An Introduction to Point-Set-Topology (Part 2)
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Lecture 11
Paracompactness

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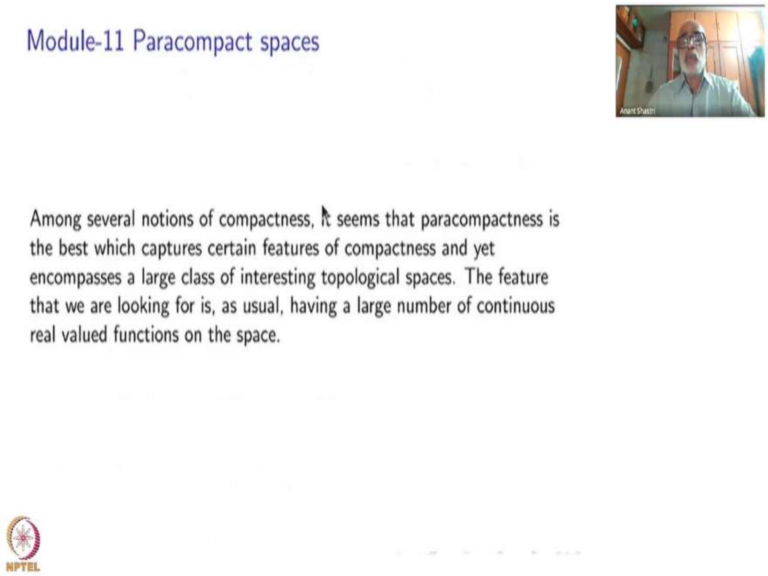

Hello, welcome to module 11 of NPTEL NOC course on point set topology part II. So, today we should pick up another important topic, paracompact spaces. Among several notions of compactness, some of them we are going to study, It seems that paracompactness is the best, which captures certain features of compactness and yet, encompasses a large class of interesting topological spaces.

The feature that we are looking for is as usual, having a large number of continuous real valued functions on a space. So, this is one of the interesting properties of paracompactness that we will prove.

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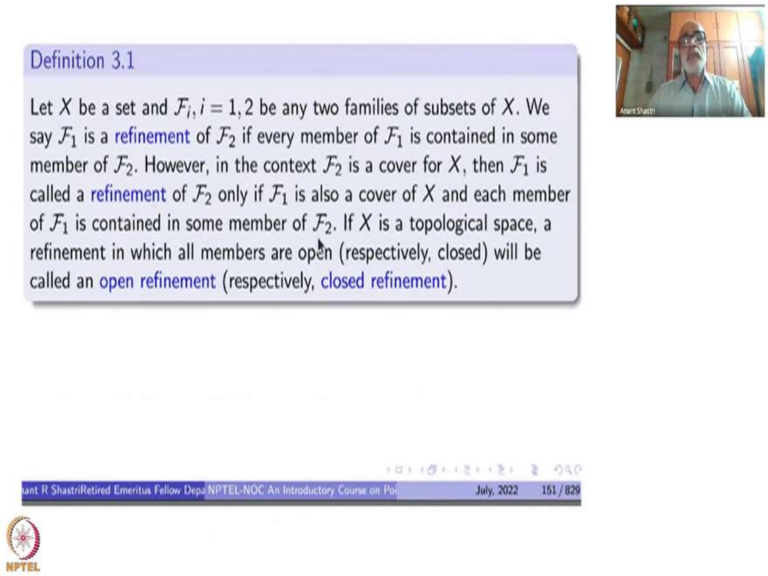

So, let us begin with a formal definition or two. Take any set X and two families of subsets of X, $\mathcal{F}_1$, and $\mathcal{F}_2$. This $\mathcal{F}_1$ is called a refinement of $\mathcal{F}_2$, if every member of $\mathcal{F}_1$ is contained in some member of $\mathcal{F}_2$. Pay attention to this one. I am not saying that the family $\mathcal{F}_1$ is a subfamily of $\mathcal{F}_2$. Not necessarily.

Sometimes that is also possible. Actually, if $\mathcal{F}_1$ is a subfamily of $\mathcal{F}_2$, then this condition fro refinement is obvious because you can take any member here in $\mathcal{F}_1$, the same member will be there in $\mathcal{F}_2$, which you can take the same member which will obviously contain itself. So, refinement is much better notion than just taking subfamilies.

Actually, it comes from the practice of giving a subdivision of a division of an interval $[0, 1]$ or any interval $[a, b]$, in real analysis. The collection of all smaller intervals in a subdivision is a refinement of the collection of all subintervals of the given division. The word `refinement' is sometimes used in Analysis also.

However, in the context of $\mathcal{F}_2$ is a cover for X, which just means that members of $\mathcal{F}_2$ together contain all the points of X, i.e., union of members of $\mathcal{F}_2$ is X, in that context, we will use the same word `refinement' with a slightly stronger meaning, namely, $\mathcal{F}_1$ is a refinement of $\mathcal{F}_2$, as before and $\mathcal{F}_2$ and if $\mathcal{F}_1$ is also a cover of X.

If X is a topological space, a refinement in which all members are open, (respectively, closed), will be called an open refinement (or a closed refinement) accordingly. So far, we have actually made four different definitions here.

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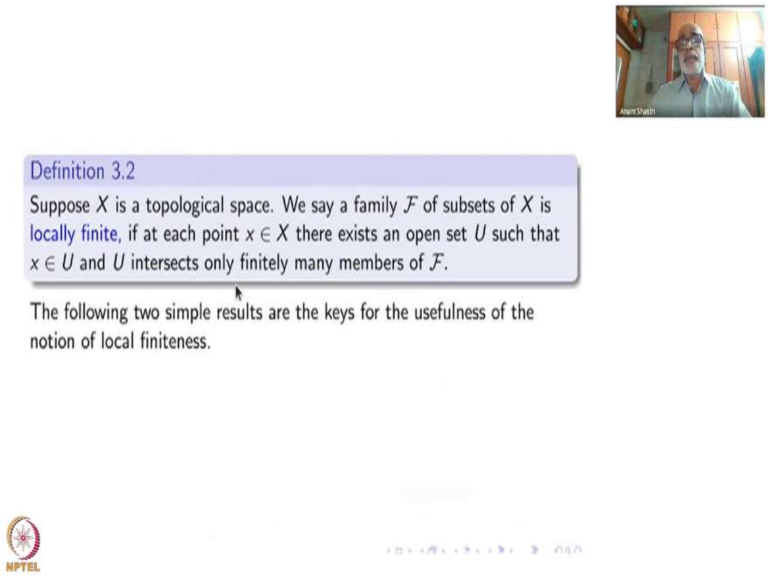

Suppose X is a topological space. We say a family $\mathcal{F}$ of subsets of X is locally finite if at each point x in X, there exists in a neighbour U of x such that U intersects only finitely many members of $\mathcal{F}$. So, this is not the property of the space X. It is just the property of the family of subsets $\mathcal{F}$.

Similarly, we will have another definition needed later on. $\mathcal{F}$ is called point-finite a given point of X can be found at most in finitely many members of $\mathcal{F}$. Clearly, locally finite implies point finite, but not the converse. Anyway, I will recall the definition of point finiteness when it is necessary. The following two simple results are the keys for usefulness of this local finiteness concept.

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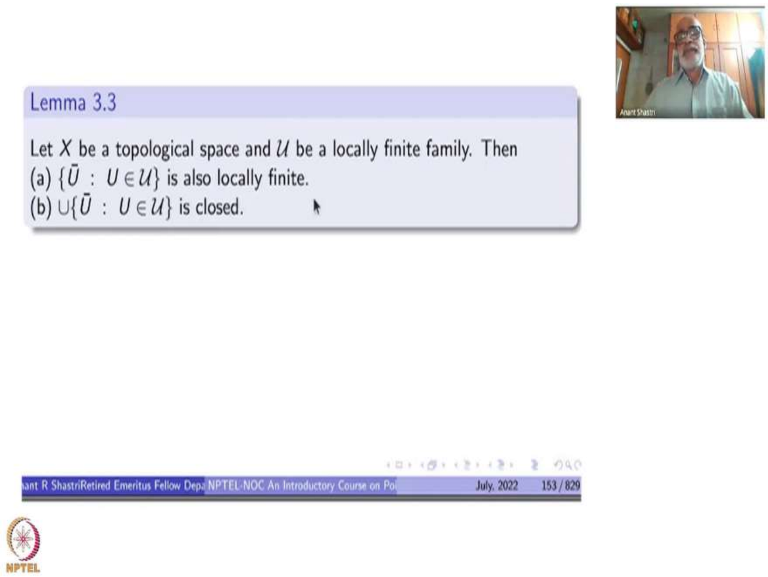

So, these are the two things here. Namely, take a family $\mathcal{U}$ of subsets of X, which is locally finite. Then the family of closures of members of $\mathcal{U}$ is also locally finite. The second thing is that the union of all these closures of members of $\mathcal{U}$ is a closed set.

Remember that if you have a finite union of closed sets, then the union as also closed. But if you take an infinite union, then this will not hold, in general. So, here is the case where this happens. So, it is in particular, it follows that the closure of the union of all members of $\mathcal{U}$ is equal to the union of the closures of members of $\mathcal{U}$. So, that is because of local finiteness.

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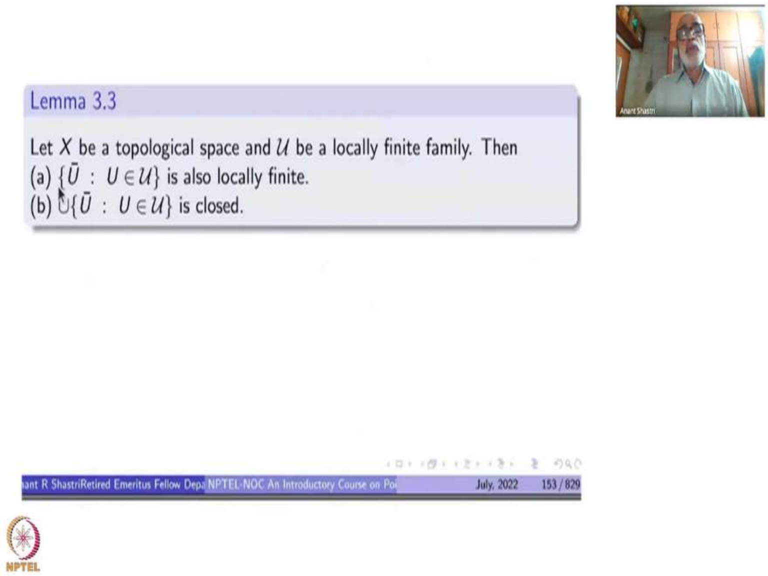

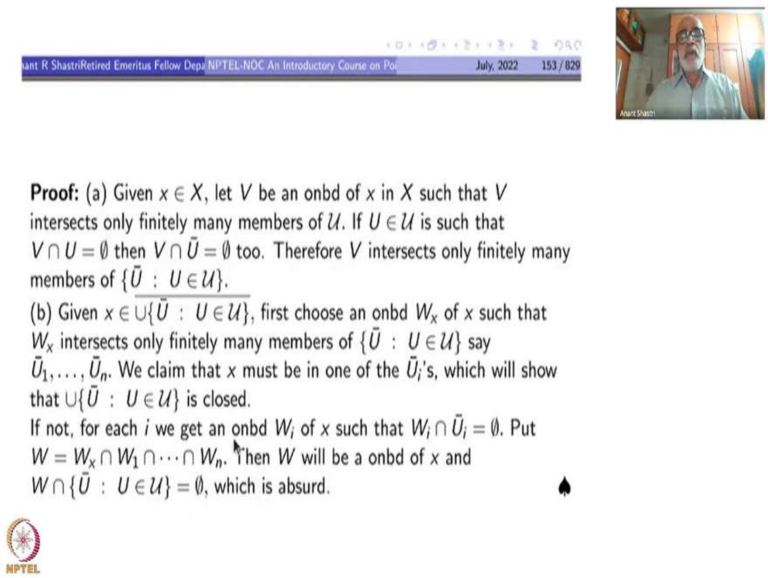

So, let us have a look at how this works. In (a), I want to show that the family $\{\bar{U} : U \in \mathcal{U}\}$ is also locally finite. Given $x \in X$, let V be a neighbourhood of x such that Vintersects only finitely many members of $\mathcal{U}$. Now if U is a member in $\mathcal{U}$ such that $V \cap U$ is empty, then

$V \cap \bar{U}$ is also empty, because V is an open subset. Therefore, V intersects closures of only finitely many members, in fact precisely the same members. Proof of (a) is over.

Now part (b). This is slightly more complicated. Actually, this we have seen in part I of the course but never mind, I will repeat it. Take a point in the closure of this union and we must show that it is in the closure of one of the members. That will prove that this union is closed.

So, first of all, choose an open neighbourhood W_x of x such that W_x intersect only finitely many members of this family say, $\bar{U_1}, \bar{U_2}, \ldots, \bar{U_n}$. We claim that X must be in one of these $\bar{U_1}, \bar{U_2}, \ldots, \bar{U_n}$. That will be enough.

Suppose, this is not the case. Then what happens? For each i, we would get a neighbourhood W_i of x, such as that, $W_i \cap \bar{U_i}$ must be empty. because x is not in $\bar{U_i}$ which is a closed set already. (I could have taken complement of this $\bar{U_i}$ for $\bar{U_i}$, that is all.) Now you take W equal to the intersection of W_i's along with the original W_x. That is a neighbourhood of x. Now it follows that $W \cap \bar{U}$ is empty for all $\bar{U} \in \mathcal{U}$, which just means that point x is not in the closure of this union. So that is absurd, because you started with the point x in the closure.

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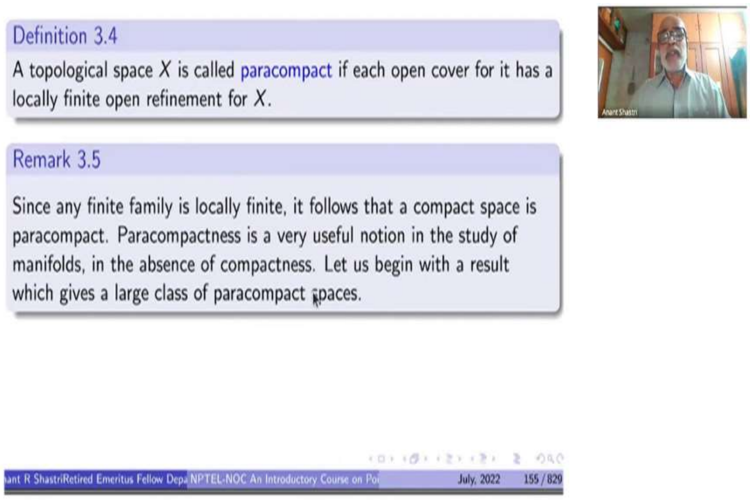

Now, let us make a definition, the one we were aiming at. A space is called paracompact if each open cover for it has a locally finite open refinement. I want to recall each of these terms once again. You have an open cover for X. Let us call this $\mathcal{F}_2$. $\mathcal{F}_1$ is a refinement of $\mathcal{F}_2$ means, first of all, that each member of $\mathcal{F}_1$ is contained in a member of $\mathcal{F}_2$ and $\mathcal{F}_1$ itself is a cover for X. All members of $\mathcal{F}_1$ are open as well. Finally, $\mathcal{F}_1$ must be locally finite. i.e., each point of X, there must be an open neighbourhood which will intersect at the most finitely many members of this refinement.

For each open cover of X, if you have a locally finite open refinement, such a space will be called paracompact. It is somewhat similar to the definition of compactness, wherein each open cover has a finite subcover. We are not insisting on a finite subcover here. Finiteness is replaced by local finiteness, which is much more general. Not only that, we are not insisting upon that this is a subfamily of the of the original cover. It is a refinement. So, you may have many more open sets here, but each new member will be smaller than some old member. So, that is the beauty of this definition.

Since any finite family is automatically locally finite, it follows that a compact space is automatically paracompact. You start with any open cover, first take a finite sub cover. That will be refinement already. Now you take a refinement. You do not have to because any sub cover is a refinement. You do not have to worry about local finiteness because this is already finite, so automatically it is finite also. So, compact spaces are paracompact. Paracompactness is very useful, really useful in the study of manifolds, in the absence of compactness.

In fact, it is also useful in the study of spaces other than manifolds, which are called simplicial sets, simplicial complexes, and what are called CW complexes and so on. So, I cannot dwell much into those objects. Those are the things which you study in algebraic topology.

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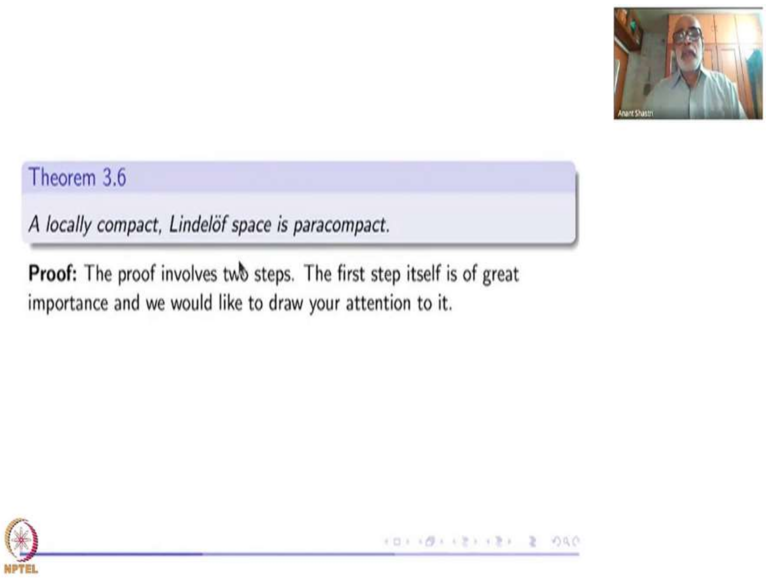

A locally compact, Lindelof space is paracompact. So, we are now up to what are the kind of spaces which will become paracompact, what kind of conditions we can impose to ensure paracompactness. Of course, definition is there finally, but how to verify whether some given sapce is paracompact or not?

So, suppose something is locally compact and Lindelof, that is paracompact. So, this is the second result of this kind here, the easy one was compact implies paracompact. The proof of this result itself is very illuminating. So, you should pay attention to that.

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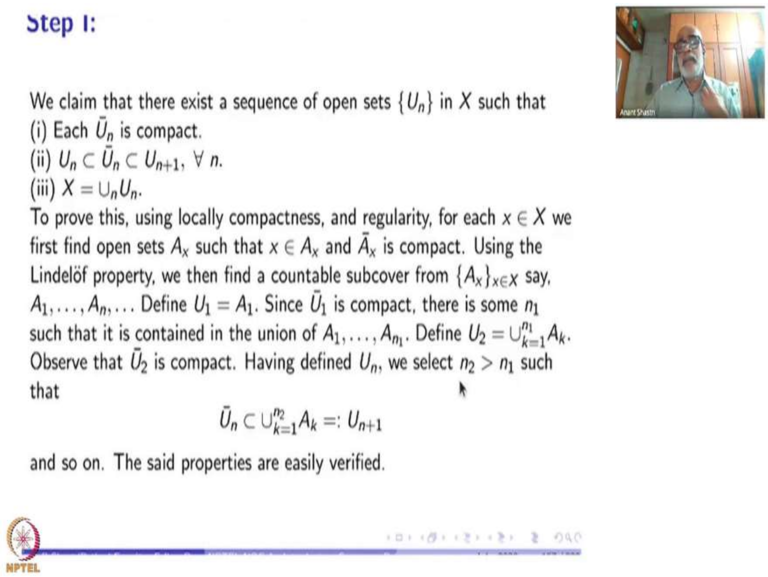

I am going to prove the result in several steps. Each step itself is worth remembering. Each step itself is a kind of property. First step is that there is sequence $\{U_n\}$ of open sets in X such that each $\overline{U_n}$ is compact, U_n is contained inside $\overline{U_n}$ contained inside U_{n+1} etc., and X is the union of all U_n's. So, every locally compact Lindelof space can be written as an increasing union of open sets such that closures these open sets are compact.

So, that is what we are going to prove. Using local compactness and hence regularity for each x belonging to X, we first find open neighbourhood A_x of x such that such that $\overline{A_x}$ is compact. Next, using the Lindelof property, we find a countable subcover from $\{A_x, x \in X\}$. Every open cover of a Lindelof space admits a countable subcover. Let us rewrite this countable subcover merely by $\{A_1, A_2, \ldots, A_n, \ldots\}$. We do not worry about which A_i comes from what point of X. We are only worried about this countable sequence of open sets whose closures are compact and their union covers X. That is what we are worried about.

So, you see part of that has already come. But now we have to arrange them so that the entire thing becomes an increasing sequence. This kind of argument is quite useful in measure theory. You start with one of them, any one of them. So, take U_1 equal to A_1. Since $\overline{U_1}$ is compact, and all these A_n is together cover this $\overline{U_1}$.

So, some finitely many of them will cover it. So, there is some integer n_1, such that $\overline{U_1}$ is contained in the union of $A_1, A2, \ldots, A_{n_1}$. Take U_2 to be the union of these $A_i, i = 1, \ldots, n_1$. Each A_k is open, remember? So, U_2 is an open set. It contains this contains $\overline{U_1}$. And if you take the closure of U_2 that will be equal to the union of the closures of A_k's, $k = 1, 2, \ldots, n_1$.

Being a union of finitely many compact sets, $\overline{U_2}$ is compact. So, what we have done is setting up an iteration process here. U_1 has been fattened to $\overline{U_1}$, which is fattened further to U_2 so that $\overline{U_2}$ is compact. Now we will repeat this. Repeat the same procedure for $\overline{U_2}$. Again, $\overline{U_2}$ is contained inside the union of $A_1, A2, \ldots, A_{n_2}$.

So, there will be some finite number n_2. Obviously that finite number can be taken bigger than 2. In some funny cases, it may happen that you get an open cover consisting of no new members at all, since our original labeling does not have any such properties. But you can always take successive n_i to be larger than i. Nobody stops you from that. That n_1 bigger than $1, n_2$ bigger that 2 and so on.

Then given any point $x \in X$, say it is in A_k. It follows that x belong to U_k. Therefore union of all the U_n's is equal to X.

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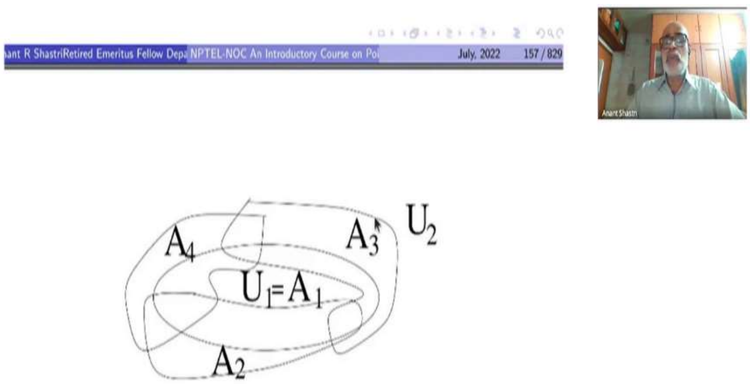

Figure 2: X is the increasing union of countably many compact sets

So, here is the picture how things are going on. You started with A_1 equal U_1. The closure of U_1 is represented by the set along with its boundary curve. A_2, A_3 and A_4 are covering the boundary that is $\overline{U_1}$. Other A_i may also be there but you do not worry abut them. Now you take the union of A_1, A_2, A_3 and A_4 as your U_2. Now look at the closure of U_2 which will be compact. So, again, some $A_4, A5, A6$, etc, will come, which will cover $\overline{U_2}$ and go on. Ultimately, all the A_i will be taken care.

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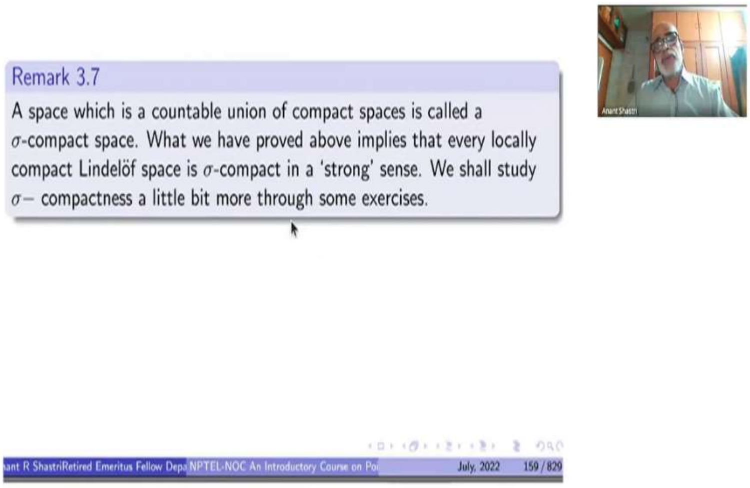

So, here is a remark before we go further. A space which is a countable union of compact spaces is called a sigma-compact space. What we have proved above implies that every locally compact Lindelof space is sigma-compact in a strong sense. What is the strong sense? That these compacts subsets are actually closures of open subsets and the open parts themselves cover the whole of X.

So, that is the strongness here. So, we shall study sigma compactness a little more, through some exercises later on. This is also an important concept, but mostly it is used in analysis.

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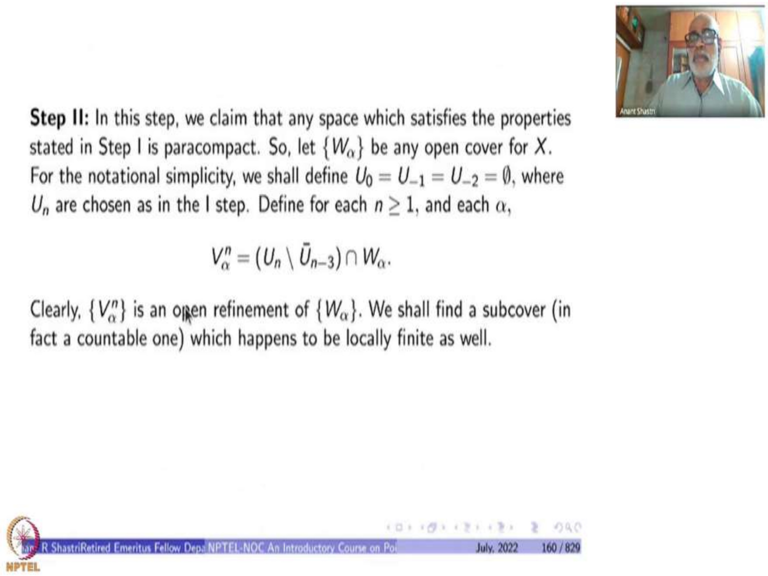

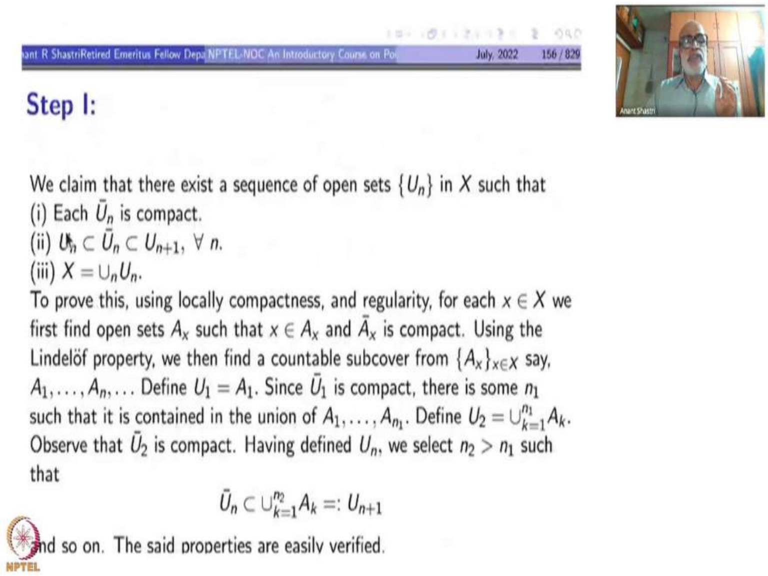

Now let us go to the step 2 toward proving that the space is paracompact. So, in this step, we claim that any space which satisfies the property stated in step 1 is paracompact. So, now we are not going to use local compactness or Lindelofness separately. We will just use the properties (i), (ii) and (iii) here of the sequence of open sets $\{U_n\}$. With such a family of open sets in X, we are to prove that the space X is paracompact.

So, here is again, another typical step which we have to learn. Start with a family $\{W_\alpha\}$, any open cover for X. We shall extract a locally finite open refinement for this. For notational simplicity, let us also define U_0, U_{-1} and U_{-2} to be empty set. See we have taken the countable family $\{U_n\}$ indexed from $n = 1$ onward. Without any harm, we extend this family with three more members, all empty sets, so that you can write down the inductive step very easily. That is the whole idea.

Define for each $n \geq 1$, and each alpha, look at $U_n \setminus \overline{U_{n-3}}$. Throw away this closed subset from the open subset U_n, that is still an open set. See, I want this to work from $n = 1$, and that is why I have introduced these harmless members U_0, U_{-1} and U_{-2} all empty sets. In the case of$n = 1$, $U_1 \setminus \overline{U_{n-3}}$ makes sense, but it is actually U_1, because this part is empty, does not matter.

What I insist is that this is an open subset, this is an open subset set, so, its intersection with W_α, denoted by V_α^n is an open subset. So, this is now defined for each α and each positive integer n. Clearly the family $\{V_\alpha^n\}$ this is an open refinement of $\{W_\alpha\}$. So, what are the things that we have to verify? W_α are open and so are V_α^n. That is fine. Each V_α^n is contained in W_α. That is also fine.

These themselves cover the whole X, why, because whatever point x you take, it is of all in some W_α. After that, it must be also in one of the U_n's. You take the first U_n for which, this happens. Possible because U_n's are increasing sequence of open sets, covering the whole of X . Then x will be in V_α^n. So, it is an open refinement. We shall now find a countable subcover of this, which happens to be locally finite as well.

So, this itself may not be locally finite. So, I will pass onto a subcover. Subcover means what? It is a cover, but only a few members will be taken. How many? Actually we will take countably many members from here. (Refer Slide Time: 26:59)

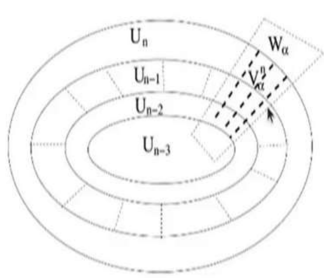

Figure 3: σ-compactness implies paracompactness

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So, this is what, the picture here says. This rectangular thing represent W_α. This is the increasing sequence of our open subsets U_n's which cover the whole of X. These W_α's also cover the whole of X. So, what I am doing here is $U_n \setminus \overline{U_{n-3}}$. The closure of U_{n-3} is being thrown away.

All these happening only inside W_α. So, you look at only this portion and subtract this portion. This heavily dotted bar is V_α^n. As alpha varies, you will get a lot points from X. After that, if you vary n as well, you will get the entire of the space X. So, that is how V_α^n's have been constructed. Now, some countable subfamily of this is going to be locally finite as well as a cover. So, that is what we have to find out.

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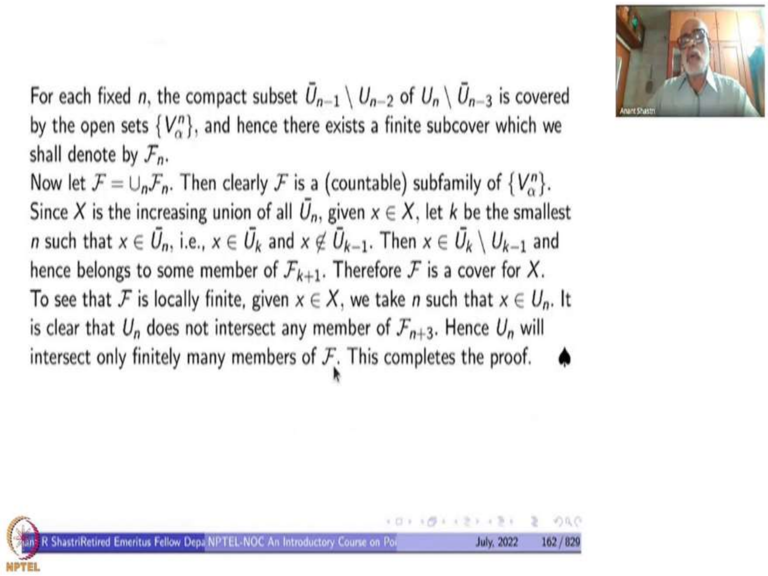

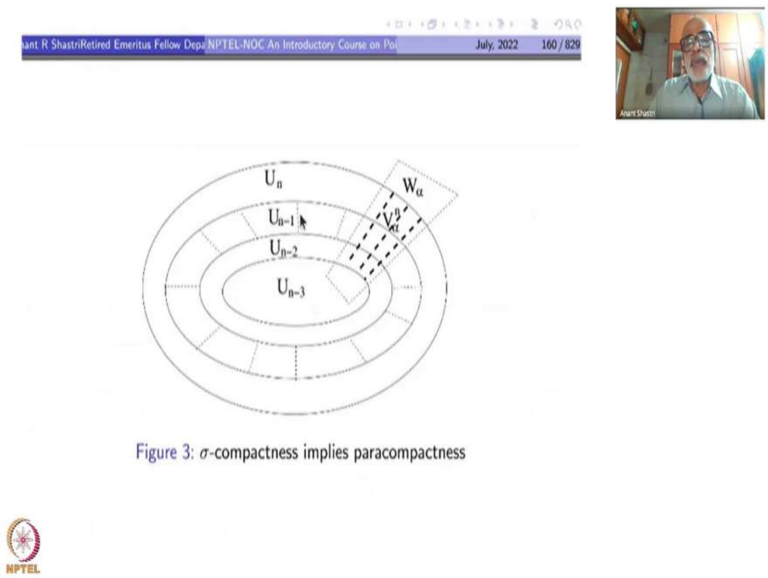

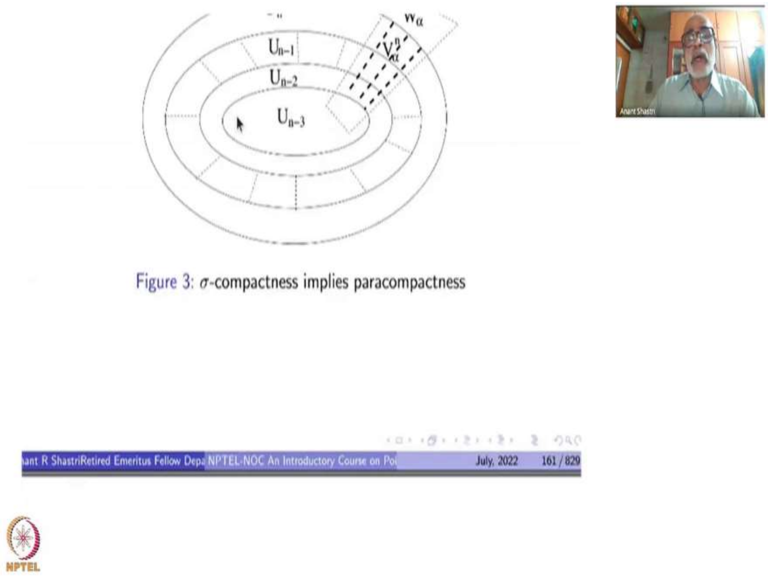

Figure 3: σ-compactness implies paracompactness

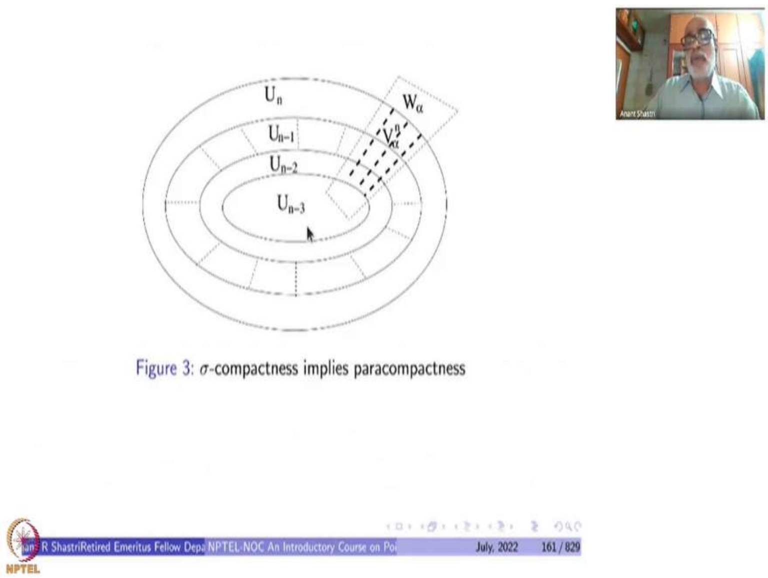

Figure 3: σ-compactness implies paracompactness

For each fixed n, look at the compact subset $\overline{U_{n-1}} \setminus U_{n-2}$ of $U_n \setminus \overline{U_{n-3}}$. Here I am not intersecting it with W_α. Look at these concentric ellipses here. Inside each open set I have a compact subset. It is covered by the family $\{V_\alpha^n\}$, where n is fixed. Therefore, we will have a finite subfamily of them covering this part, call that finite family $\mathcal{F}_n$.

What is the purpose of this one? Members of $\mathcal{F}_n$, you see, they will not intersect $\overline{U_{n-3}}$ at all. They are contained somewhere here somewhere like this. The construction is over. All that I do now is put $\mathcal{F}$ equal to union of all $\mathcal{F}_n$'s. Each $\mathcal{F}_n$ has finitely many members. So, the union will be countable. Each member of $\mathcal{F}_n$ is a member of $\{V_\alpha^n\}$. So, $\mathcal{F}$ is a subfamily of $\{V_\alpha^n\}$ and hence a refinement also.

Since X is an increasing union of $\overline{U_n}$, given any $x \in X$, let k be the smallest n such that X is in $\overline{U_n}$, which means that X is in $\overline{U_k}$, but not in $\overline{U_{k-1}}$. So, that is the smallest k. Then x will be inside $\overline{U_k} \setminus U_{k-1}$, therefore belongs to some member of F_{k+1}. Therefore, every member of X is inside some member of $\mathcal{F}$. So this $\mathcal{F}$ is a cover for X.

Finally, why $\mathcal{F}$ is locally finite? Given any $x \in X$, we take n such that x is inside U_n. It is clear that U_n does not intersect any member of F_{n+3}. So, in this picture, it is the other way around. If you take something here in U_{n+3}, then members of $\mathcal{F}_n$ will contain that element. So, when the index n is high enough, the point will not be in $\mathcal{F}_n$. U_n may intersect only members of $F_k, k = 1, 2, \ldots, n+3$. How many members are there? Finitely many members. So, the family $\{U_n\}$ itself will serves the purpose of local finiteness. Therefore, $\mathcal{F}$ is locally finite. So, that completes the proof of this theorem.

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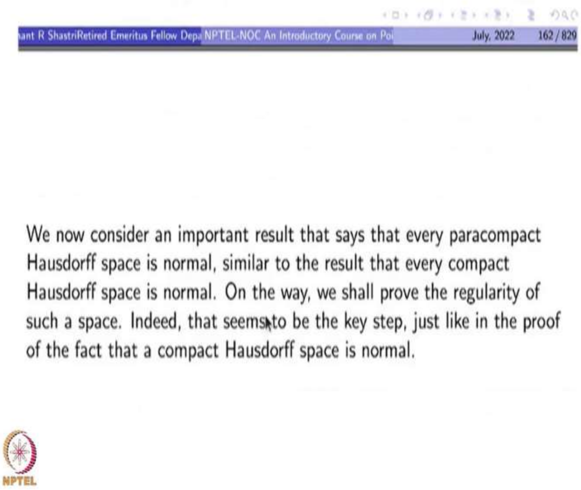

We now consider an important result that says that every paracompact Hausdorff space is normal. Similar to the result that every compact Hausdorff space is normal. On the way, we shall prove that it is regular also. So, the proof is somewhat similar to what we have seen earlier, which seems to be the key step to prove normality. We first have regularity here, just like in the proof of the fact that compact Hausdorff space is normal, which we have proved a couple of days before.

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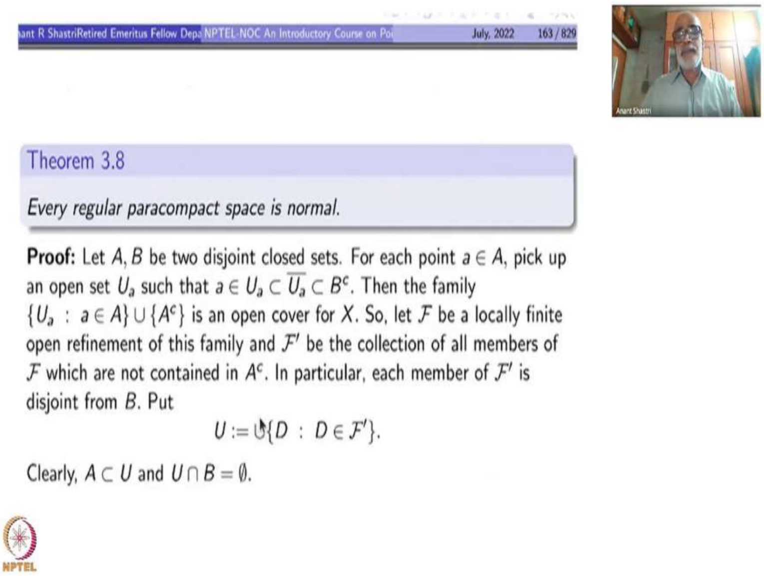

So, let us go through the proof of this one. Every regular paracompact space is normal. Start with any two disjoint closed sets A, B. Pick up a point $a \in A$, pick up an open neighbourhood U_a of a such that $\overline{U_a}$ contained in the compliment of B which is an open set. Then the family U_a such that a belongs to A along with A^c is an open cover for X. So, we have just used regularity here.

A and B are disjoint closed sets. Now we have got an open cover for A. Now we use paracompactness. So, let $\mathcal{F}$ be a locally finite open refinement of this family and let $\mathcal{F}'$ be the collection of all members of $\mathcal{F}$, which are not contained in A^c. Being a refinement, every member of $\mathcal{F}$ will belong to $\mathcal{F}'$ or is contained in A^c. I do not want those which are contained in A^c.

I want to concentrate on this part. That is $\mathcal{F}'$. In particular, each member of $\mathcal{F}'$ is disjoint from B, because it is not contained in some $\overline{U_a}$ which is contained in B^c. So, all these opens subsets, which are in $\mathcal{F}'$, none of them intersects B. So, if U is equal to the union of members of $\mathcal{F}'$, then U will not intersect B and of course, U contains A.

So, A is contained in U and $U \cap B$ is empty. I repeat here, look at members of $\mathcal{F}'$. They are contained in this part. And U_a is in the complement of B, so, that is why they do not intersect B that is all. So, we have got one part, namely, an open subset around A, which does not intersect B.

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Note that $\mathcal{F}'$ is locally finite. Therefore for each $b \in B$, there is an onbd V_b of b such that only finitely many members, say, $D_1, \dots, D_n$ belonging to $\mathcal{F}'$ will intersect V_b. Let $D_i \subset U_{a_i}$. Consider

$$W_b = (\cap_{i=1}^n \overline{U}_{a_i}^c) \cap V_b.$$

Then W_b is a nbd of b and is disjoint from U.

Every regular paracompact space is normal.

Proof: Let A, B be two disjoint closed sets. For each point $a \in A$, pick up an open set U_a such that $a \in U_a \subset \overline{U_a} \subset B^c$. Then the family $\{U_a : a \in A\} \cup \{A^c\}$ is an open cover for X. So, let $\mathcal{F}$ be a locally finite open refinement of this family and $\mathcal{F}'$ be the collection of all members of $\mathcal{F}$ which are not contained in A^c. In particular, each member of $\mathcal{F}'$ is disjoint from B. Put

$$U := \cup\{D : D \in \mathcal{F}'\}.$$

Clearly, $A \subset U$ and $U \cap B = \emptyset$.

Theorem 3.8

Every regular paracompact space is normal.

Proof: Let A, B be two disjoint closed sets. For each point $a \in A$, pick up an open set U_a such that $a \in U_a \subset \overline{U_a} \subset B^c$. Then the family $\{U_a : a \in A\} \cup \{A^c\}$ is an open cover for X. So, let $\mathcal{F}$ be a locally finite open refinement of this family and $\mathcal{F}'$ be the collection of all members of $\mathcal{F}$ which are not contained in A^c. In particular, each member of $\mathcal{F}'$ is disjoint from B. Put

$$U := \cup\{D : D \in \mathcal{F}'\}.$$

Clearly, $A \subset U$ and $U \cap B = \emptyset$.

Now, look at $\mathcal{F}'$. It is locally finite, because it is a subfamily of $\mathcal{F}$ which is locally finite. Therefore, for each $b \in B$, there is an open neighbourhood, V_b of B such that only finitely many members let us say $D_1, D_2, \ldots, D_n$ belonging to this $\mathcal{F}'$ will intersect V_b. So, look at these D_i's, since they are members of $\mathcal{F}'$, there are $a_i \in A$ such that D_i is contained in U_{a_i}.

Now you look at W_b equal to the intersection of this V_b with the complement of the closures of $U_{a_i}, i = 1, 2, \ldots, n$. But if you take finite intersection of the complements of the closures, that will the open subset, So W_b is open.

So, W_b is neighbourhood of b for each $b \in B$ and disjoint from U. So, let me prove this one. Why this is a disjoint from U? Once you have done that we have finished the proof roughly by taking V equal to union of all W_b', we get U and V will be open subsets which are disjoint.

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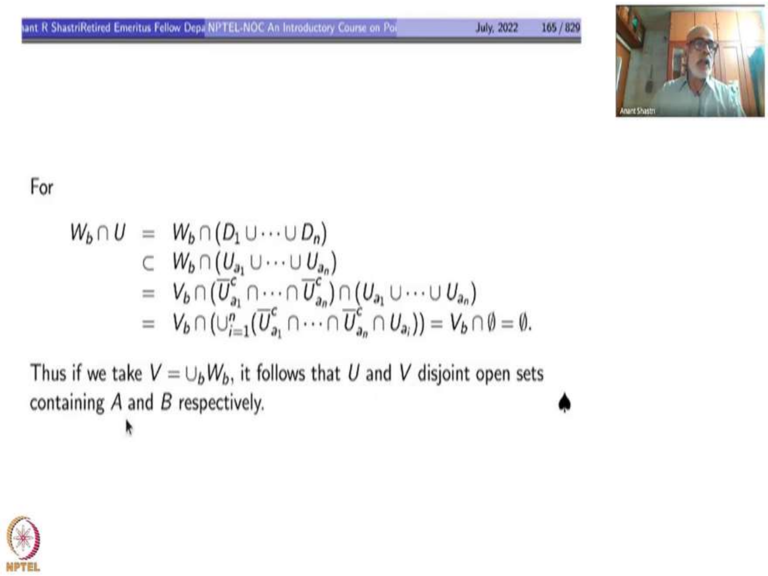

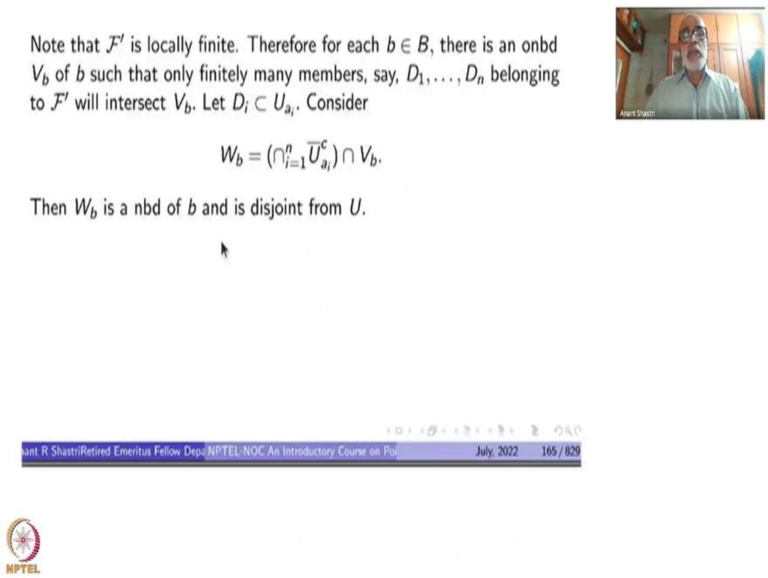

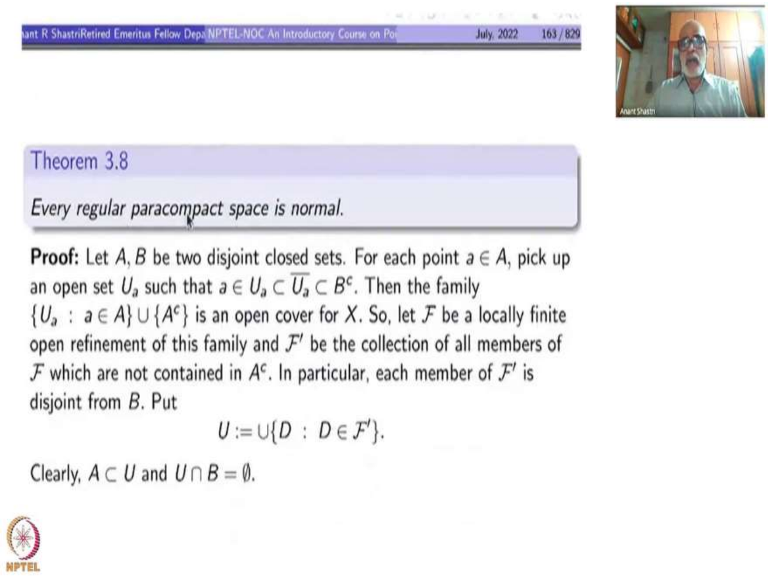

So, let us see why $W_b \cap U$ is empty. $W_b \cap U$ is $W_b \cap (D_1 \cup D_2 \cup \cdots \cup D_n)$. That is contained in $W_b \cap (U_{a_1} \cup \cdots \cup U_{a_n})$. But W_b is nothing but $V_b \cap (\overline{U_{a_1}}^c \cdots \cup \overline{U_{a_n}}^c)$. See the brackets are important. Here, I am taking unions here, this intersection here, here I am taking intersection, this is also intersection. The intersection of three sets. But the second set itself is a finite intersection and the third one is a union. Therefore, it is equal to the intersection of V_b with the union over i ranging from 1 to n, of the second set intersected with U_{a_i}. Since the second set intersected with any U_{a_i} is empty, we get the whole thing equal to V_b intersection with the empty set.

So, we have found out two disjoint open subset U and V respectively containing A and B. That completes the proof that, a regular paracompact space is normal.

Added by the reviewer: Alternatively, you can directly take V equal to the complement of the closure of the union of members of $\mathcal{F}'$. Since, $\mathcal{F}'$ is locally finite, we have closure of U is the same as the union of closure of D where D ranges over $\mathcal{F}'$. Since each such D is contained in some U_a, closure of D is contained in closure of U_a and hence is disjoint from B. Therefore, their union is also disjoint from B. That means V contains B. Clearly, V is disjoint from U.

The next step will be there to prove that Hasudorff paracompact space is regular.

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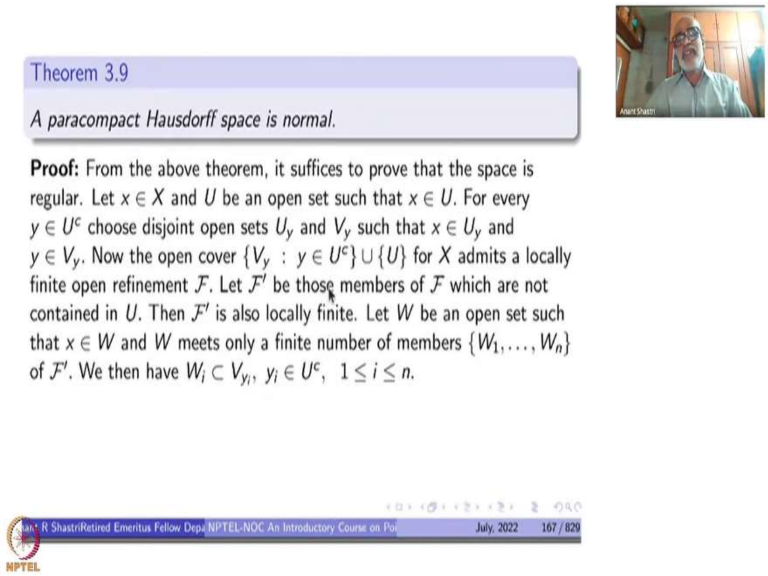

So, that is the next step. A Hasudorff paracompact is normal. What we have to do? You have to just prove regularity and use the previous theorem. So, let x belong to X, U be an open subset such that x is inside U. For every y in the complement of U, choose a disjoint open sets U_y and V_y such that x is inside U_y and y is inside V_y, x and y are distinct and then I can do this because X is a Hasudorff space.

Now, as we vary y over the compliment of U, these V_y's will cover the compliment of U. Along with U, that will be an open cover for X, say $\mathcal{F}$. So, this admits a locally finite open refinement say $\mathcal{F}'$. That is where paracompactness of X is used. Again, as in the previous step, take only those members of $\mathcal{F}'$ which are not contained in U here. They may be coming from here, they have to come. If it is not coming from here, it must be a subset of some member here. Obviously, this being a sub family, it is also locally finite. Take W be an open set such that x is inside W and W meets only finitely many members of $\mathcal{F}'$, which we will label as $W_1, \ldots, W_n$. Each W_i will be contained in some V_{y_i}, V_{y_i} are subsets of U^c. See, these steps are identical, almost identical as in the proof of a previous theorem. At least the idea is identical.

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Put

$$N = W \cap U_{y_1} \cap \cdots \cap U_{y_n}.$$

Clearly N is an open set containing x. We claim $\overline{N} \subset U$. (This will complete the proof.) Let $V = \cup\{A \,:\, A \in \mathcal{F}'\}$. Then V is an open set and $U^c \subset V$. Therefore, it is enough to show that $N \cap V = \emptyset$. Now

$$\begin{aligned} N \cap V &= W \cap U_{y_1} \cap \cdots \cap U_{y_n} \cap V \\ &= W \cap U_{y_1} \cap \cdots \cap U_{y_n} \cap (W_1 \cup \cdots \cup W_n) \\ &\subset W \cap U_{y_1} \cap \cdots \cap U_{y_n} \cap (\cup_{i=1}^n V_{y_i}) \\ &= W \cap (\cup_{i=1}^n (U_{y_1} \cap \cdots \cap U_{y_n} \cap V_{y_i})) = W \cap \emptyset = \emptyset. \end{aligned}$$

This completes the proof. ♠

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finite open refinement $\mathcal{F}$. Let $\mathcal{F}'$ be those members of $\mathcal{F}$ which are not contained in U. Then $\mathcal{F}'$ is also locally finite. Let W be an open set such that $x \in W$ and W meets only a finite number of members $\{W_1, \ldots, W_n\}$ of $\mathcal{F}'$. We then have $W_i \subset V_{y_i}$, $y_i \in U^c$, $1 \le i \le n$.

ant R ShastriRetired Emeritus Fellow Depa NPTEL-NOC An Introductory Course on Poi July, 2022 167 / 829

Put

$$N = W \cap U_{y_1} \cap \cdots \cap U_{y_n}.$$

Clearly N is an open set containing x. We claim $\overline{N} \subset U$. (This will complete the proof.) Let $V = \cup\{A \,:\, A \in \mathcal{F}'\}$. Then V is an open set and $U^c \subset V$. Therefore, it is enough to show that $N \cap V = \emptyset$. Now

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Theorem 3.9

A paracompact Hausdorff space is normal.

Proof: From the above theorem, it suffices to prove that the space is regular. Let $x \in X$ and U be an open set such that $x \in U$. For every $y \in U^c$ choose disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Now the open cover $\{V_y \,:\, y \in U^c\} \cup \{U\}$ for X admits a locally

finite open refinement $\mathcal{F}$. Let $\mathcal{F}'$ be those members of $\mathcal{F}$ which are not contained in U. Then $\mathcal{F}'$ is also locally finite. Let W be an open set such that $x \in W$ and W meets only a finite number of members $\{W_1, \ldots, W_n\}$ of $\mathcal{F}'$. We then have $W_i \subset V_{y_i}$, $y_i \in U^c$, $1 \le i \le n$.

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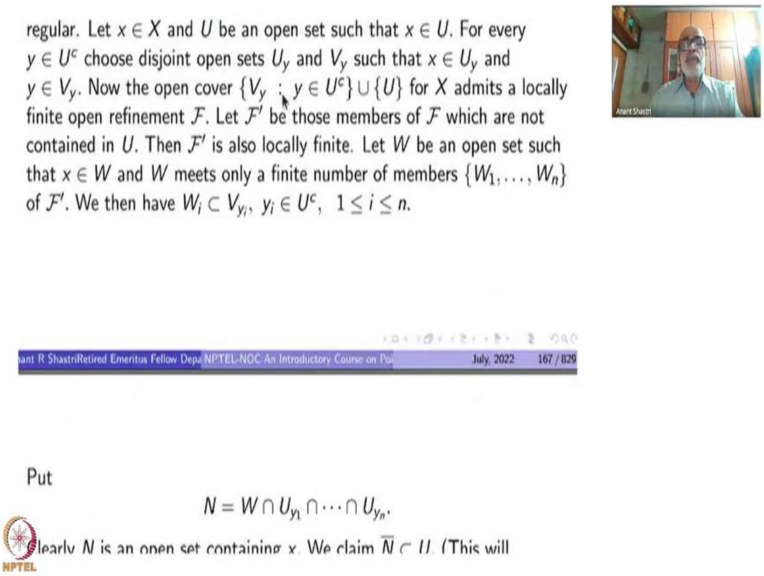

Now take N to be intersection of W with $U_{y_1}, \dots, U_{y_n}$, where y_i are chosen as above. U_{y_i} are disjoined from corresponding V_{y_i}. I do not have to take the complement and closures and complements and so on here they are readymade open subsets which are disjoint from that one. Clearly, this N is an open set containing x, because each U_{y_i}'s are all neighbourhoods of x. We claim that the closure of N is inside U. That will complete the proof of regularity of X. Given $x \in U$, you and found N such that x is in N and closuer of N contained in U. That is the regularity.

So, let V which union of all A, where A ranges over $\mathcal{F}'$. This V is an open set and U^c will be contained inside V, because together with U^c, they cover the whole of X. After all, we started with $\mathcal{F}$ as a covering of X.

Therefore, it is enough to show that $N \cap V$ itself is empty. Since V is an open subset, this will imply that $\bar{N} \cap V$ is empty. So, $\bar{N}$ will be contained inside U. So, look at $N \cap V$. Same argument as in the proof that $W_b \cap U$ is empty, yields that this set is empty.

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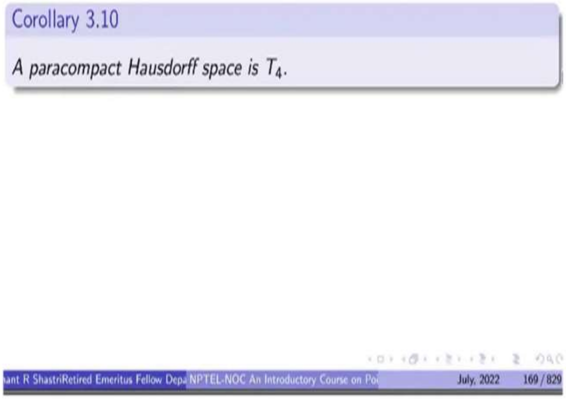

Module-12 Partition of Unity

So, in conclusion, paracompact Hasudorff space is normal. Plus T_1 is already there, so, it is T_4. So, we have done some major topological properties of paracompact spaces. The next thing is functional properties. Namely what are called partition of unity. Partition of unity is one of the major purpose of introducing paracompactness by the great mathematician, Dieodonne. So, thank you. Let us, do this partition of unity next time.