An Introduction to Point-Set-Topology – Part II
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Lecture 07
Local Compactness

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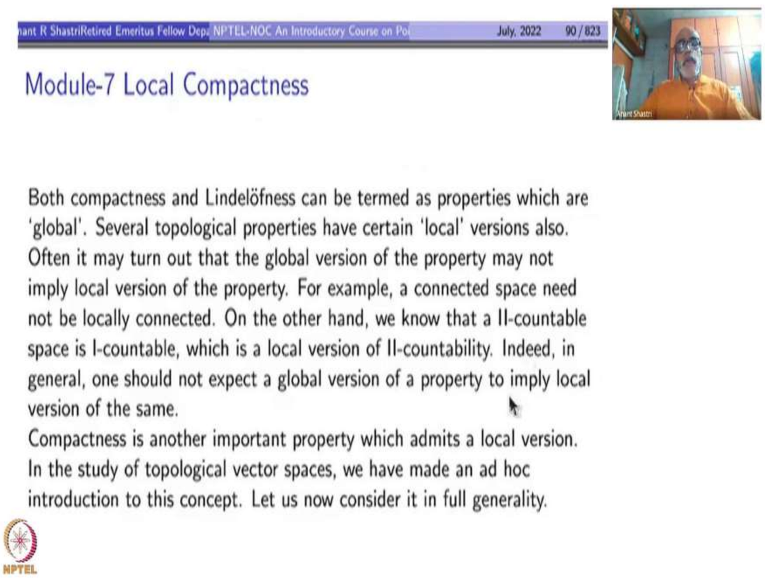

Hello, welcome to NPTEL NOC an introductory course on Point Set Topology part II. So, today we will do local compactness. Both compactness and Lindelofness can be termed as properties which are global. Several topological properties have certain local versions also. Often it may turn out that global version of a property may not imply local version of the same property.

For example, we have studied connectedness and also locally connectedness. Connectedness need not imply local connectedness. On the other hand, you know the first countability can be thought of as the local version of second countability, but second countability implies first countability. So, do not jump to the conclusion either way. That is all I want to say. In general, one should not expect a global version of the property to imply a local version of the same.

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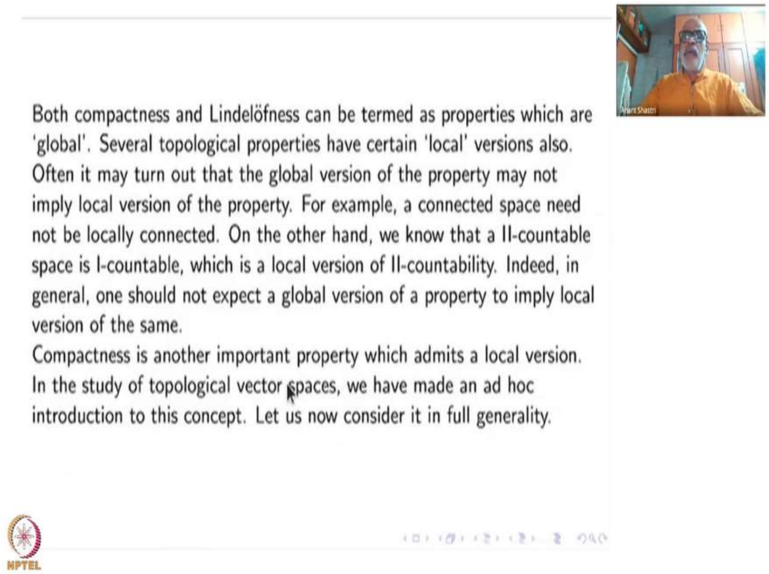

Compactness is another important property which admits a local version. In the study of topological vector spaces, we have already made an ad-hoc introduction to this concept in part one. Let us now consider it in full generality. So, whatever we did in the study of topological vector spaces that was an ad hoc thing. That will be recovered automatically, we are not abandoning it.

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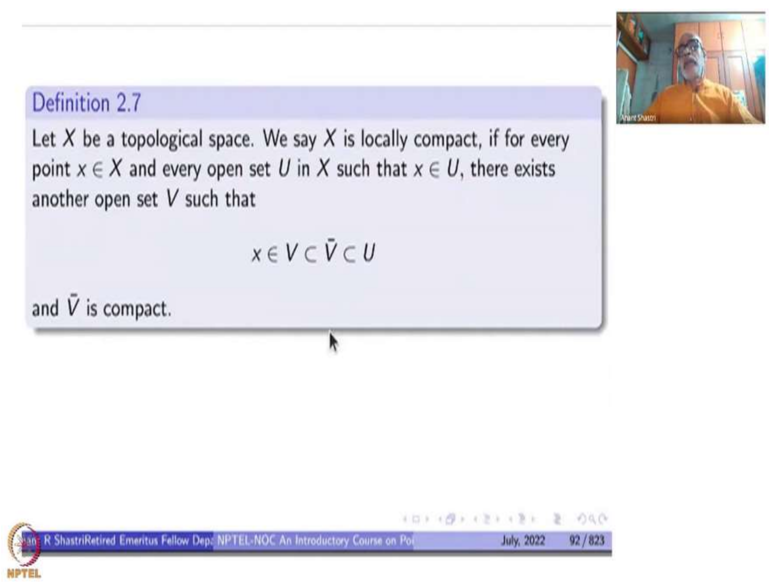

Let us begin with a definition. Start with a topological space X. We say X is locally compact, if for every point $x \in X$ and every open subset U in X such that x is inside U, there exists an open set V such that x belongs to V contained in $\bar{V}$ contained in U and $\bar{V}$ is compact.

So, at each point and each open set, this should happen. $\bar{V}$ can be thought of as compact neighborhood of x. Why this is a neighborhood? Because it contains an open subset around x and we have assumed that it is compact. The compact and closed neighborhoods. $\bar{V}$ is closed also. So, compact and closed neighborhoods of a point they form a fundamental system of neighborhoods at that point. It just means that for each open subset containing x, there is a smaller one in the system, that is the meaning of fundamental system of neighborhoods.

This is just another way of saying that X is locally compact. It should happen at all the points of X. So, the definition of local compactness is some what like the definition of continuity. For a function, we first define continuity at a point and then continuity on the whole space means continuity at each of these points. It is similar to that. So, that is the definition of local compactness. Now, I want to tell you that there are other definitions slightly weaker and weaker and so on.

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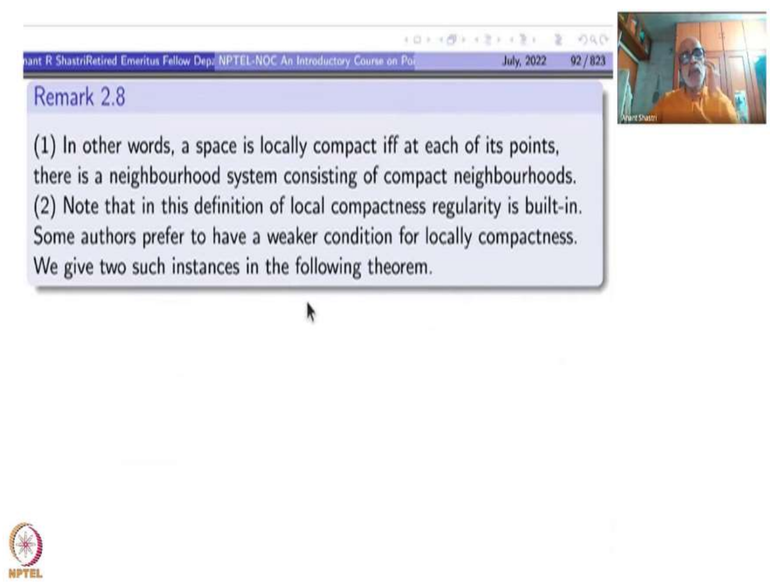

So, let us be careful about this definition. So, I repeat, a space is locally compact if and only if each of its points, there is a neighborhood system consisting of compact neighborhoods and closed neighborhoods. Note that this definition of local compactness regularity comes automatically. See that at each point of X, there is a fundamental system of closed neighbourhoods (not necessarily. compact ones) is the same as saying X is regular.

So, perhaps my definition of local compactness is too strong. Anyway it is not my definition. It is there in the literature. Out of the several definitions I have adopted this definition of local compactness. So, I want to tell you that some authors prefer to have a weaker condition for locally compactness. Below, we give two such instances which are very much common. There are many others. We cannot go on dealing with all of them. These two are quite important also, so, I would like to incorporate them here in a theorem.

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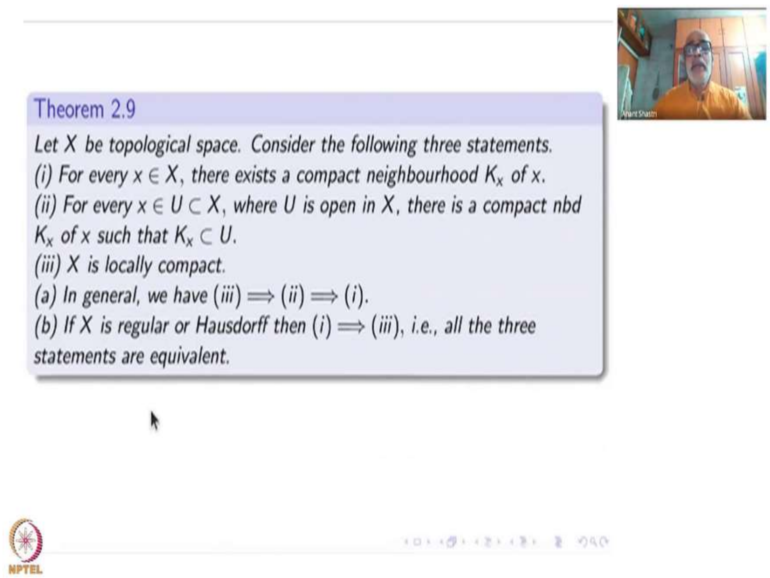

Start with a topological space. Consider the three conditions here, three different statements. Final conclusion is that the third one implies the second one implies the first one. If X is regular or Hausdorff then all of them are equivalent; (i) implies (iii) as well. What are these statements?

The first one is very simple namely, for every point of X there exists a compact neighborhood, which you may write K_x, that is all.

The second statement is slightly more elaborate: for each point $x \in U$ where U is open in X, there is a compact Neighborhood K_x of x such that K_x is contained inside U. So, this you can reformulate by saying that x belongs to V, V contained inside K_x, K_x contained inside U and K_x is compact.

The third one is our definition of local compactness.

Note that the difference between (ii) and (iii) is that x belongs to V, V contained inside $\bar{V}$ contained inside U and $\bar{V}$ is compact is (iii) whereas, in (ii) some compact set comes between V and U; $\bar{V}$ itself may not be compact. So, that is the difference between (ii) and (iii) you can see. Thus (iii) will automatically implies (ii) because I can take $\bar{V}$ in place of K_x.

(ii) implies (i) is also automatic because for each point you can take U to be the whole of X, then there is a compact neighbourhood. So, three implies two implies one these are very easy.

So, what I want to show you that under Hausdorffness or regularity, statement (i) which is the weakest here implies (iii), namely our definition of local compactness. So, these three things go hand in hand.

Quite often the underlying topological space is regular or Hausdorff, there are two schools of topologists. Some of them take regularity all the time, some of them take Hausdorffness all the time. (There are very few people who do not deal with either regularity or Hausdorffness.) So in that case, all these three definitions are equivalent. This is very, very important to notice. So, that is why I put it as a theorem here. So, let us give the proof.

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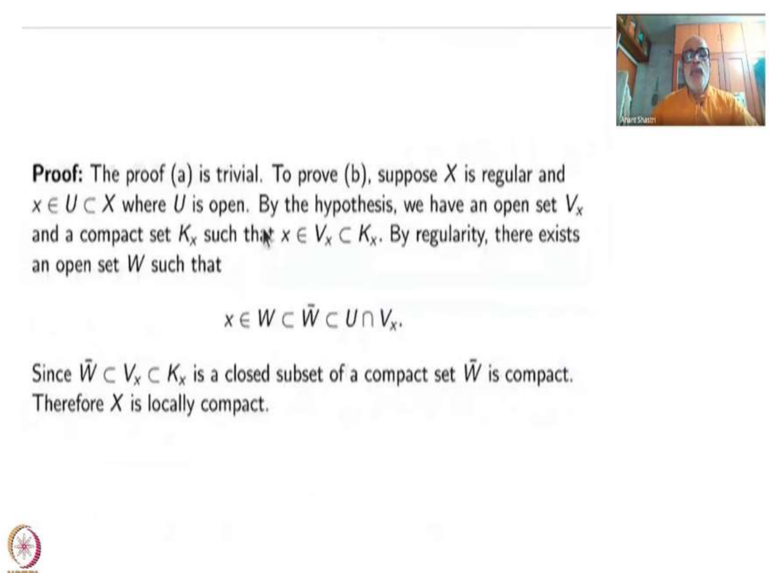

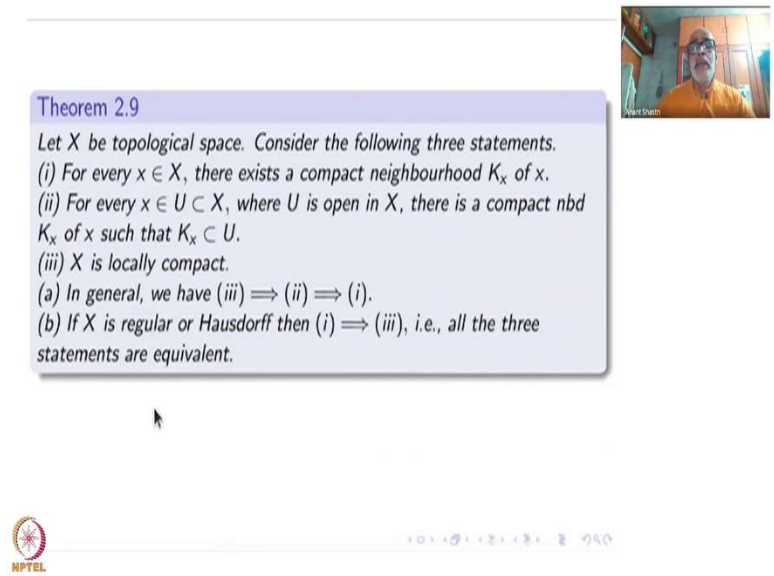

The proof of a part is what that is what if I told you that is three implies two implies one. Now, let us come to prove of (b). Under regularity, take x belonging to U where U is open in X. By the hypothesis (i), we have an open set V_x and compact set K_x, such that x belongs to V_x which is contained inside K_x. So, this is the hypothesis (i).

Now, I use regularity. There exists an open set W such that x is inside W contained in $\bar{W}$ contained inside this open set with $U \cap V_x$. U is open subset around x, V_x is also an open set around x, so $U \cap V_x$ is an open subset around x. So, regularity gives you an open set W with $\bar{W}$ lying in between. Now, $\bar{W}$ obviously is inside V_x also, and V_x is contained inside K_x. Therefore $\bar{W}$ is a closed subset of K_x. But K_x is compact. Therefore, $\bar{W}$ is compact.

So, this implies (iii). So, we are done with one part of part (b). The second one is instead of regularity, assume Hausdorffness.

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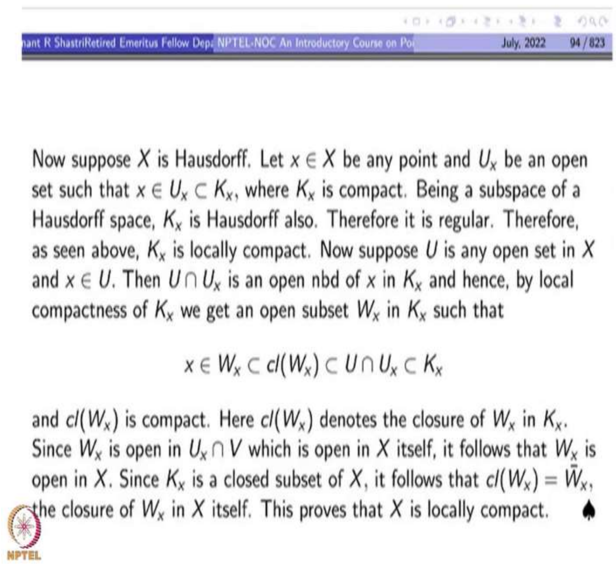

Now suppose X is Hausdorff. Let $x \in X$ be any point and U_x be an open set such that $x \in U_x \subset K_x$, where K_x is compact. Being a subspace of a Hausdorff space, K_x is Hausdorff also. Therefore it is regular. Therefore, as seen above, K_x is locally compact. Now suppose U is any open set in X and $x \in U$. Then $U \cap U_x$ is an open nbd of x in K_x and hence, by local compactness of K_x we get an open subset W_x in K_x such that

$$x \in W_x \subset cl(W_x) \subset U \cap U_x \subset K_x$$

and $cl(W_x)$ is compact. Here $cl(W_x)$ denotes the closure of W_x in K_x. Since W_x is open in $U_x \cap V$ which is open in X itself, it follows that W_x is open in X. Since K_x is a closed subset of X, it follows that $cl(W_x) = \bar{W}_x$, the closure of W_x in X itself. This proves that X is locally compact. ♠

Now, suppose X is Hausdorff, start with a point $x \in X$ and let U_x be an open set around x and U_x contained inside K_x which is compact, just like in the previous case. Being a subspace of a Hausdorff space, K_x is also Hausdorff. Now, K_x is compact and Hausdorff. Therefore from one of our previous results, it is regular. So, now K_x is regular. Therefore, as seen above K_x is locally compact. So, I am using this earlier part here so that I do not have to work again too much.

So, I can assume X itself is regular and then by the previous part it is locally compact. However, if you are not convinced of this argument as to why I can assume X itself is regular, let us prove (iii) directly. Now, suppose U is any open set in X and x is inside U. Then $U \cap U_x$ is open inside K_x and it contains x. Hence by local compactness of K_x, we get an open set W_x in K_x such that x is in W_x contained in the $cl(W_x)$ contained in $U \cap U_x$ contained in K_x.

So, here I use the local compactness of K_x and $U \cap U_x$ is neighborhood of x. I am writing this $cl(W_x)$ here just to indicate that I am taking the closure inside K_x. And of course, $cl(W_x)$ is compact. That is important. If I can show W_x is open in X and its closure $\overline{W_x}$ in X is equal to $cl(W_x)$, then I am done. That then the proof is over and that is precisely what I want to indicate now.

Now W_x is open in $U \cap U_x$, which is open in X itself. Therefore W_x is open in X itself. Next, K_x is a closed subset of X because it is compact subset of a Hausdorff space X. It follows that

$cl(W_x)$ is also closed in X. Therefore, $\overline{W_x}$ is contained in $cl(W_x)$. In any case, $cl(W_x)$ is a subset of $\overline{W_x}$. Therefore, the two are equal.

So, this is the fact which allows us to say instead of writing this new symbol here, I could have taken $\overline{W_x}$ that is a justification. Once you observe that K_x itself is locally compact you can go ahead and replace it by X itself. This is the justification. Some justification was needed, because closures are being taken in different subspaces. Take an arbitrary set A subset of Y subset of X, the closure of A in Y and closure of A inside X they can be quite different. In general what we have $cl_Y(A)$ is a subset of $cl_X(A)$. So, that is why I needed to justify that step. That is all. This completes the proof of the theorem.

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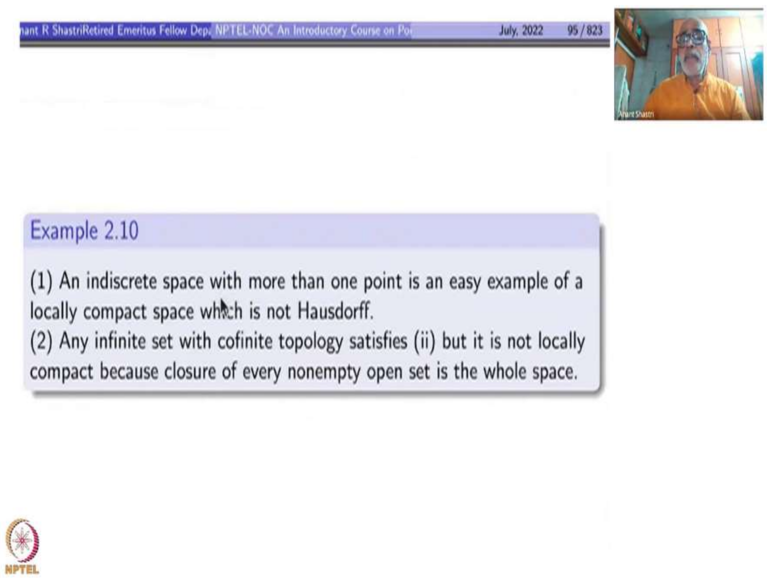

So, here are some example. (i) an indiscrete space with more than one point is an easy example of a locally compact space which is not Hausdorff. An indiscrete space is actually compact also because what are the open sets there? the only open sets are empty set and the whole space. So, these spaces are non Hausdorff spaces.

(ii) Any infinite set with co-finite topology satisfies property (ii) but it is not locally compact, why? Because the closure of every non empty open set in the co-finite topology the whole space. So, every non empty open set is dense. So, x belongs to U, where U is proper open set. Then no matter what neighbourhood W_x you choose, $\overline{W_x}$ will not be contained U. It will be the whole of space X. So, any infinite set with cofinite topology satisfies (ii) but does not become locally compact in our definition. So, under Hausdorffness/regularity, the three conditions are equivalent but this co-finite topology is neither Hausdorff nor regular. That is the problem. But there are such interesting examples wherein property one property two property three maybe all different, property one is true for all indiscrete spaces.

(iii) An open quotient map preserves (i) because all that you need is a compact neighborhood. An open map preserve open sets Being continuous also, it preserves compact sets also. So, you will get a compact neighborhood of the image points. Similarly, an open quotient map will preserve (ii) also. But not necessarily (iii), namely our definition of local compactness.

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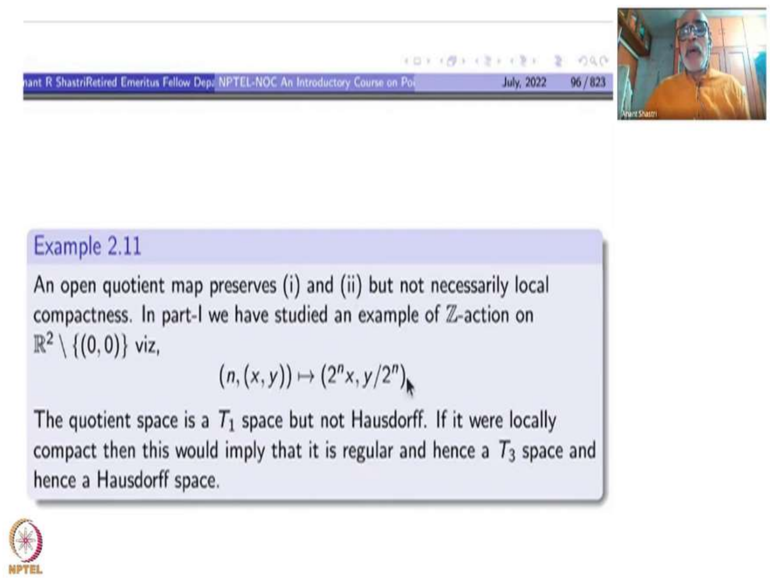

So, in part I we have studied an example of infinite cyclic group acting on $\mathbb{R}^2 \setminus \{(0,0)\}$. Namely, the action was given by $(n,(x,y)) \mapsto (2^n x, y/2^n)$, where n is an integer. So, x coordinate is multiplied by 2^n, y coordinate is divided by 2^n. This is a topological action of the discrete group $\mathbb{Z}$. So, this action was used to give several counter examples in part I. This quotient space is a T_1 space, but it is not Hausdorff. The images of $(1,0)$ and $(0,1)$ in the quotient space cannot be separated by open sets. That is what we have seen anyway. So that is why it is not Hausdorff.

If it were locally compact in our definition, then it would imply that it is regular and regular T_1 will be Hausdorff. But we know it is not Hausdorff. Therefore this quotient space is not locally compact. The $\mathbb{Z}$ action automatically gives you open quotient. So, this is an example of a locally compact space, $\mathbb{R}^2$ is locally compact, $\mathbb{R}^2 \setminus (0,0)$, beign an open subset is also locally compact. This we know. This is very easy to see directly also. Given any point not equal to $(0,0)$, you can take a closuer of an open ball around that point not containing $(0,0)$. The closed balls in $\mathbb{R}^2$ are compact. That is what we have seen. So, $\mathbb{R}^2 \setminus (0,0)$ is locally compact. But this quotient space is not locally compact in our definition.

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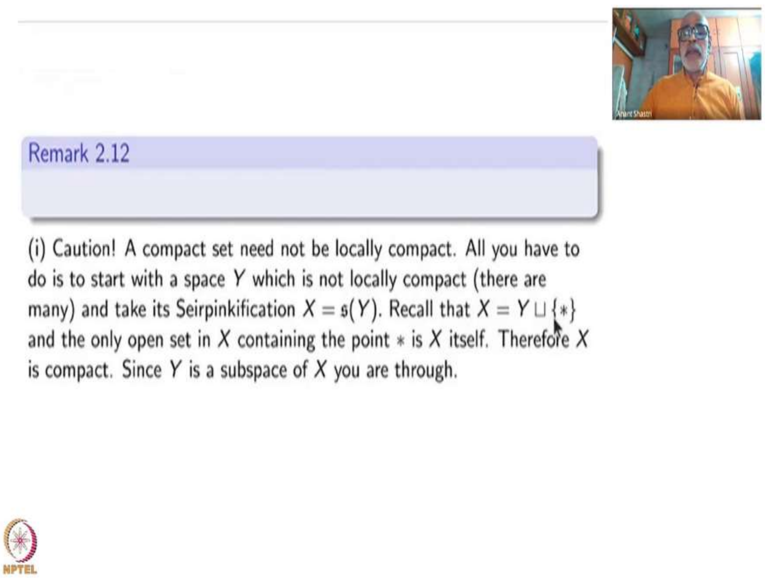

Here is a remark. A compact space need not be locally compact. In the definition (i) it is true because for each point you can take the whole space as a neighborhood which is compact. But in our definition, it does not work all. To get a counter example what you have to do is to start with a space Y which is not locally compact in our definition such as the ones which I have given many examples here already, there are many and then you add one more point to get X. I have introduced this idea in part I very meticulously, because I want to keep using it. This is called Seirpinskification. So, this Seirpinskification has the property, it adds one more point to the underlying set, but what are the neighborhoods of this point? only the whole space X that is all. All other open subsets are open subsets of Y they are there, that is a topology on X.

Obviously, any open set which contains this extra point must be the whole space. Therefore, this is automatically compact no matter what Y is. Y is a subspace of this but I started with a non locally compact space at all those bad points of Y, local compactness will fail inside X also, because Y itself is an open subset of X.

So, this way you can produce compact spaces which are not locally compact by converting one into a compact space by just putting an extra point.

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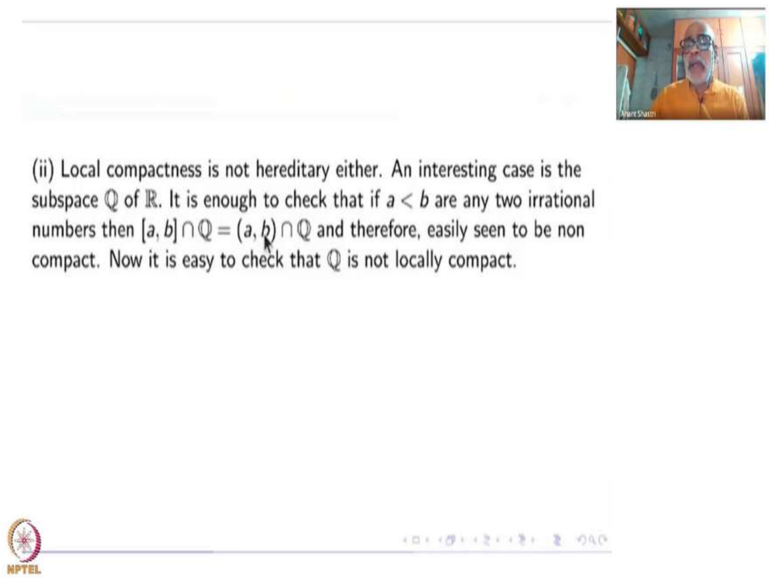

The local compactness is not hereditary either. Recall hereditary means, whenever a space is locally compact, all its subspaces must be locally compact. That will not be the case here. Again, very easy to see this. So, here is an interesting example first of all before going further, take the very interesting case namely rational numbers inside $\mathbb{R}$, very popular example. I want to say that $\mathbb{R}$ is locally compact that we know because for every open set around a point you can take a closure of an open interval around that point and contained in the open set. On the other hand, we want to show that $\mathbb{Q}$ is not locally compact. What is the meaning of that? For some point and some neighborhood of that point the criterion fails. Here, we are going to show that this fails at every point and for every neighborhood. You can choose any point and any neighborhood of that inside $\mathbb{Q}$.

What is a neighborhood? A neighborhood must be a neighborhood inside $\mathbb{R} \cap \mathbb{Q}$, only rational numbers inside an open interval. Take such a thing take a point for example 0, take an open interval around 0 and then take all the rational points in that that will be the subspace open subspace of $\mathbb{Q}$. I want to say that there is no compact neighborhood of that one.

For there were such compact neighbourhood, then there would exist irrational numbers $a < b$ such that the closure of (a, b) intersection with $\mathbb{Q}$ will be contained in such neighborhood and hene compact. But closure of $(a,b)\cap\mathbb{Q}$ is the same as $[a,b]\cap\mathbb{Q}$ and hence closed in $\mathbb{Q}$. However, it is easy to see that $(a,b)\cap\mathbb{Q}$ is not compact. Because a and b are irrational numbers and so I can take a nested sequence of open subintervals (a_n,b_n) of (a,b) with rational coordinates, which cover (a,b).

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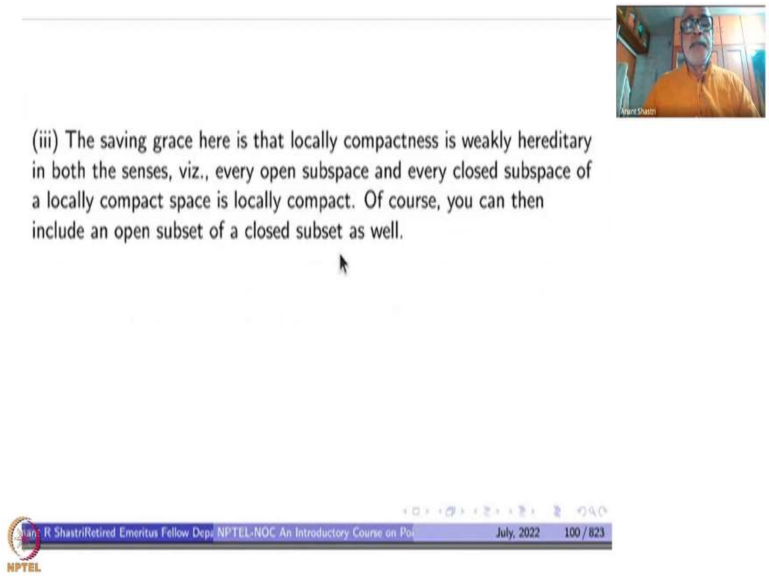

However, the saving grace is that local compactness is weakly hereditary. Weakly hereditary means every closed subset will have that property or sometimes every open subset will have that property, so there are two versions of weakly hereditariness. Here, both cases will work. Every open subspace and every closed of space of a locally compact space is locally compact. So, then you can combine them also, suppose, you start with a locally compact space and then take an open subset that is locally compact.

Now, we will take a closed subset of that. That will also locally compact. So, you can play this game. So, you get a family of subspaces, called locally closed subsets, very nice terminology which means subsets which closed inside an open set or open inside a closed set. So, such subspaces of locally compact space will be also locally compact.

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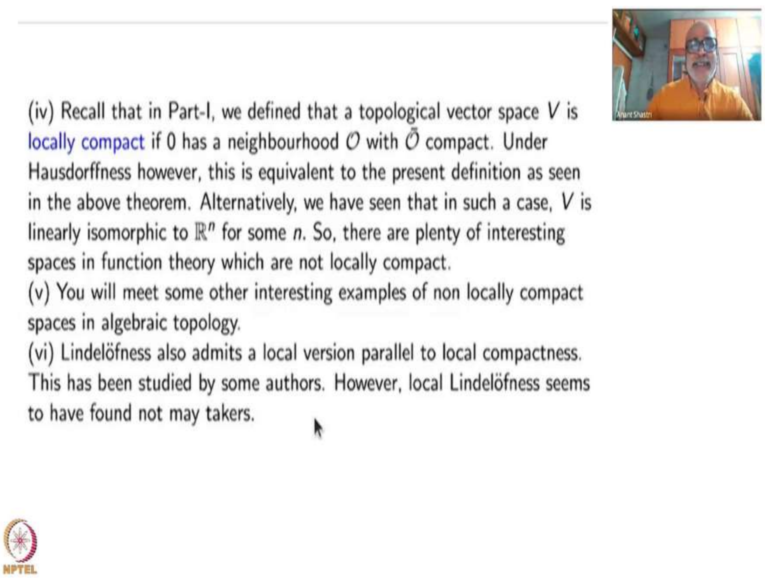

So, I want to call recall a few things here. In part I, we define a topological vector space V to be locally compact just by the small condition namely 0 has a neighborhood $\mathcal{O}$ with its closure compact. So, this was even weaker than our definition (i). Just that at one single point if (i) is staisfied, we are done. But in the case of topological vector spaces, something happens at 0, it will happen at all the points by taking translation.

If there is an open subset around 0 translate it by any other vector, so that will be an open subset around that vector, with its closure being the translation of the closure of the original open set. That is because translations are homeomorphisms. Therefore, it will work for all the points. That is not all. It is so strong that we have been able to prove that under this condition the topological vector space is actually linearly isomorphic to some finite dimensional Euclidean space.

So, this was one of the important results that we have proved in part I. In particular, $\mathbb{R}^n$, is locally compact in our sense also, in the strongest sense. To sum up it was enough to assume such a weak condition to get the strongest form of local compactness for a topological vector space.

You will meet some other interesting examples of non-locally compact spaces in algebraic topology, wherein you have to take unions of infinitely many spaces perform some identification. So, I will not be able to go through that here.

Lindelofness is also one property which admits local versions, exactly the way we have done above, namely in three different ways if you like, just like for local compactness. This has been studied by some authors. There are papers you can Google-search them.

However, local Lindelofness does not seem to have many customers. So, we will not discuss it any further. So, that is enough for today. There will be many more things about local compactness. We will discuss it next time. Thank you.