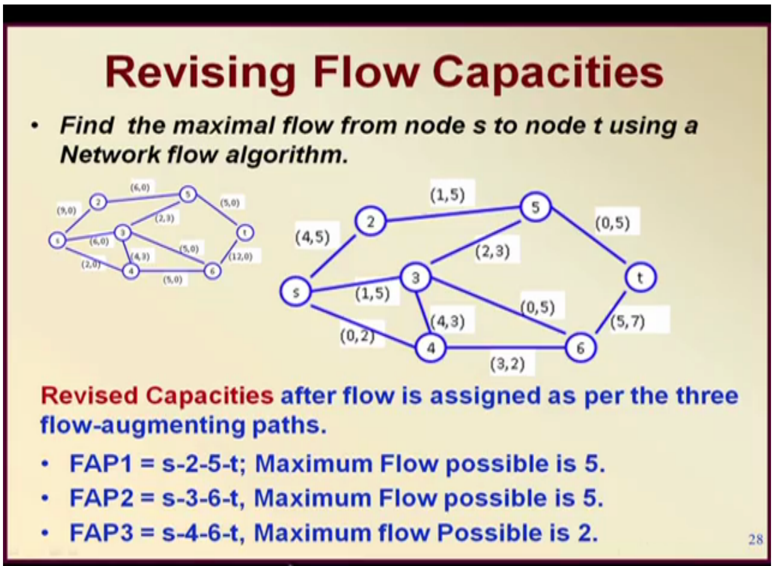
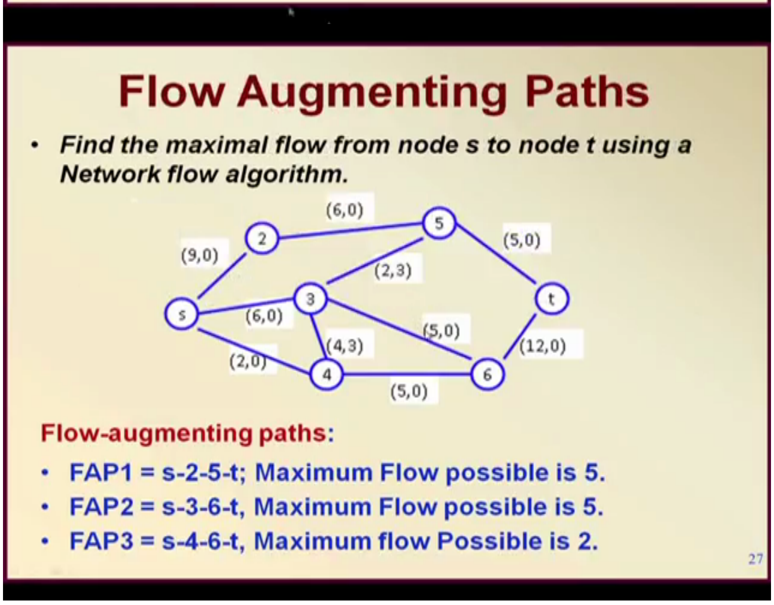
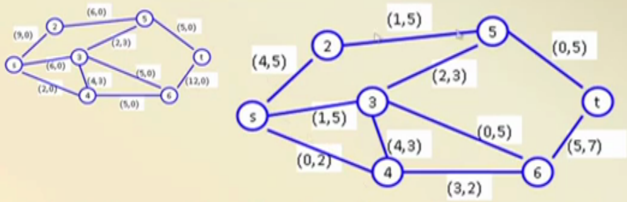




Revising Flow Capacities

- *Find the maximal flow from node s to node t using a Network flow algorithm.*

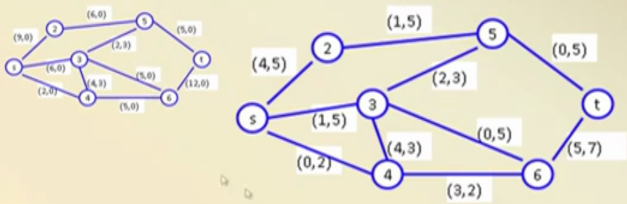


Revised Capacities after flow is assigned as per the three flow-augmenting paths.

- FAP1 = s-2-5-t; Maximum Flow possible is 5.
- FAP2 = s-3-6-t, Maximum Flow possible is 5.
- FAP3 = s-4-6-t, Maximum flow Possible is 2.

New Flow Augmenting Paths

- *Find the maximal flow from node s to node t using a Network flow algorithm.*

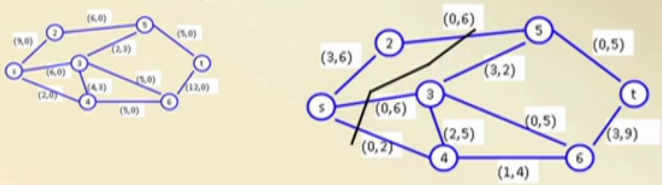


New Flow augmenting paths:

- FAP4 = s-2-5-3-4-6-t; Flow = 1.
- Also, FAP5 = s-3-4-6-t; Flow = 1

Final Revised Capacities

- *Find the maximal flow from node s to node t using a Network flow algorithm.*



Flow Capacities after assigning Flow augmenting paths:

- FAP4 = s-2-5-3-4-6-t; Flow = 1.
- Also, FAP5 = s-3-4-6-t; Flow = 1

No more flow is possible!

Check this from the cutset shown.

So quick summery of what we have done look at here we have started here, this is the network flow diagram we identify the flow augmenting paths and we found how much flow is possible then we assign those flows as we assign we revise the flow capacities say from (9,0) it is (4,5) from (12,0) it is (5,7) etcetera, as we assign the flow capacities then the flow capacities are now revised, so at this stage 3 are identified further we identified 2 more and like s-2 3-4-6-t and s-2-5-3-4-6-t and after assigning all this flow augmenting paths you know we find a cutset is exhausted and this is a same cutset which we obtained from the maximal flow minimum cut theorem, is all right, so no more flow is possible, because the cutset is exhausted. So these are the final revised capacities.

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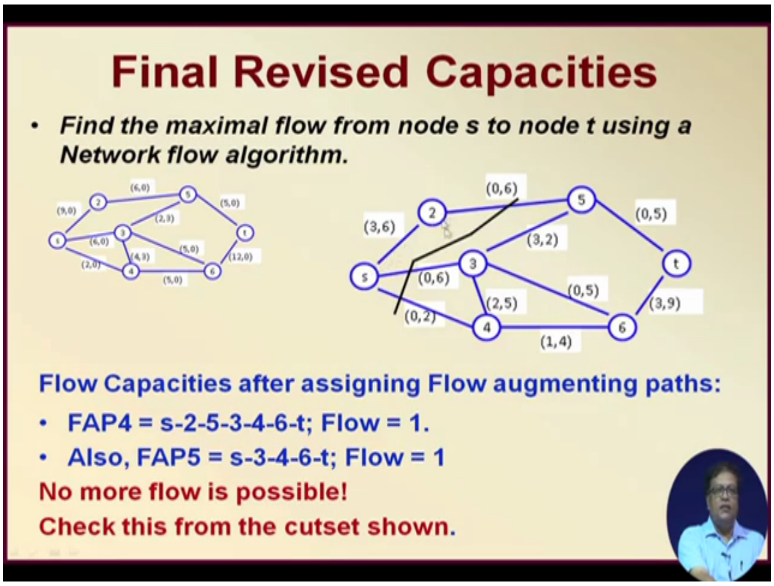

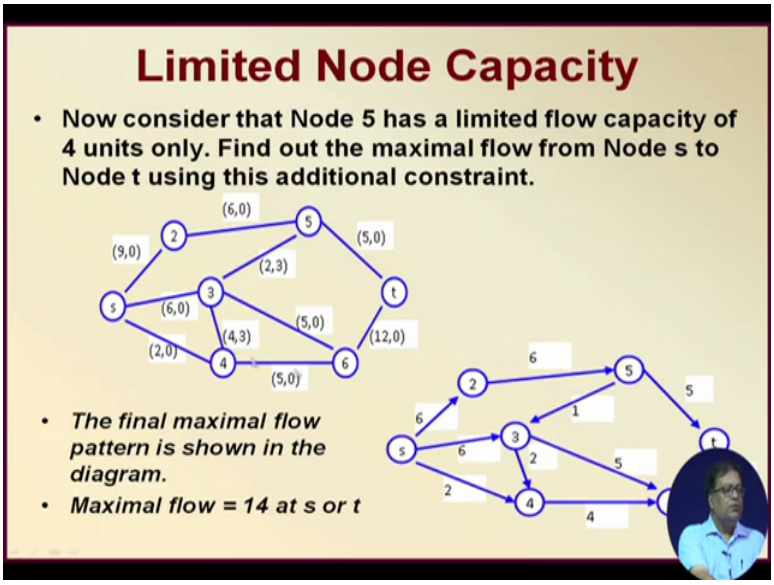

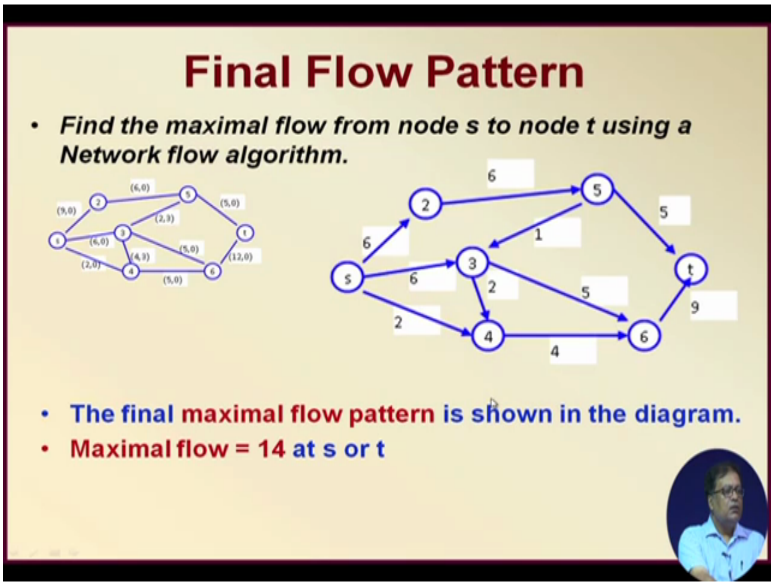

And here is the final flow pattern also. How do we find look here, this is (3,6), so it was (9,0) it has become (3,6). So forward flow capacity reduce from 9 to 3. So a flow of 6 is assigned. So exactly sorry this is what we do the flow is 6 flow is 2, flow is 6, 1 is going in the reverse direction flow of 2,5,4 and 9, right. So 6,6,14 this 6 goes this way splits into 5 and 1, this 6 splits into 6 plus 1,7 is split into 2 and 5, this 2 and 2 adds become 4, 4 and 5 adds become 9 and this 5 and 9, 14 goes to the sink. So this is how we do the maximal flow problems with actually solve.

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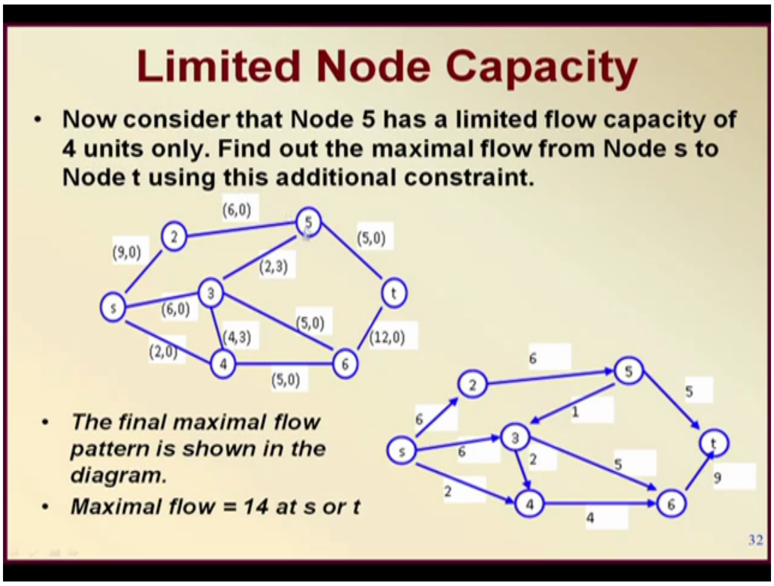

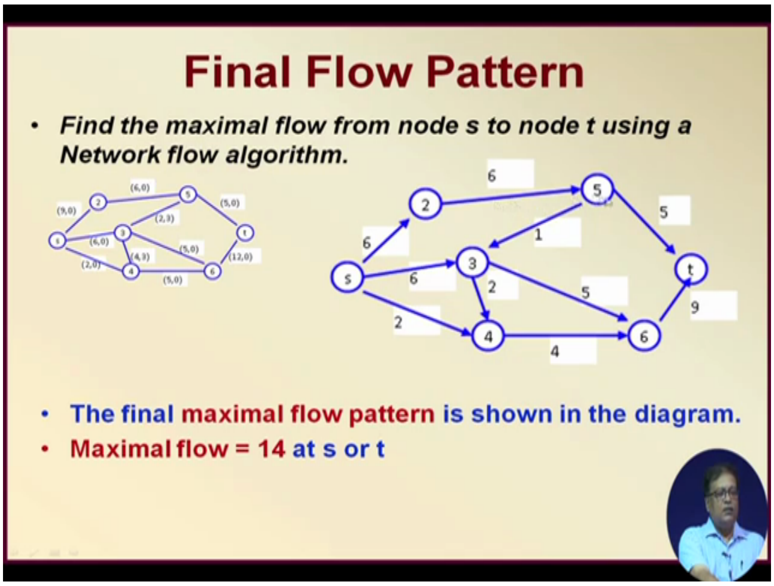

Now very quickly let us see an example of how a limited node capacity can actually be working out. Now consider that node 5 has a limited flow capacity of 4 units only, look at these particular network, the solution was through 5 how much flow is going through? 6 is coming 1 is going out 5 is going out. A flow of 5 is taking place through node 5. Now if we say that node capacity flow of 6, 6 is going through the node 5.

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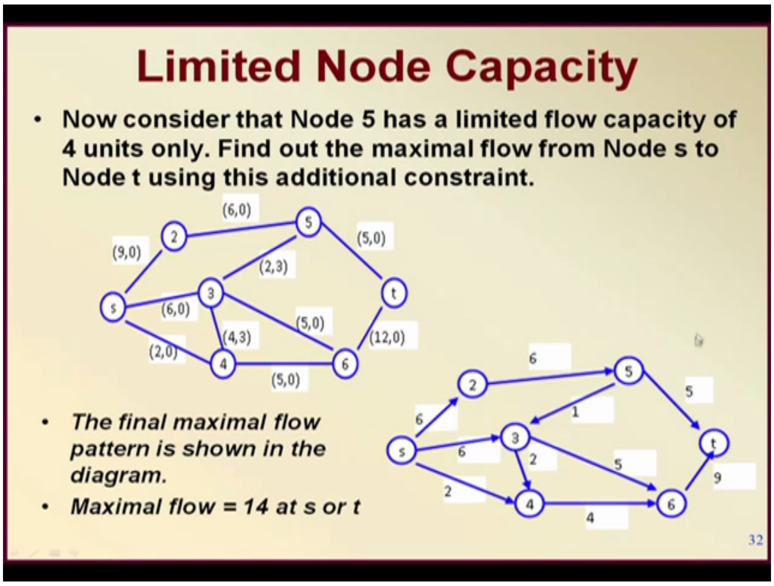

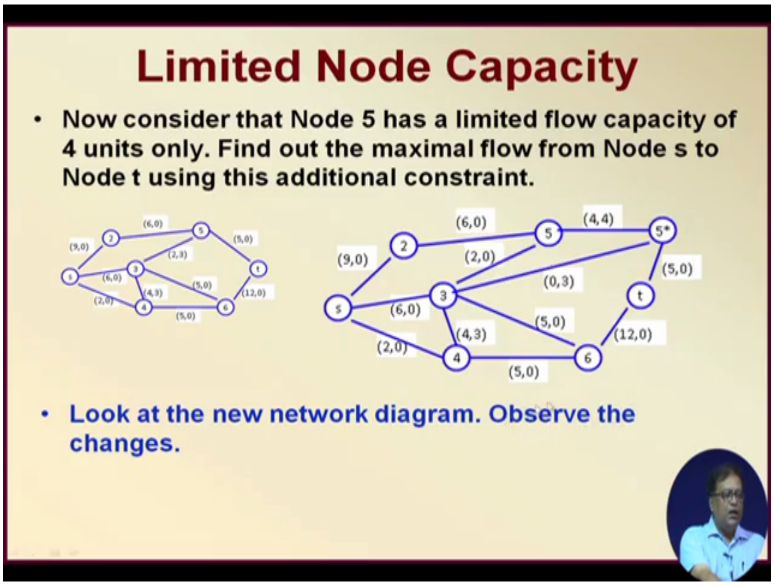

Now if we say that node 5 has a limited flow capacity of 4 only right then obviously, these flow patterns is not possible. This is not possible, because flow of 6 will not be going through this. So how do we implement this? What we have to do we have to really split node 5 into 2 nodes, one is 5 another is 5 dash. So look here that is what is done here, the node 5 is split into 5 and 5 star and the capacity is shown as both sides 4, right both forward and reverse flow capacities as 4. So rest of the network will be as it is, all those things that were coming to 5, they are join to 5 and all those they are going away from 5 they are now connected to 5 star, all right. So this is the new network flow diagram.

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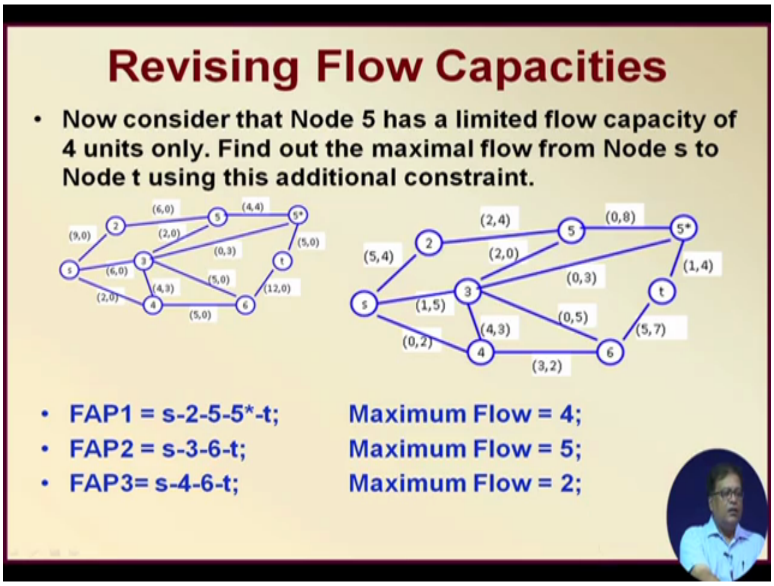

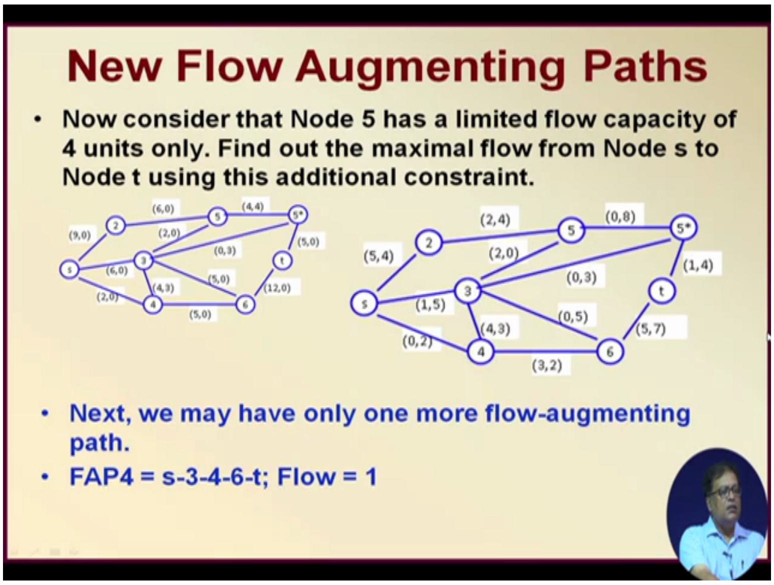

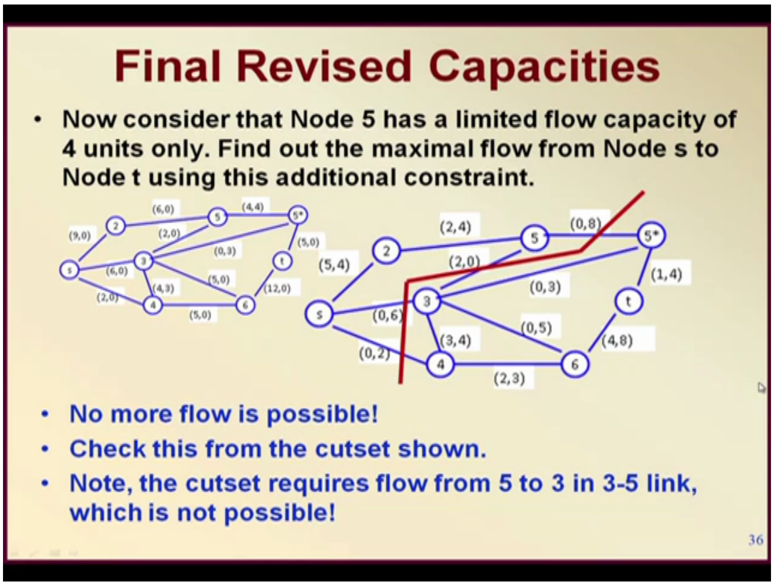

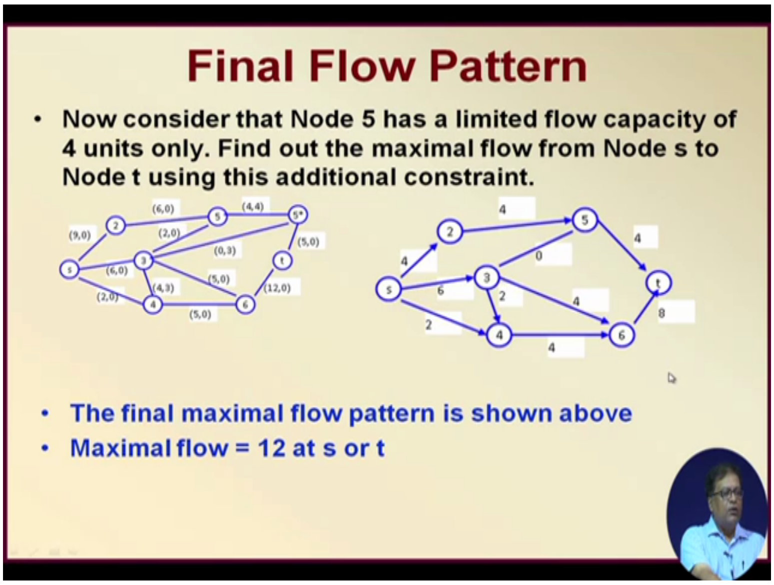

So again we follow the same process, we find the flow augmenting paths, we find the maximal flows, right and then revise the capacities then we find the no flow augmenting paths and then we see a cutset is exhausted. Which cutset gets exhausted? You know you can very easily understand that it will involve 5 and 5 star right 5 and 5 star, so you can see this; this is the cutset that gets exhausted right. There is no more flow is possible, so as you can see that then you can you can see that this is the cutset that gets exhausted then because 0 here 5flow to 3 is 0, this is 0 and this is 0. So this cutsets is exhausted and no more flow is possible and then this is the final answer, right. So this is how and now you have seen that node 5 is having only a flow of 4, right. So this is how you solve maximal flow problem.

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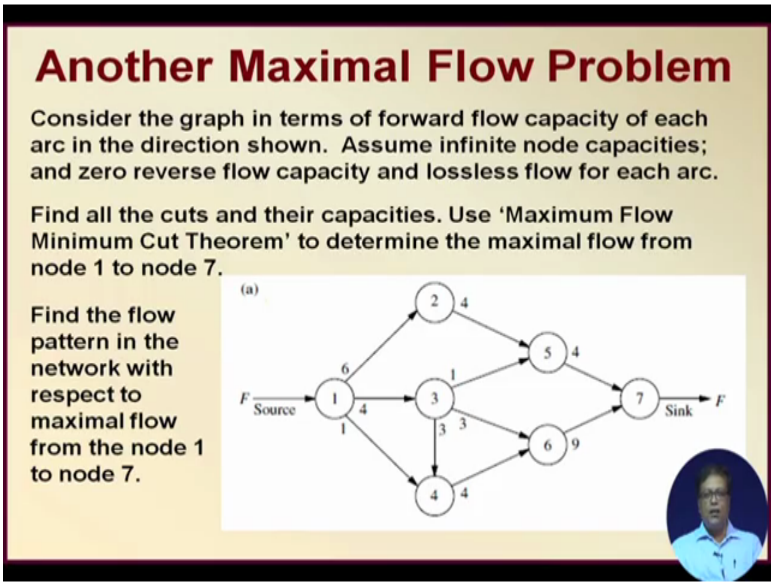

Here is an exercise you can do that the another maximal flow problem that example is given and you can try out find out the maximal flow pattern by using first maximal flow minimum cut theorem and also find flow pattern by the maximal flow algorithm from source to sink and do this exercise and in our next class we consider what is known as the shortest path problems, right. So thank you very much.