Decision Making Under Uncertainty
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Lecture - 23
Safety Stock: Example and Derivation

This lecture is an example of the safety stock that we talked about in the previous lecture. And I have also derived the expression for the safety stock that $z_{1-\alpha}$ expression I will talk.

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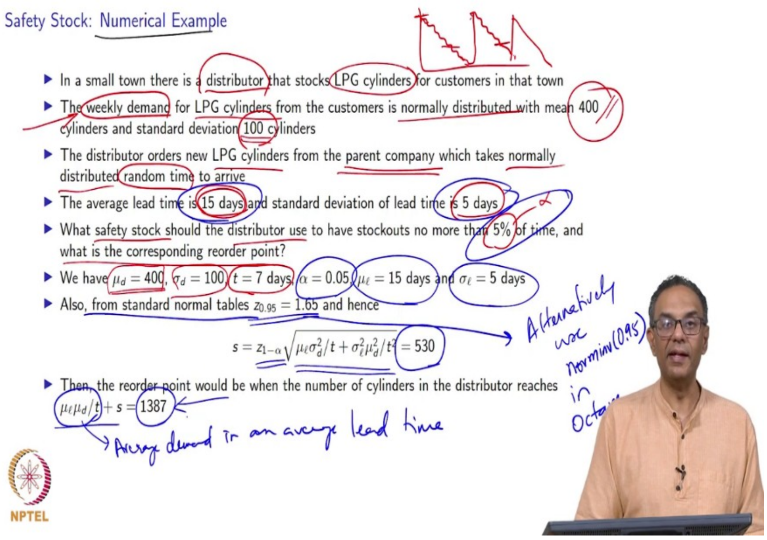

Let us do a quick numerical example. So, think of a small town where they have a distributor that sells LPG cylinder- this is the Liquefied Petroleum Gas cylinder. These are the cylinders that we use in our homes to for cooking. So, you get those two cylinders, most people actually already planned for uncertainty by having two cylinders in their house so that when one cylinder finishes off, they would make a call. And then they would get a new cylinder after a few days and by the time the second cylinder will be useful.

We are not actually talking about households, we talking about the distributor who is going to also be facing some uncertainty because the distributor if you think about it gets a fresh cylinders, which have been filled with liquefied petroleum gas from elsewhere. So, that comes in a large truck, get dropped off at the distributor location and people also when the demand comes from people that is the random demand, the number of cylinders goes down.

So, if you look at the distributor itself, we have tons of cylinder as the number of cylinders will keep going down and then the distributor reorders, somewhere here and then the reorder comes in and then keeps going down. So, this is what happens at infinitum- we will reorder and every time these level goes below a certain threshold. So, for this we assume that the distributor has information about the demand distribution from the customers as well as the lead time for replenishing this cylinder.

So, when the person comes for the replenishment by the way all the empty cylinders also will be given back. So, were not worried about that part of it, we just worry about whenever someone bring new cylinders that will get put in their floor and they will take away the old cylinder. So, this is an example of a single item distributer type of system that we were talking about. I just wanted to emphasize that this is not an unheard of example.

Now, we are going to assume that the LPG cylinders demand is normally distributed that is the weekly demand. In 1 week, we assume that on average 400 cylinders are demanded and the standard deviation is 100 cylinders. So, that is the demand. The demand could be not exactly 400, but anywhere from 0 to about 800 for all practical purposes; so that would be the weekly demand.

Now, the distributor orders new cylinders from the parent company. So, if there is a parent company that will deliver and takes a random amount of time to show up. Now, the average lead time is 15 days and the standard deviation of lead time is 5 days. So, we assume that the lead time is uncertain with a certain mean and certain standard deviation, we also assume that it is normally distributed. Remember that the lead time as well as the demand are both normally distributed.

Now, the demand is after all a discrete quantity here and in the previous example. However, we went ahead and assumed normal, especially in the cylinder case when the demand is as high as 400, these are practically continuous. So, we do not have to worry about the discreteness of this. So, we go ahead and use it; if that was a question that came up, that is a wonderful question, do not worry about, it would work out reasonably well. The question is- what safety stock should the distributor use so that your alpha, the stock is there at most 5 percent of time. Also, what is the corresponding reorder point? This is the threshold at which they should order.

Now, remember that the lead time is 15 days. So, now, let us put down some numbers, the demand during a week. Now, t is 7 days and I am going to use days as my unit. Therefore, t which is the demand cycle, if you look at it, it says the weekly demand right here. So, the demand is 7 days that is why t is 7. Now, μ_d in 7 days the demand is 400 on average, the standard deviation is 100 on average.

And then, α like I said is 0.05, μ_l is 15 days, remember the average lead time is 15 days and the standard deviation of lead time is 5 days. So, we have all these numbers given to us. So, we could just go ahead and use the standard normal table and write down $z_{0.95}$ is 1.65, it is a little bit lower than that, but if we forgot how to compute it, we could use a software like octave.

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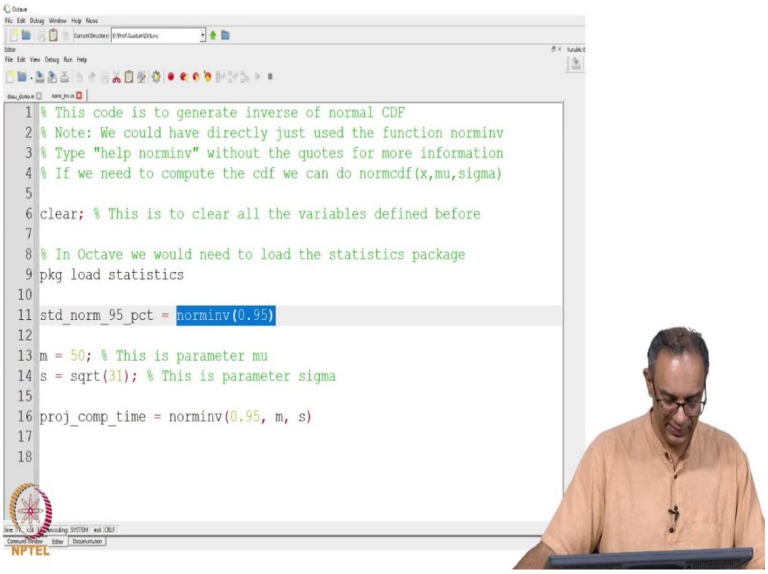

We could directly just use this *norminv.m* program we had in the previous topic. However, all you have to probably do is to say: what is the norminv of 0.95. So, this is norminv 0.95 in octave, which is right in the screen and in a second it is basically going to give me the inverse of this standard normal, and this is 1.6449 is roughly 1.65.

So, one way to do this is to use a table and it will give you this result. You could also alternatively use *norminv* of 0.95 in octave and that will give you roughly the same number- it should be 1.644 something like that. So, either ways that is going to give you the value of $Z_{1-\alpha}$. You plug in the numbers for μ and σ and once the smoke clears, we could write this down in octave as well, it is not that difficult, you just put down this formula and it will give you 530.

So, safety stock is 530. Now, if you think about it- 530 is actually quite a large amount. In fact, it is more than the average demand in 1 week. Of course, it is less than the average demand two weeks, our's more than the average demand in 1 week.

Now, the reorder point; now this is interesting, when do you reorder? So, this is when the distributor says: "ok. I have reached my threshold, I need to order". So, the reorder point is going to be this guy: the average demand in an average lead time. Like, the average lead time is 15 days, the average demand is 400. So, this is the demand that you are going to see in 15 days, but we want to be sure that our units are matching. So, you want to be sure that you get this as a number.

So, therefore, you might divide it by t and thus how much should we have. So, when the total inventory level reaches 1387, which is way more than 2 weeks of demand, 2 weeks of demand is 800, this is way more than 2 weeks of demand and this is how much of inventory you will carry.

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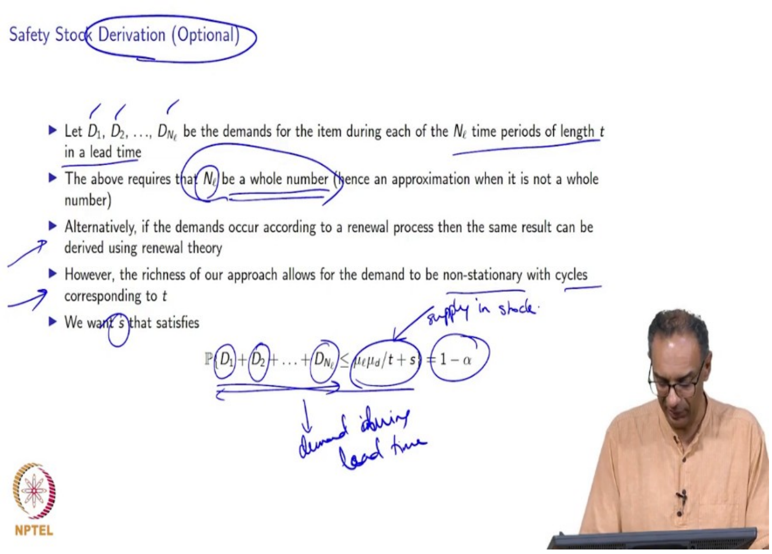

Now, we will do an optional derivation. Again, the reason I call it optional is because I am not going to expect you to be able to derive this in the exam. However, it is important for you to understand how these things are computed.

So, now, the way we are going to do this is: now were going to use that N_l is the time periods in a lead time. So, how many ever time periods in a lead times, are going to be in a lead time. So, D_1, D_2 and so on up to the D_{Nl}, which is a demand during each of the N_l time periods in a lead time. Now, we do require this N_l to be a whole number. So, this is really an approximation. It is not that big a deal if there are multiple time periods of N_l if it is somewhat reasonably large. It is not that big a deal. If it is tiny, it is kind of an issue. So, pretty reasonable approximation.

Now, we could do something different. Although that could also be an approximation is; if you had arrivals according to a renewal process, there is another way to solve this. Renewal processes is beyond the scope of this course. So, we will not go there, but that is another way. However, that has a requirement that the demand process be independent and identically distributed one after the other. That do not does not have to be the case. Let us say your demand is for 1 week, maybe in the weekend you had a higher demand, week days you had lower, it did not matter, your period is 1 week.

So, in some sense this is a decent approximation of thinking of this as a whole number. And on the other hand, if your demand is- does not matter which day of the week it is, which sometimes happens in cases like the LPG gas cylinder, where you have to cook food every day at home. Anyway does not matter whether it is a weekday or weekend, maybe it does not change a whole lot and you could perhaps do a renewal process type of argument. That is another route that one can take, I am sure there are papers out there that talk about this in the literature.

Now, this is what I just said which is that, we do not need now and we allow with non-stationary as long as we have cycles: weekly cycles or something. Now, we want an s that satisfies this equation. Now, let us just look at this really carefully. This is the demand in the first, in the second, and in the last of the time periods during lead time. The total demand during a lead time- during all the periods must be smaller than what you had in hand.

At least this fraction of time. So, the probability that the demand during the lead time is smaller than what you have, we want that one to be smaller than your supply in stock. So, how much you have in stock must be larger than your demand and lead time, that probability needs to be less than or equal to $1-\alpha$.

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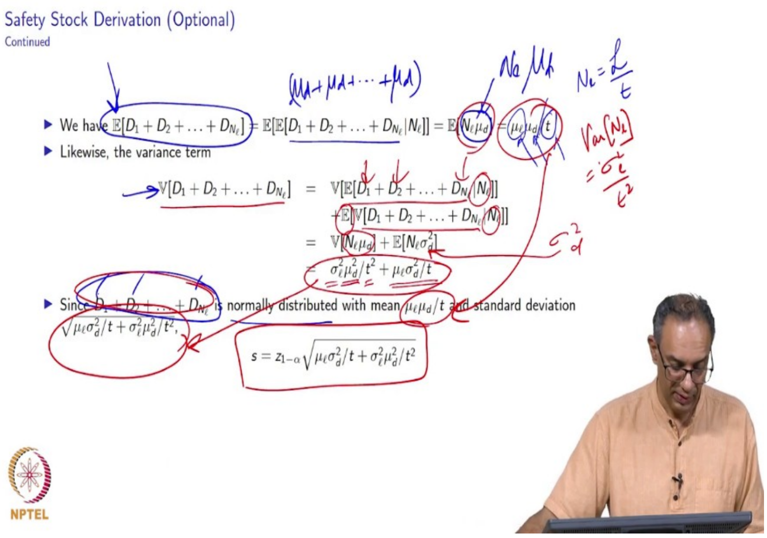

$E[D_1 + D_2 + D_3 + \ldots + D_{Nl}] = E ¿$

$$¿ E[N_l \mu_d] = \mu_l \mu_d / t$$

Now, I might use this a little bit later to explain a few things if necessary. I am skipping that page. So, now, we have derived and we have essentially shown that D_1, D_2, D_3 and so on we want to compute the probability. But, we know that the sum is normally distributed. Why it is normally distributed? It is normally distributed because each one of them is normal and therefore summation of normal is also normally distributed.

So, two questions we are going to ask is: So, what is the mean and what is the variance of that? So, the mean of the sum of this, we use our standard expected value of a function of a random variable and compute this. So, we condition on N_l being equal to value N_l. Then, this sum will be just $\mu_d + \mu_d + \ldots + \mu_d$, each of them is identical. So, all of them are equal to μ_d. How many of them we have?

Well (Refer Slide Time: 12:08), we have N_l of them. Therefore, what inside this bracket is: $N_l \mu_d$, and the expected value of that is going to be μ_l over t. Remember that, N_l equals l divided by t. So, the expected value of N_l is the expected value of l divided by t, t is a constant. So, expected value of l is μ_l and this is the μ_d, we have and this is divided by t, all right. Now, the variance term, this is another result that we had before. So, maybe I will use this here.

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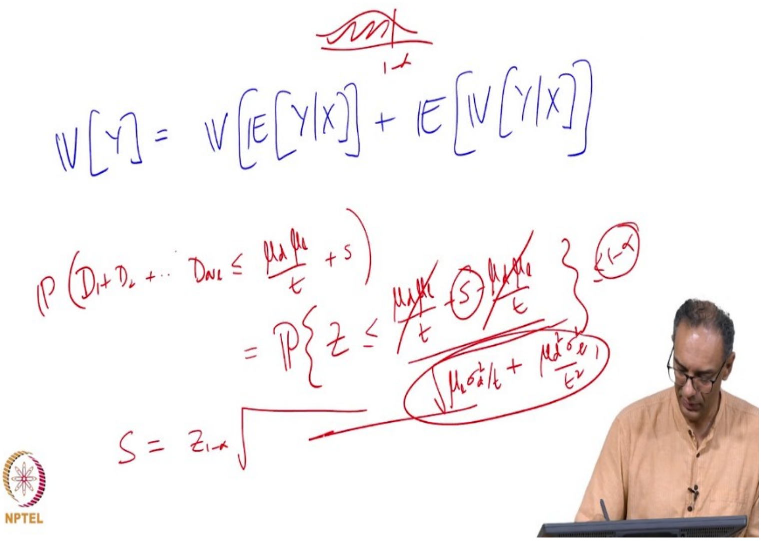


$$V[D_1+D_2+D_3+\ldots+D_{Nl}]=V¿$$

$$+E¿$$

$$¿V[N_l\mu_d]+E[N_l\sigma_d^2]$$

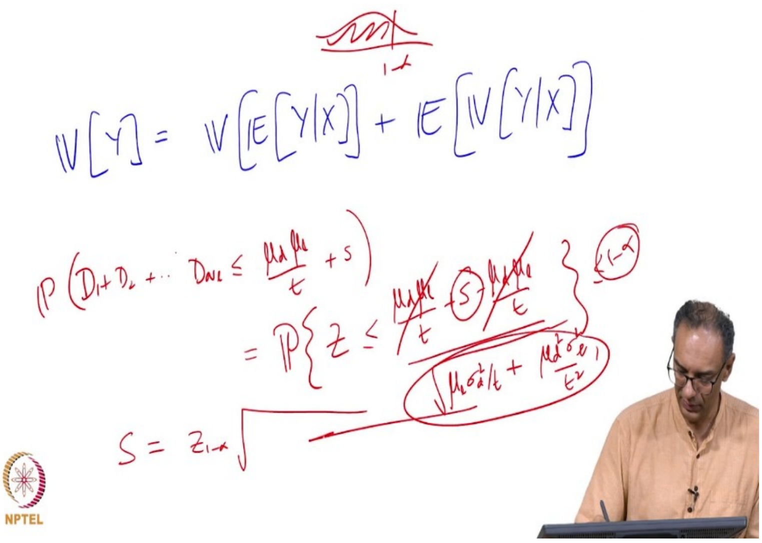
$$¿\frac{\sigma_l^2\mu_d^2}{t^2}+\frac{\mu_l\sigma_d^2}{t}$$

The variance of Y is equal to the variance of the expected value of Y given X plus the expected value of variance of Y given X. We are using that result in the next derivation right here and for that if you look at it. So, the variance of the sum is equal to the variance of the expected value that given N_l, but the expected value of the variance of that given N_l. So, we do both. We compute the inside part first, so this is N_l times μ_d because the same argument that we had before right here; so the $\mu_d+\mu_d+\ldots+\mu_d$ so on.

Now, the variance part- each of them has a variance of σ_d^2, there are N_l of them therefore that what gives us N_l times μ_d^2 because that is a variance. So, that is variance is μ_d^2. Now, the variance of this- remember that the variance of a constant multiplied by random value is the square of a constant that is what gives me the μ_d^2 times the variance of N_l.

So, the variance of N_l is just the variance of l which is sσ_l^2 divided by t^2 and that is what I have here, $t^2$and σ_l^2. And this one is the expected value of N_l times σ_d^2, $\sigma_d^2$is a constant that comes out and that multiplied by the expected value, which we had before which is $\frac{\mu_l}{t}$.

Now (Refer Slide Time: 14:22), that this is normally distributed with this mean that we have from here and this standard deviation that we have from here, we get this result, that we have here. Now, what this one tells me is- since this guy is normally distributed, let me the previous one, $D_1+D_2+D_3+\ldots+D_{Nl}$ is normally distributed.



The probability that $D_1+D_2+D_3+\ldots+D_{Nl}$ is less than or equal to $\frac{\mu_d \mu_l}{t}+s$ is equal to the

$$P\left(z \leq \frac{\frac{\mu_d \mu_l}{t}+s-\frac{\mu_d \mu_l}{t}}{\sqrt{\frac{\mu_l \sigma_d^2}{t}+\frac{\sigma_l^2 \mu_d^2}{t^2}}}\right).$$

Now, we want this to be less than or equal to *1-α* which is exactly, so these two gets cancelled, which is exactly the same as saying: “my value of s is equal to $z_{1-\alpha}$”. This is the value where the area to the left is equal to *1-α*. So, the area to be left is *1-α*. So, $z_{1-\alpha}$ times the square root ($\sqrt{\frac{\mu_l \sigma_d^2}{t}+\frac{\sigma_l^2 \mu_d^2}{t^2}}$) that is sitting here. So, you have that here and that is what is used in deriving this result,

$$s=z_{1-\alpha}\sqrt{\frac{\mu_l \sigma_d^2}{t}+\frac{\sigma_l^2 \mu_d^2}{t^2}}$$

So, that brings us to the end of this presentation.

Thank you.