Decision Making Under Uncertainty
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Lecture - 20
Newsvendor Problem: Example and Proof

In this lecture we will look at an example- a numerical example and we will also look at the proof for this result.

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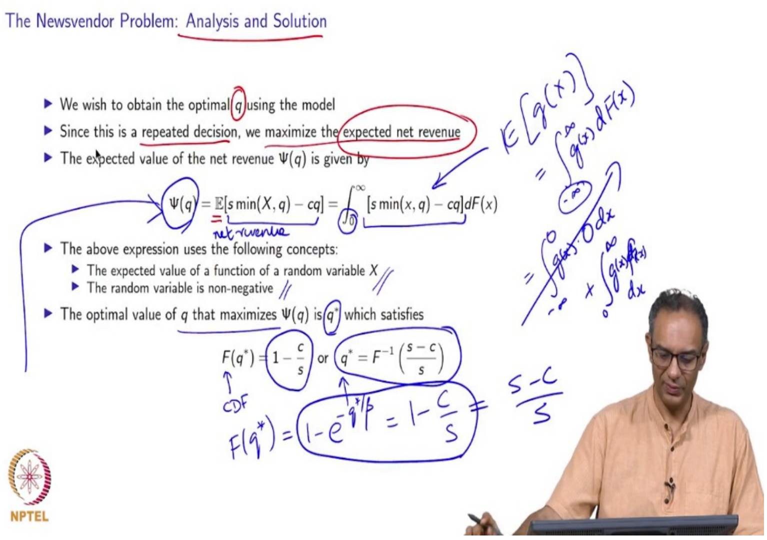

So, we only presented this result in the previous slide of this F(q*); we will derive this result.

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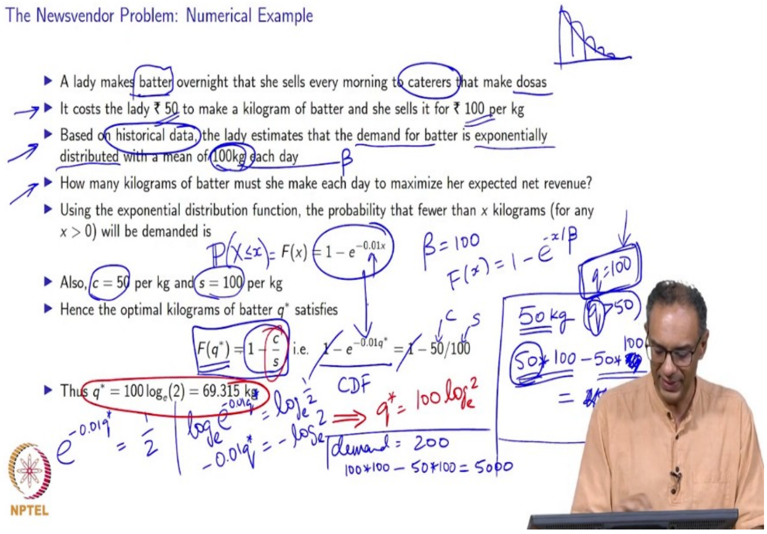

So, first let us look at this numerical example; let us look at a lady who makes batter for dosas. Some people call it dosai, some people call it dosa. I am going to call it dosa for the rest of this talk. If you have never heard what a dosa is; I am really sorry! You missed out something very important in life. But, you know I would still recommend to at least look it up on Google or something like that.

So, this person give us dosas; It kind of looks like a pancake and you have a batter and it is not a pancakes. So, I do want to make that plug for it. So, this lady makes this batter. It has to be made overnight unlike a pancake, which you can whip right away. This has to be made overnight. What she does is: she gives this to a bunch of caterers. And these caterers use this batter to make their dosas for various events like a wedding or various events that are actually going to cater this kind of food.

Now, turns out that the lady has to decide the night before: how much batter she need to make and give it to them. And I know that this batter only lasts for exactly one day. So, every single day she needs to decide: how much she needs to soak and sell the next morning. And there are bunch of caterers not just one, there a handful of caterers, who come and buy dosas from her. So, let us look at the cost. Remember we had the two letters: c and s; it costs the lady 50 rupees to make 1 kilogram of batter. So, that is her cost for various things such as the ingredients and the electricity price and so on and she sells it at rupees 100 per kg.

So, for every kg of batter, she makes a profit of 50 rupees. She spends 50 rupees making the batter and she gets 100 rupees for selling the better per kilogram- per kg. Now, the lady has historical data and she knows the demand for batter and let us say for the purposes of this problem that the demand is exponentially distributed.

Remember this is one of the continuous distributions and we use the parameter beta for the mean. So, this is a parameter beta, which is the mean of the exponential distribution. So, on average her demand is 100 kilos. So, 100 kilos of batter; she would have to sell on average. However, remember the exponential distribution kind of looks like; if you look at histogram, it kind of looks like this and has this shape (Refer Slide Time: 3:18). So, it is possible that her actual demand is much smaller than 100. It could probably be 20 kilos or it could be as high as 300 kilos. So, we could go anywhere from one small value to a fairly large value.

So, the question is: how many kilograms must she make of batter? Now, I do want to say one other thing: it is really important that she uses either historical data or some way to predict the demand- the demand distribution. Now this is an important part; whatever distribution you pick, you can do the analysis. And now that is not crucial, but the ability to actually go ahead, collect data and fit a distribution is critical to make this decision; like I said we need three things: the distribution, the value *c*- the cost price, and the value *s*- the selling price per unit.

Now, turns out that if we use the exponential distribution, we get the CDF as this (Refer Slide Time: 4:22); remember is the probability that the random variable is less than or equal to little x that is given by $P(X \leq x)$. So, here our beta is 100 and that is what we had earlier. So, if you remember for all positive values of x, $F(x) = 1 - e^{-x}/\beta$. So, 1/100 = 0.01 and that is how we get this 0.01. Remember that her cost is 50 rupees per kilogram and her selling price is 100 rupees per kilogram. So, we have all these three things known to us. So, c is 50: the cost price, s is 100: the selling price, both these are per kilogram. So, let me do a quick illustrative example.

Let us say her demand is 50 kilograms (Refer Slide Time: 5:24). So, if her demand is 50 kilograms then the amount of revenue that she would get is (50 X 100) and she would have spent (50 X 50). So, now for this to be the case, she needs her q to be greater than 50.

So, let us say q is 100. She had made 100 kilograms of batter. However, her demand is only 50. So, she would sell 50 X 100 = 5000 rupees. This is money made and she would have used up 5000 rupees to just buy the ingredients for the batter.

So, her net profit is 0. So, she basically breaks even. So, if the demand is only 50 she would just barely break even. Ok. However, if the demand is 200 then she would only sell 100 because that is how much she has; she does not have 200 and she would have bought 100. So, her profit is going to be 5000.

So, 5000 rupees is the profit she would make if the demand is anything above 100 because she will sell 100 kilos and the other people unfortunately will not get the batter that they need. So, those caterers will go home without any batter. Now, what is the optimal value of q? A lot of times when I post this problem, a lot of times people think the optimal value is 100 because on average my demand is 100.

So, I might as well store 100. This is an important point; Turns out, that is not the optimal value that you should store. Although the average amount of dosa batter, you do not actually store 100; you might store a different amount. So, let us check what you should. So, remember the optimal solution is given by this (Refer Slide Time: 8:13)

$$F(q^*)=1-\frac{c}{s}$$

So, remember the exponential distribution is given by this CDF at q star. I could write this down as: $F(q^*)=1-e^{-0.01q^*}$ and we want $F(q^*)=1-\frac{c}{s}$ or $F(q^*)=1-\frac{1}{2}$. So, I can cancel the 1 on both sides and I write down the equation as: $e^{-0.01q^*}=\frac{1}{2}$

So (Refer Slide Time: 9:20), I take the log, natural log on both sides. I get log to the base e of this- of this quantity on both sides: $\log_e e^{-0.01q^{¿}} = \log_e \frac{1}{2}$. This can be written as $-0.01q^{¿} = -\log_e 2$. That is how the log works!

This one gives me $q^{¿} = 100 \log_e 2$, which is exactly the result we have. Now, turns out her optimal amount is actually 69.31 kilo. Ok. 69.31 kilo. Let us see a little picture of how this happens ok.

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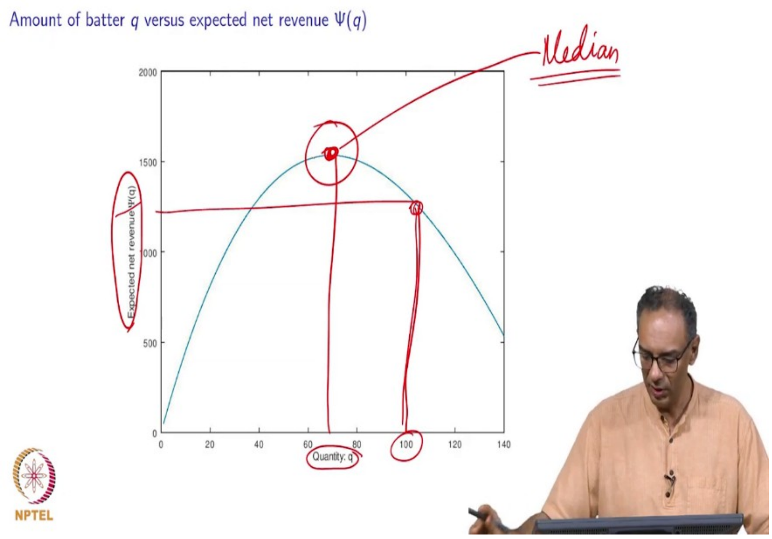

So, this is the quantity and this is the expected net revenue. This is the $\Psi(q)$. So, if you think of having 100; this is the expected profit- your expected profit is sitting here. However, the best that this person can do is this- remember 69.315. So, this would be roughly almost here, some somewhere here. So, that is the optimal number or optimal value.

So, notice that this quantity is not at the mean. Now, this is a mistake that a lot of people make. This is the mean of the exponential distribution. So, you do not store exactly the average. The reason lot of people think that way is because we are always conditioned to think of like a normal distribution, where everything is symmetric about the mean and therefore, you know.

So, if you had the exact average amount, you would do well and the reason you do well here is because your profit is such that your selling price is twice the buying price or cost price.

And therefore, for every unit that you sell, you make a profit of 50; for every kilo you do not sell you make a loss of 50. And therefore, it balances out. So, if you had a normal distribution then you would actually be right at the mean. However, where we are here interestingly because it is half, if you look at this number: $1-\frac{c}{s}=\frac{1}{2}$; so we really are exactly at the median.

We are at the median and that is because the probability that the random variable is less than or equal to the median is half. And that is the reason you get the median results; that is because this quantity 50 over 100 is exactly half. And the optimal value actually occurs at the median.

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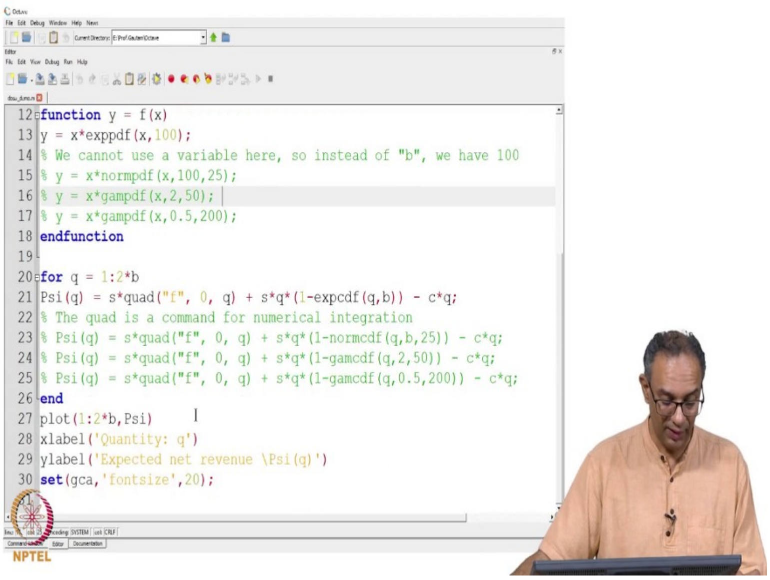

Now, I want to step aside and go to octave and I want to show you how to compute these quantities. And will do a little experiment and then you will have a better appreciation for where things stand.

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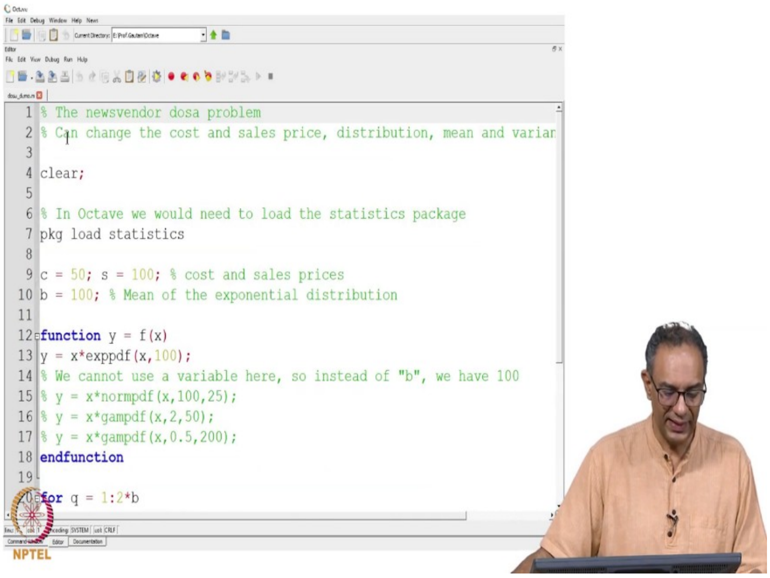

So, in this example, this is the newsvendor dosa problem. So, like I said you are welcome to play with this and by changing the cost price, the sales price, maybe the distribution; we are going to do all of that right now.

We could also change the mean and the variance of the distributions and so on. Of course, in the exponential only have one parameter you can only change the mean, which automatically changes the variance. As usual, I would like to do clear- just so that we are not using any parameter we had used before. We again want to load the statistics package because we are going to do some statistical type of generation.

So, now what happens is; now remember that let us do the basic problem, where the cost price is 50 rupees per kilo, and the selling price is 100 rupees per kilo, and the average demand with the exponential distribution is 100. So, that is the situation we have. I know I have a bunch of things commented out here; let us ignore that for now. I am going to do the following.

So, now this is an important part (Refer Slide Time: 13:32). I want you to see what we are doing here because what I am going to plot a graph of the Ψ quantity- remember that this quantity that we have in this graph. I am going to try to plot and show you and then I am going to try some different values. If you look at the Ψ function from before: it is essentially the expected value; it is this function: $s\,min(x,q)-cq-dF(x)$.

So, I am going to write that down explicitly and I am going to write that in the following way.

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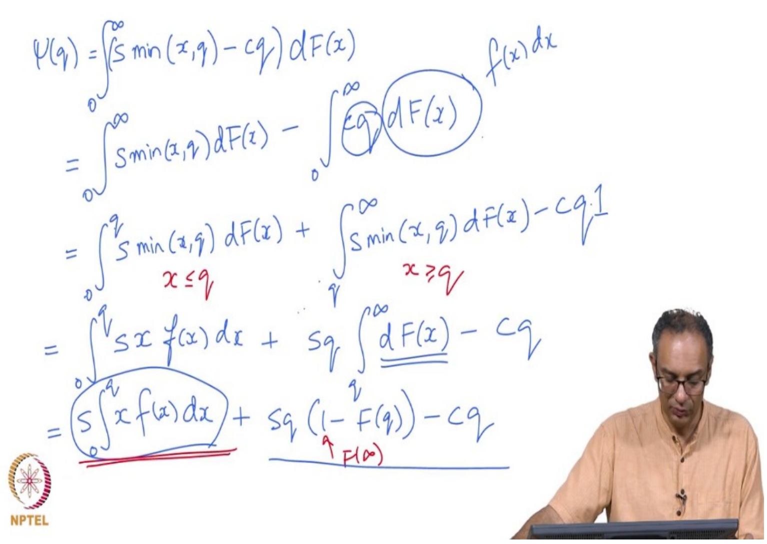

So, I write this down as $\Psi(q)=\int_0^\infty (s\,min(x,q)-cq)\,dF(x)$. We will do this one more time next, when we derive this result mathematically. You could write this down as

$$\Psi(q)=\int_0^\infty s\,min(x,q)\,dF(x)-\int_0^\infty cq\,dF(x)$$

Now, turns out that we could write this down as:

$$\Psi(q)=\int_0^q s\,min(x,q)\,dF(x)-\int_q^\infty s\,min(x,q)\,dF(x)-¿cq¿$$

Why am I doing this? Because between x values of 0 to q, we know what is the minimum of x and q. And then I will integrate from q to infinite s times minimum of x and q times dF of x minus c times q.

Why is that: c is times q? Because integral of $dF(x)$ is basically f(x) dx. So, this is the area under the pdf curve; the area under the pdf curve if you remember is always equal to 1. This is one of the properties of continuous random variables and I can take the cq outside and integrate that and I will get 1. So, multiplied by 1 is what I get.

Now (Refer Slide Time: 16:08), notice that this guy; this is the last step that I am going to do.

$$\Psi(q)=\int_{0}^{q} sxf(x)\,dx-sq\int_{q}^{\infty} dF(x)-¿cq¿$$

0 times q, the minimum of x and q is just x that is because for all values between 0 and q; x is smaller than q in all these values. I am going to say: x all over in the first term of the equation is smaller than or equal to q, x all over in the second term of the equation is greater than or equal to q.

Now, I could write this guy as:

$$\Psi(q)=s\int_{0}^{q} xf(x)\,dx+sq(1-F(q))-cq$$

Now, first term in the right hand side of the equation is not easy to do unless it goes to infinity. Plus second term, which is s times, q times, 1 minus the CDF; why is that 1 minus CDF? Because dF(x) = f(∞). If you remember another property- was that F of infinite was equal to 1. So, I am writing down that $1-F(q)$. And then minus the, cq.

So (Refer Slide Time: 17:53), the only part that is difficult is first term in the right hand side of the equation. So, we want to do this integration in octave, I am going to use a numerical integration to perform this integral; and then I will subtract this. I am going to try all values of q to plot this graph i.e., we want to plot this graph for all values of q. Now, the only tricky part there is how I go about computing this integral; this integral here is a bit tricky. So, that is my first computation, then the second computation- I am going to put in the other part, which is basically a direct function.

So (Refer Slide Time: 18:40), now let us move to the octave software. So, you look at the function in the octave program. The way you write down this octave program is write y = f (x) that is how they want you to write. And y = x times the pdf that is what we want to integrate. So, let me just show that to you one more; this is x here and this is the pdf of the exponential distribution. So, x times the pdf of the random variable. So, we write down x times the pdf. So, that is my function that I want to integrate and then we write s multiplied by quad; quad performs a numerical integration between 0 and q.

It performs the numerical integration of this function f that we defined here; y = f (x). We defined it on the top; it is going to perform a numerical integration from 0 to q and that is exactly what we want: 0 to q, s times the function; that is what we want to integrate. That is precisely what it does, plus s times q, times 1 minus the CDF at q, and this is the exponential distribution with parameter beta.

And therefore, I write this as q and this is a parameter; notice that we also did the pdf. However, I did not put the parameter beta here. I wrote directly the value of 100. That is one of the issues with octave; octave will not let you put the letter b there. Everything other than the x need to be a numerical value. It cannot be a function like b or a symbol like b or something like that and therefore, we need to put down a number. But, here I could have just written this as 100. I just chose to write b minus c times q; I just want to verify that- s times, q times, 1 minus the CDF, minus c times, q.

So, we have that function right there and the quad basically performs a numerical integration. So, let it do the integration and if you do the plot; you will see a plot exactly like what we had. So, let us first reproduce the plot that we had earlier in the file.

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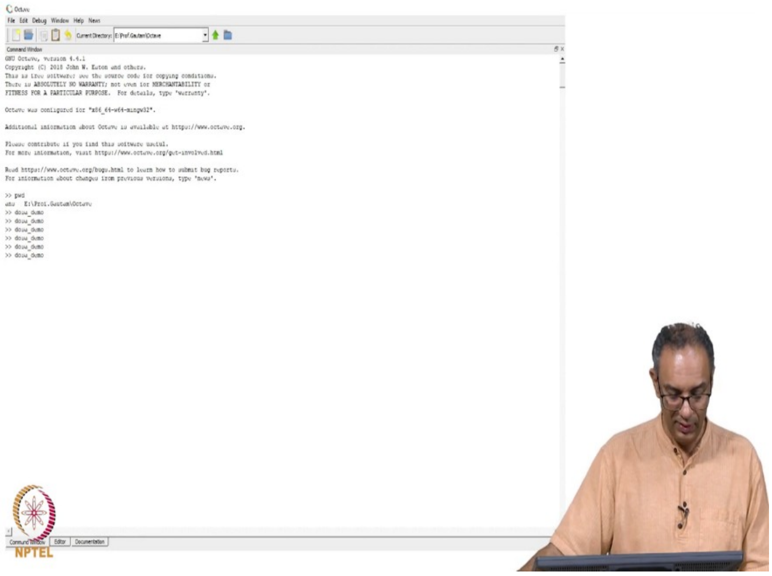

So, this program is called dosa underscore demo. So, I am going to write: dosa underscore demo. That is all I am typing. I am not doing anything else and it will plot a graph in just a second as soon as it is able.

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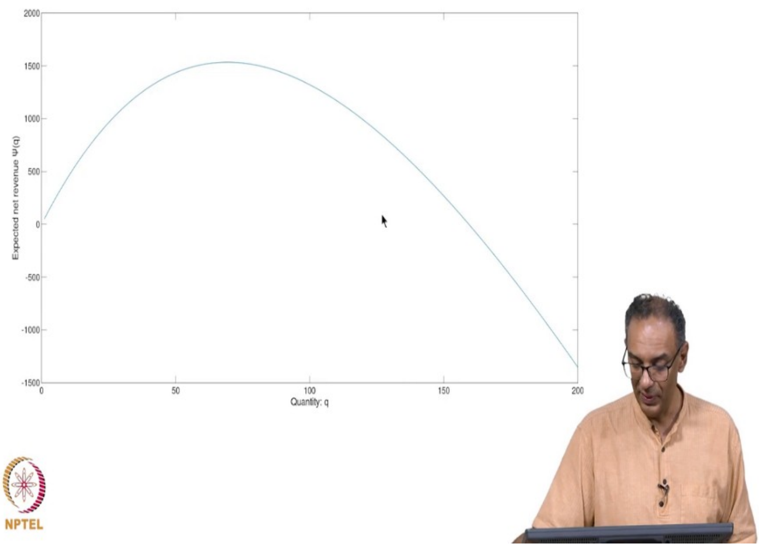

And I am going to make this a bit larger and I am going to write auto scale. So, you can see. So, this is precisely the same graph; however, the x axis is a little bit different. I am going all the way to 200, this is not what we did here. Here we only went till 140. So, so if you look at this graph, it goes. So, we only went to 140. So, it is kind of somewhere here. So, this is the exact same graph and if you see here the optimal value is 69 point something. So, that is what we got. Now let us look at this, let us look at a few other minor changes and I want you to feel free to play with this.

So, now, instead of exponential let us say we do normal distribution and see what happens. So, I am commenting that out and I am uncommenting the normal part. So, let us say we do the normal distribution and let us see how the numbers look like for the normal case. ok. So, I picked up the normal and then I will save this: and control s, and then I am going to run the program again; dosa demo- this is for normal distribution; let us see what happens.

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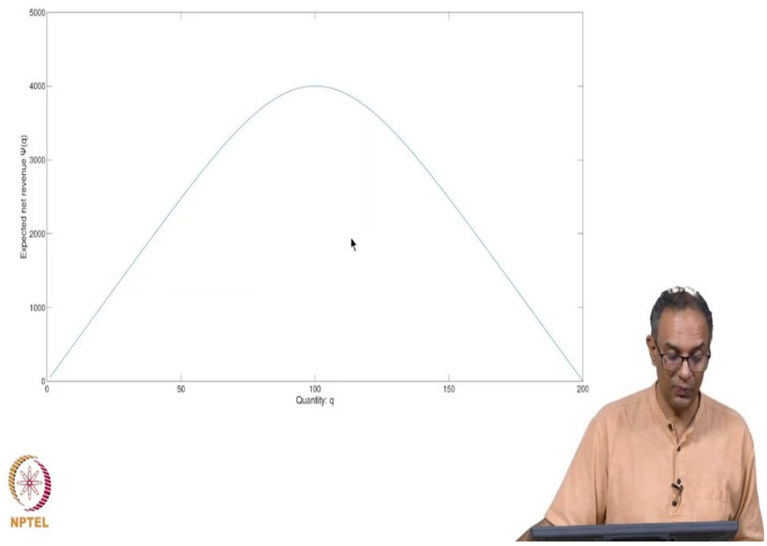

Now, notice what we said before; in the normal distribution since the mean, and the median, and the mode are all at 100, the optimal solution is also 100. This is something we expected for normal. So, it looks nice like this. Now let us try in another example, this example, we are going to look at the gamma distribution. There are two types of gamma distribution: one with a parameter where the variance is not as high as exponential, but not as low as normal; its somewhere between the two.

So, will try one example like that. Then, will try another example where the gamma CDF has a variance that is higher than even exponential; lot more variability. So, if you notice- exponential the optimal solution was 69 for the same, remember the mean is the same. If you look at the normal distribution that also has a mean of 100. The exponential also has a mean of 100. I am telling you that this gamma also has a mean of 100; 2 times 50 is the mean. So, that is also 100; the same mean.

We are only changing; only change the distribution and we are keeping the means the same. Of course, the variance will change, that is mainly what is causing it. So, I am keeping the means the same. Let us see what happens in this gamma situation, I am saving it and I am running it. Next.

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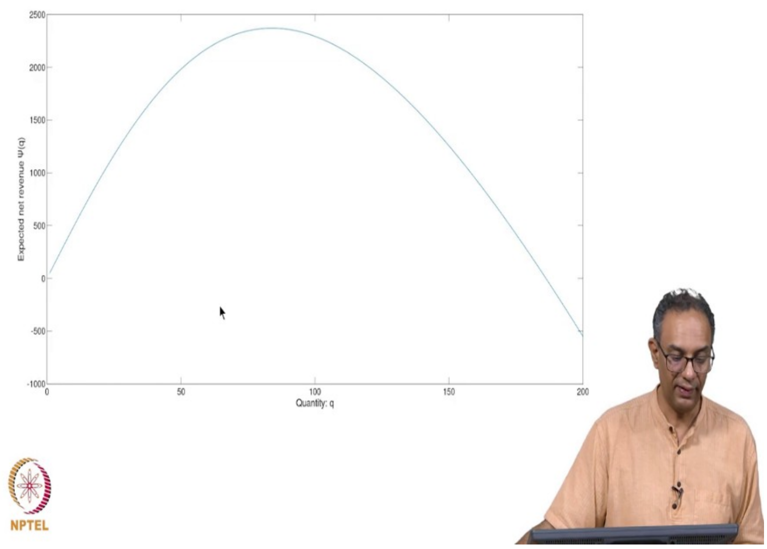

So, let us see what happens- it will give you a distribution and it looks like slope. So, now, notice that now the value is higher than 69 because the variability is not as much as exponential, but not as little as normal. It is somewhere in between. So, the value looks closer to about 80 or something like that. I could actually compute what exactly it is going to be.

Now, let us try a different distribution. Now, let us try a more diabolical gamma distribution. In fact, one in which the mean is still 100, but the variance is a lot more than even the exponential. So, let us see what happens and then once we save it, and run it, let us see what happens to this. As soon as it finishes running we will see what value gets. I think since it is a quite a diabolical function it took a little bit more time.

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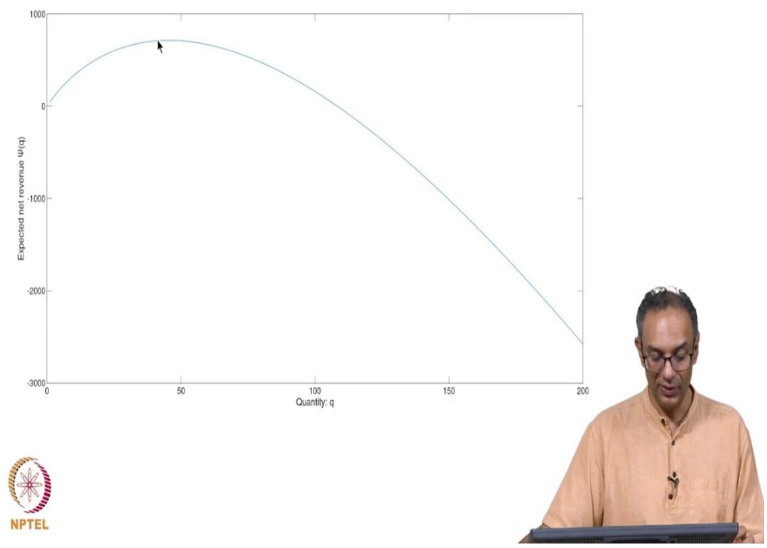

Now, notice that this is much smaller. So, for exponential we had it at 69 when the variance became even smaller then it went down to about 40’s.

So, it is 40’s, and then almost 70 for exponential, and then about 80 for gamma, and 100 for normal. So, as the variance or as a standard deviation became smaller and smaller, we are saving more and more. I want to take a brief moment to say a little bit about that. Importantly, notice that, you know, as the variability increases what happens is you have a lot of small and small values; once in a while you have a large value that is what is the meaning of having the same mean, but having different variances.

So, when the variances are really small, all your numbers are going to be right around the mean. If your variances are large, it will be very away. In particular, you will see a lot of small values and once in a while and you will get a large value; remember all the values are positive. So, to in order to get a larger variance, you need to have a lot of small values. So, more often than not, the amount of dosa batter that gets demanded is going to be a small number. That is why as the variance becomes higher, you are holding lesser and lesser batter in the morning. So, really important for you to understand what is going to be the variability in your demand in order for you to figure out- how much to keep.

So (Refer Slide Time: 26:01), that is the essential lesson here. Now, I do want to take a moment to also tell you what happens if I even change the number and kept our favourite distribution and see how it looks like. So, we will go ahead and continue with normal; this is the one we have a better intuition for. We will keep the normal example here; that we have a better intuition for. However, I am going to change the costs. I am going to say- well, what if it costs 80 rupees for the person to make, and selling price is 100, continues to be 100, then what happens. So, I just want to check to see everything is normal. And I want to run it one more time and see what happens and if I look at it and see what is my picture looking like.

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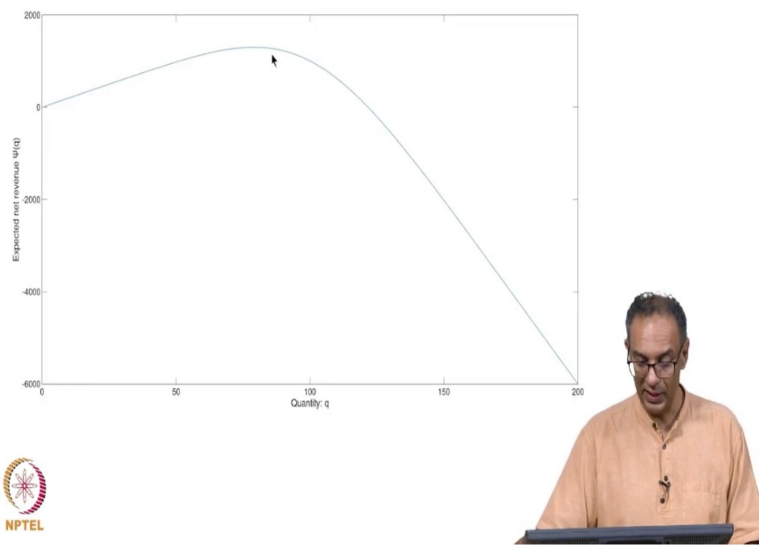

Now, if you see I am actually storing less than 100 that is because my cost price is 80, very close to the selling price. ok. So, I want to be a little bit conservative. I do not want to keep all 100 because I would get into a huge amount of losses. Now, let us see the other, the flipside- if my cost is much smaller. Let us say the cost is 30, let us see what happens to the problem. Now, let us see what happens to the picture; this would hopefully result in a solution.

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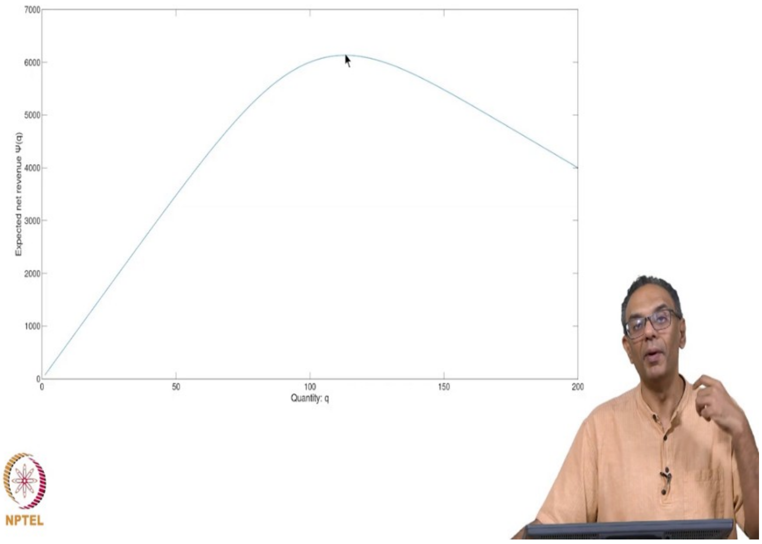

Where it is higher than 100 because what happens is- my cost is really small when compared to the profit. So, therefore, I would want to be a little bit aggressive and have a little bit more than 100, something like 110 or 120. So, you would change it depending on what is your actual demand. So, that is what I wanted to say in terms of a demo. Now, I want to go back to this file and talk about the proof of the result and let us look at the proof of how all this works out.

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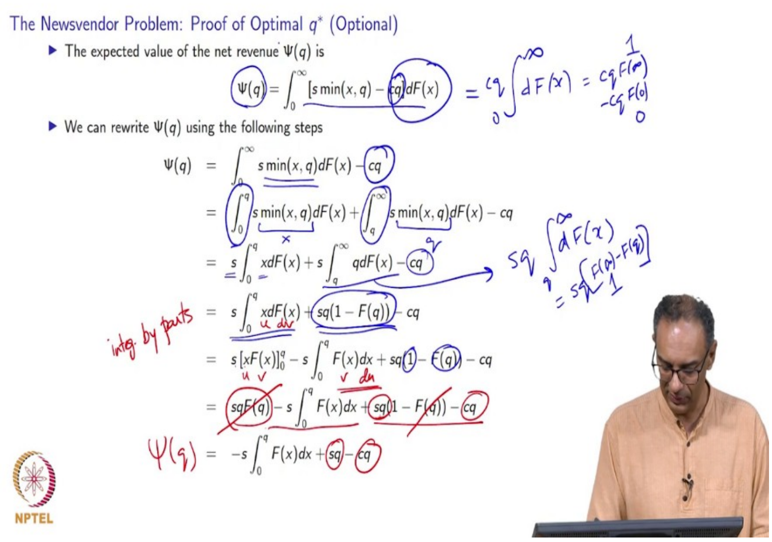

So, remember that the expected value of the net revenue

$$\Psi(q)=\int_{0}^{\infty}(s\min(x,q)-cq)dF(x)$$

We saw this earlier, I am not going to rewrite this and we did most of the calculation already. So, I am not going to repeat my calculations. Notice that I have taken the cq outside because it can be rewritten as:

$$\Psi(q)=\int_{0}^{\infty}cq\,dF(x)=cq\,F(\infty)-cqF(0)=cq(1)-cq(0)=cq$$

F of infinite is 1; F of 0 is 0. So, therefore, that just gives me cq, which I pull out by itself. Next, we do the same step we did earlier.

$$\Psi(q)=\int_{0}^{q}s\min(x,q)\,dF(x)+\int_{q}^{\infty}s\min(x,q)\,dF(x)-¿cq¿$$

$$\Psi(q)=s\int_{0}^{q}x\,d\,F(x)+s\int_{q}^{\infty}q\,dF(x)-¿cq¿$$

$$\Psi(q)=s\int_{0}^{q}x\,dF(x)+s\,q(1-F(q))-cq$$

$$\Psi(q)=s¿$$

$$\Psi(q)=sqF(q)-s\int_{0}^{q}F(x)\,dx+¿sq(1-F(q))-cq¿\Psi(q)=-s\int_{0}^{q}F(x)\,dx+¿sq-cq¿$$

We take the integral from 0 to q because that way I know how to compute the minimum of x and q. Because between 0 and q, the minimum of x and q is always equal to x; and that is because in that range all values of x, between 0 and q is always smaller than q. And in here, the minimum of x and q is always q that is because between q and infinite, x is always larger than q. So, the minimum is q.

Now, I go here and I write down s multiplied by x from here times dF(x), plus s times, q times, dF(x) minus the cq that we had before. Now, we wrote this step before. So, I am just repeating this one more time- s times integral 0 to infinite x times dF of x. And this step we had before in our slide that I hand wrote. So, it is exactly the same step; you are welcome to just go back and see. So, basically what we brought, what we are doing here is: we taking this guy and writing this as s times, q times, q to infinite dF of x, which is equal to s times, q times, F of infinite, minus F of q. F of infinite like we said before here- is equal to 1 and therefore, I get 1 minus F of q and multiplied, the whole thing multiplied by s times q.

Now this is the part (Refer Slide Time: 30:50); I am sorry! I meant that to say that here. Now, this is a part that is a little bit interesting. So, now, we do integration by parts. So, I am going to change my colour now to red and I am going to call this guy as u, this guy as dv. So, therefore, it is

$$\int u\,dv = uv - \int v\,du$$

That is the standard integration by parts. So, I do x times F of x, 0 to q, minus s times integral.

So, essentially this is just standard integration by parts: uv minus integral vdu. Now, this one is straightforward because if I plug in x equals q, I get s times, q times, F of q which is this guy. If I plug in 0; I get 0. So, I would not even writing it down because 0 times F of is 0; F of 0 is 0 as well, x is 0 as well. So, 0 minus vdu expression, which is a minus s times integral 0 to q, F of x times dx, plus the remaining terms are the same as before. Now, this sq cancels with this $sq(1-F(q))$. s times, q times, F of q gets cancelled and therefore, we are left with sq and cq, and this expression: negative s times, 0 to q, F(x)dx. So, we have that expression.

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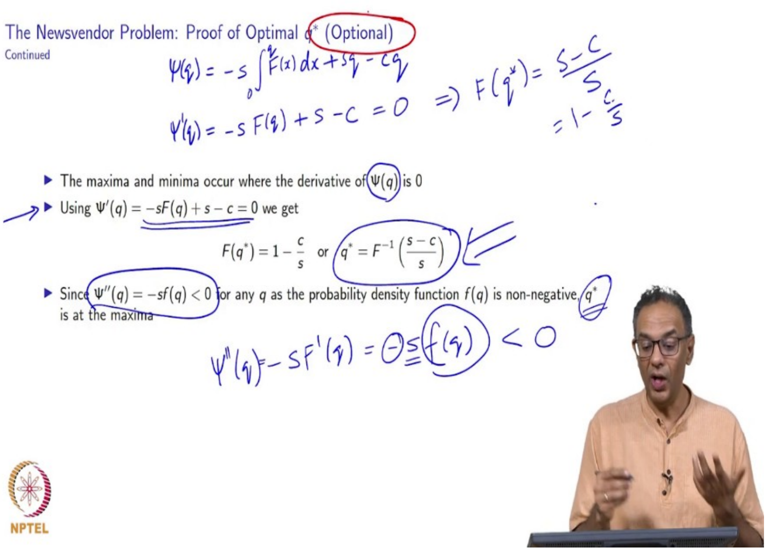

Now, that is all we have. So, this is my $\Psi(q)$; I want to maximize this function. So, what I do is: I take the derivative of this guy. I am going to write this out really quickly for you all.

$$\Psi(q) = -s\int_0^q F(x)\,dx + ¿\,sq - cq\,¿$$

Now, we know that to compute the maxima, we take the derivative of $\Psi(q)$ and equate it to 0. So, $\Psi'(q) = -sF(q) + s - c = 0$

So, we had an integral with no q inside. Then, the integral is just exactly equal to F(q) because you just take the derivative of an integral- it is basically the function that is sitting inside, plus the derivative of sq is s, the derivative cq is c. We want this guy to be equal to 0 that is exactly the expression. For any maxima or minima we take the derivative of function equated to 0. I want to just say one quick thing- notice that for this reason we want this function F(x) to actually be a CDF.

And what I mean by that is we do not want the random variable to be discrete; otherwise taking the derivative is difficult. So, we pretend like q is continuous quantity and this is where we really use it; take its derivative and equate it to 0. Now, if we equate that to 0- if you look at this expression, which we just have on the top. If you solve for F(q), which we write as F(q^*) star,

$$F(q^{¿}) = 1 - \frac{c}{s}$$

$$q^{¿} = F^{-1}\left(\frac{s-c}{s}\right)$$

So (Refer Slide Time: 32:10), that is what we have and if I write F of q star, then I write q star as f inverse, s minus c, over s.. Now, I want to take the second derivative and make sure that the second derivative is negative. So, if you take the second derivative of this guy, you essentially get

$$\Psi''(q) = -sf(q) < 0$$

Plus the other things vanish; F prime of q is nothing, but the pdf. The pdf is always a number that is positive multiplied by a positive selling price s with a negative sign. So, this is guaranteed to be less than 0 and therefore, we are looking at q star being the maximum. We saw the graph and that also explained us that it was a maximum. So, the best q star is indeed given by the inverse. So, notice how the inverse of the function at 1 minus c over s.

So, notice how this function itself is a very nice crisp expression. It looks really pretty you could nicely use this in a variety of situations. Now, I do want to say one other thing before I wrap this up- notice that this item is optional. What I mean by that is I have mainly presented this so that you know how these expressions are derived. I will not ask you a question in the exam or in the test, where we are looking at how to derive this. So, this is something that is purely as an optional item. But, I would highly recommend actually going through this carefully. We will stop here and start with the next topic later.