MCDM Techniques using R
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Lecture – 7
Analytic Hierarchy Process (AHP) – Part IV

Welcome to the course MCDM Techniques using R. So in previous few lectures, we have been discussing AHP that is Analytic Hierarchy Process. Specifically, we have being doing a modeling exercise in R, so we will get back to that point where we stopped in the previous lecture.

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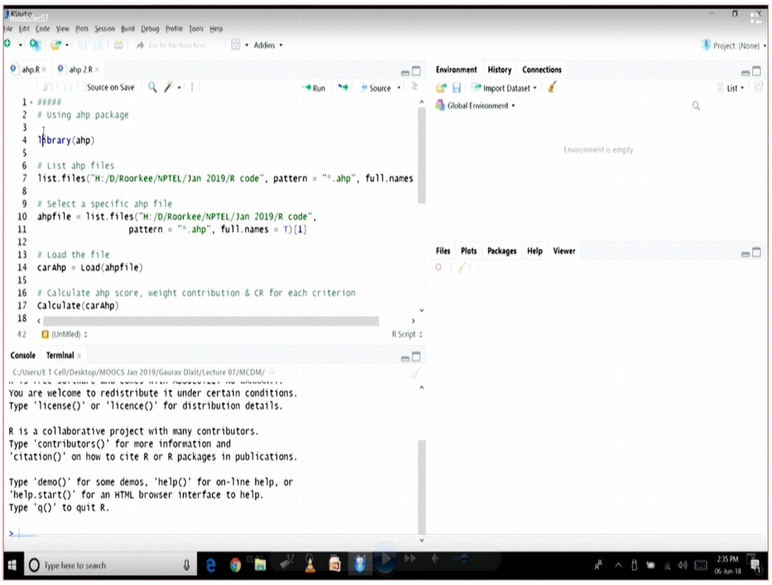

So let me open RStudio. So we were doing this choice of a small car or buying a small car goal that were doing. So we talked about two specific packages in R that are available for modeling AHP. So first package that we talked about in in the in previous lecture was the MCDA package and the second package which is slightly more advanced is ahp 2 package. So first package MCDA package we had already covered in the previous lecture and we were in the middle of using this second package that is ahp package.

So we will restart some of the steps that we have done in previous lecture. So again we will start with loading this library ahp and then this particular file we need load again we need to copy just like in the previous lecture. We need to copy the location of this file. So after loading this ahp library, we will locate our ahp files which we talked about in the previous lecture. So, we will use this particular line of code and you can see there are two ahp files. So as we did in the previous lecture, we will select this one car2.ahp so let us select this file.

Once this is done, the first thing that we are going to do is load this ahp file here using this load function and you can see in the environment section few things are changing. Here you can see ahp file and car ahp, these are two variables that have been created and the same are visible in this environment panel. Now next thing is we need to calculate ahp scores, weight contribution and CR that is consistency ratio. So, all that we can do using this one function calculate. So this one function does most of the computations that required under this package.

So we will run this code, and once this is done if at this moment you would like to visualize the tree diagram, then visualize function is available and we will just have a click here. We will see this in the viewer panel here, you would see this tree has been loaded. So this particular thing also we did in the previous lecture. Now, we come to the next line of code, so here we are using this analyze function and the same argument car ahp is being fast. So actually this particular variable is dynamically linked.

So therefore each function is using this variable and doing its own computation and we are just using the function and displaying the results. So let us run this code and you would see. So in the previous lecture, I think we stopped at this point. So we produce the results, so if you see look at the results and compare it with what we did in previous lecture using the other package that is MCDA package. We just got the weights there.

The final globalized priorities for different alternatives and those priorities could be used to produce the ranking, but in this particular package, we can actually have much more details as you can see here in the console panel in the results. So you can see here that we have these four alternatives, for this example these four cars, and you can see the priorities in the percentage format, you can see here the results are similar and you can also see the inconsistency which is displayed here.

So one of them is for the criteria, then the others there were four comparison matrices and for each one of them this is displayed. Now you can also see that against these criteria reliability, style and fuel, we have these scores which are actually the priorities for these criteria and within these criteria for different alternatives, we can this percentage scores, which are the priorities for these alternatives with respect to these criteria. So with respect to criteria reliability, we get these priorities for alternatives.

With respect to criterion style, we get these priority scores for the alternatives, and then with respect to fuel criterion, we get these priority scores for our alternatives. These scores as I have talked about under the weight column, so these are actually the priority scores for criteria right, and these 4 values that we see here, these are the global priority scores. So these are the ones actually which can be used to produce the ranking. These 4 values were actually produced in the decimal format in the other package that is MCDA that we had used.

In in this package, you can see here the inconsistency that is also displayed here in the percentage format, and as we have talked about for a particular comparison matrix to be acceptable, this value should be less than 10%. So therefore, we can find out now for one of the matrix this particular value is actually greater than 10%, so therefore this needs to be reconsidered. So now let us move on So there is another function that is available under this package which is called analyze table.

So this function can also be used to display the same results, but in a more beautiful format. So, we will see that in the viewer tab here where we have this hierarchical structure for our model. We will see another output there. So let us run this code and in this viewer panel, you would see the same numbers which we saw the in the console, now here they are being displayed in another fashion. So this is about the AHP package and the different results that we can produce.

So in this particular package as you can see here, we can produce the priority scores for criteria, alternatives and the global priorities, the inconsistency scores. So, all those scores can be produced. There are few more functions available under this package which can also be used for certain information. So name of these functions are for example if you want to get priorities, so there are different methods to actually calculate priorities. So, we are going to talk about those methods later in this lecture.

So how these priorities are derived, what is the underlying mathematics behind these priorities calculation, so that we will talk about later in the lecture, but for the same, we have few functions also under this package. So as we will see later on, we will talk about 3 popular methods. One is Eigenvalues method. Then there is another method which is called the approximate method, here it is just being called the MeanNormalization in this function, and then there is GeometricMean method.

So these are three mean methods which are actually used to perform the main core mathematics in AHP method. So what these method will do for a given matrix. We would be able to generate priorities. So for this example that we talked about, we had 4 comparison matrices, and so for each of those matrices using these particular function. For example typically Eigenvalue method is typically used in most of the packages including most of other softwares including this R environment and these packages as well.

So we can always use eigenvalues method and the input is going to be the comparison pairwise comparison matrix and we will get the priorities. So if you are just interested in finding out what are the priorities and how they are being computed, what are the form the pairwise matrix, then we can use this function. More information on this functions and any other function that you have seen in this lecture, you can always go into the help section.

You can always type the names of these functions and full detail would be displayed to you and you would be able to find out more specific details. So what we are going to do is we will run these codes. So as you can see before we can even run this code, we need to create this matrix style that is criterion. So for this, we will go back to our previous RScript where we have actually created this matrix. So let us run this code.

So if you remember, this particular matrix is for comparison of alternatives with respect to style criterion, so these are the values. So we want to, given these values, these pairwise comparisons given in this matrix form if want to compute the priorities from this, then we can go back to this second script. The first here, this particular line of code and this function, if we just use this function, we will get the weights. So you can see again here for each of the alternatives, we have got these priorities in decimal format and you would also see consistency scored there.

So the previous output that we had seen right 16.4%, this score for inconsistency, and the same thing you can see here. So these functions again can be used to compute the priority scores given a particular matrix, and then if you are not interested in Eigenvalue method, then of course you can go for MeanNormalization which is also called approximate method or for GeometricMean. So, we will also run these lines of code. Now you would see that in MeanNormalization, you would see the weights here.

If we compare the results of these two functions, so you can see the weights, these scores that we have got using Eigenvalue and using this MeanNormalization which is also the approximate method, they are slightly different. So, why this difference is there. Because essentially they are different methods, so therefore the computations that are performed the mathematics that done which is slightly different. So later on in this lecture, we will talk about what these methods are and how this calculation is done.

For right now, as you can see in this R code that different methods can be used to compute priorities from pairwise comparison matrices and the scores are slightly different, but if we look at from the perspective of what is the final output that is ranking, then probably if the matrix is consistent from all these methods, we will get the same output. Now in this method, in the second method the MeanNormalization method, you would also see that the consistency score is not given so that is mainly because consistency score cannot be calculated under this method.

We will talk about this in more detail later on later in this lecture. Now the third function that you can see here in the script, this is called GeometricMean, you can see here GeometricMean, for this function is called GeometricMean, and again on the same matrix, we will apply this and again you can see the scores which again you can see slightly different from previous 2 methods. So if you compare, there is slight difference in terms of scores that we have got here.

Here again in this method also, you would see we would not have any consistency scores. So we want to have a look at the consistency, we want to perform the consistency check, then Eigenvalue is the only method that is going to give us that. Now, let us go back to our discussion, there we will talk about few more important aspects of AHP. So let us get started. So I will directly go to the part where we will talk about these methods, so this was the part where we were discussing. So now we have completed our modeling exercise in RStudio for choice of a small car.

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ANALYTIC HIERARCHY PROCESS (AHP)

- Further Comments on AHP
  - Problem structuring
    - A structure must be formed through
      - Brainstorming sessions,
      - Analyzing similar problem studies, and
      - Organizing focus groups etc.
    - Structuring of decision elements into a hierarchy is an important step
      - A different structure may lead to a different final ranking
      - Criteria with a large number of sub-criteria tend to receive more weight than when they are less detailed

Now we will move to few more points on AHP. So let us start with problem structuring, the first step. So the few points we have already discussed in previous lectures, few important points have also been mentioned here, for example how the structure has be formed, so brainstorming sessions, analyzing similar problem studies because you remember when we talk about formulating a goal, we also need to consider which are the criteria which are going to be part of solving the decision problem.

So therefore for that, you might require brainstorming sessions, sometimes analyzing similar problem studies can be helpful because you can directly infer some of the important criteria from there. Organizing focus groups, this is another way to do the same thing. So in problem structuring phase, some of these methods can be applied before we can move ahead. Now the next important aspect is structuring of decision elements into hierarchy.

So this is important step because if we happen to create one structure out of one particular brainstorming session and another focus group discussion or brainstorming session, we happen to create another structure. So if the structure is different, then the final output is going to be different, so the output that we are going to get from our AHP modeling is going to be dependent on the structure that we create, the hierarchical structure that is part of the AHP technique. So, a different structure will lead to different final ranking.

At this point, I would also like to remind you another important aspect of AHP. So when we consider criteria, under each criteria, we consider number of sub-criteria. So typically what happens is some criteria they have more number of sub-criteria. So the way AHP is done the way whole modeling and mathematics is performed, typically criteria with large number of sub-criteria, they will receive more weights this is just because the more number of criteria that they have.

So both these points convey the sense that the structure that we create, the brainstorming that we do for a decision problem and the hierarchical structure that we create that is going to determine the kind of output that we get. So let us move forward. Now few more points on judgment is scale. So the scale that we typically use in AHP, I have already talked about this particular aspect in previous lecture, that more often than not we use the most popular which is fundamental 1 to 9 scale that we use.

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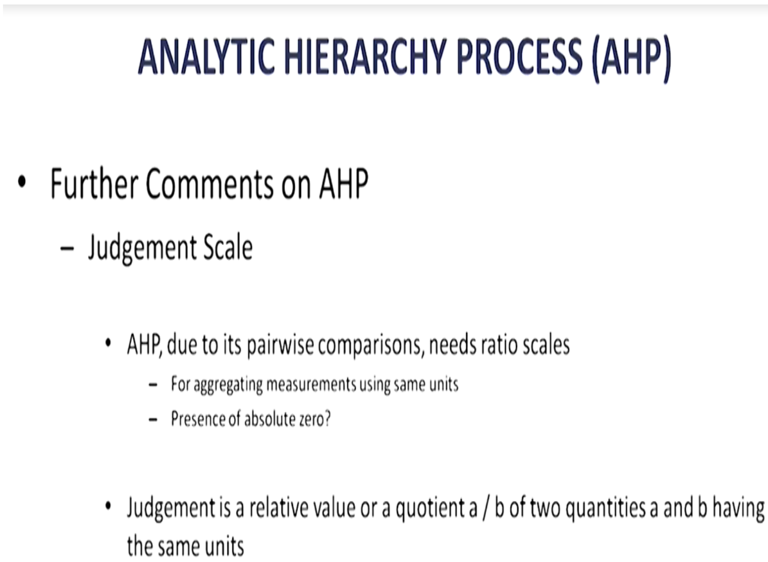

Now in AHP as we have talked about due to pairwise comparison, we seek pairwise comparisons from decision makers, so these pairwise comparisons actually need ratio scales. So, why need ratio scales this is basically because later on, we will be aggregating measurements and for that we require same unit. So some researchers have argued about why this scale is being used because there is presence of absolute zero that is questioned.

However, because the same units are to be used for aggregation later on the aggregation step that is part of AHP technique, for that purpose we need ratio scales because if we happen to use interval scale then if this unit is changed, then that can impact the result. So therefore, ratio scales are referred now in terms of judgment. This is a relative value or a quotient a divided by b of two quantities a and b having the same units. Now let us talk about few more points on consistency.

So till now we have talked about the importance of consistency check in our RStudio exercise. We have also done that particular part.

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ANALYTIC HIERARCHY PROCESS (AHP)

- Further Comments on AHP
  - Consistency
    - A matrix filled by the pairwise comparisons is called consistent if the transitivity and the reciprocity rules are respected
    - For a perfectly consistent matrix, comparison matrix element, a_{ij}
      - Which is the comparison of row i criterion/alternative with column j criterion/alternative

    obey the following equality:

$$a_{ij} = \frac{p_i}{p_j}$$

Where p_i is the priority of the criterion/alternative for row i and pj is the priority of the criterion/alternative for column j

Now what do we mean by consistency in AHP. So here it is defined, a matrix filled by the pairwise comparisons is called consistent if the transitivity and reciprocity rules are respected. If these two rules are respected, then we call that the matrix is consistent. If the matrix is perfectly consistent, then there are some other properties which become part of it. For example given a comparison matrix element aij, so this is typical notation that we use in the matrix algebra where aij i is denoting the row and j is denoting the column.

The aij is the element within the matrix for row i and column j. So this particular value if aij, this is comparison of row i criterion or alternative with column j criterion or alternative. So we have the pairwise comparison matrix. So any value within that matrix, it will be comparison of row i criterion or alternative with the column j criterion or alternative. So if the matrix is consistent that means it is following these two properties transitivity and reciprocity.

If these two properties are being followed, then you would also see that this value aij would follow this equality, aij is going to be equal to pi divided by pj where pi is the priority of the criterion or alternative for row i and pj is the priority of the criterion or alternative for column

j. So typically, the whole method that we apply AHP the pi priorities are not known rather they are calculated when we apply this method.

However, if these priorities are known, then we would see that if the matrix is consistent, this equality would be obeyed and instead using of aij, we can use these ratio of pi and pj and we will be able to construct perfectly consistent matrix and that is how we know that this particular property is there. So let us move forward.

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ANALYTIC HIERARCHY PROCESS (AHP)

- Further Comments on AHP
  - Consistency
    - Most commonly used method is : consistency ratio (CR)

      Developed by Saaty (1977)

      Related to the eigenvalue method for priorities derivation

      $$CR = CI/RI$$

      Where CI is consistency index and

      RI is the random index (the average CI of 500 randomly filled matrices)

      $$CI = \frac{\lambda_{max} - n}{n - 1}$$

      Where λ_{max} is the maximal eigenvalue and n is the dimension of comparison matrix

Now as we have talked about for performing consistency check, we use certain matrix and one of the most popular matrix is consistency ratio. So how this consistency ratio is derived. So developed by Saaty and this is related to Eigenvalue method. So in the RStudio environment and in Wireman exercise that we did, we talked about 3 functions and we saw that consistency score was calculated only in one of the method, that is eigenvalue method.

So this particular consistency ratio is also related to this eigenvalue method, which we are going to talk about later, so related to eigenvalue method for priorities derivation. So this consistency ratio CR can be defined as CI divided by RI where CI is the consistency index and how this index is derived. So this particular aspect is also related to eigenvalue method. Now you can see the formula for CI is written there CI is lambda max minus n divided by n minus 1 where lambda max is the maximum eigenvalue and n is the dimension of comparison matrix.

The other component is RI that is random index. So as we have talked about in the previous lecture also, so this average CI of 500 randomly filled matrices so that is taken and we get the random index. So you will get more clarity on how this consistency ratio is actually derived once we discuss the eigenvalue method, but at this point, you can see that the CI the consistency index that you are seeing here, formula that you are seeing here, here the lambda max - n is actually indicating the inconsistency that is there in a particular matrix.

If the matrix will be perfectly consistent, then lambda x value would be equal to n, otherwise there would be difference and that can be indicated here in the consistency index and then we divide it by RI which is the random index and we get the consistency ratio.

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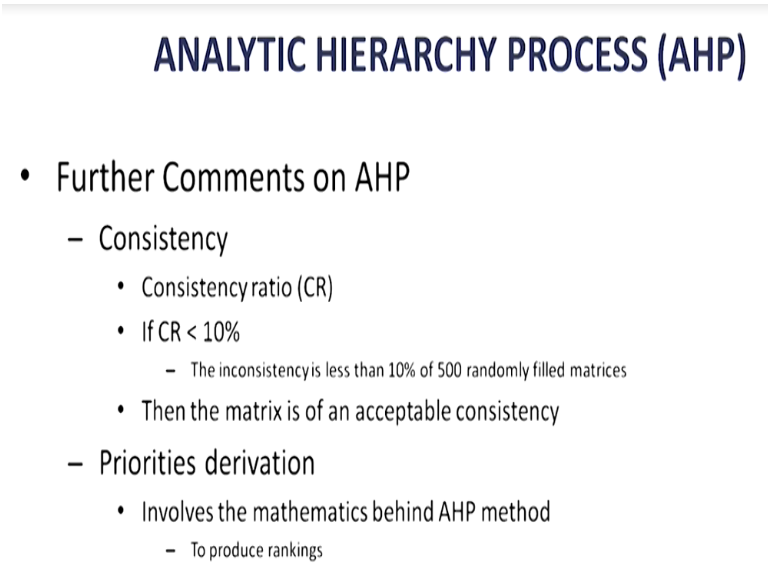

So if we have the score for consistency ratio if it is less than 10%, so as we have talked about, then this is acceptable consistency, otherwise we will have to reconsider the comparison matrix. Now let us talk about the next step that is priorities derivation. So as I have talked about this contains the mathematics behind the AHP method and which is finally used to produce rankings.

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ANALYTIC HIERARCHY PROCESS (AHP)

- Further Comments on AHP
  - Priorities derivation: three common methods
    - Approximate method
    - Eigenvalue method
    - Geometric mean
  - Each method calculates identical priorities when matrices are consistent

So under this, we will talk about 3 methods; approximate method, Eigenvalue method and Geometric mean. So some of the discussion that we did in this lecture that would be more clear once we discuss eigenvalue method. So as I talked about if the matrix is consistent, then all these methods would actually be producing the identical priorities. So let us understand these methods one by one. So first one is approximate method there are two main steps.

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ANALYTIC HIERARCHY PROCESS (AHP)

- Further Comments on AHP
  - Steps in Approximate method
    - For each matrix row i, summation of the row elements

$$r_i = \sum_i a_{ij}$$

Where a_{ij} is the comparison of row i criterion/alternative with column j criterion/alternative

- Normalization of the sums from previous step to calculate priority p_i for each row i,

$$p_i = \frac{r_i}{\sum_i r_i}$$

- Approximate method does not calculate the consistency of the matrices

So first of all each matrix row i, we perform this summation of row elements. So row would be corresponding to either criterion or alternative and those comparisons values are going to be there along the row. So in the first step, we are going to summate all those values and after that normalization is going to be performed. So in the second step, you can see the sums, if suppose the matrix is 3 by 3, so therefore we will get 3 sums for each of 3 rows.

Then normalization step is going to be performed using this formula simply pi = to ri divided by summation of ri. So, this normalization itself will also give us the priority score. So these are the two main steps, very simplest steps, which are part of approximate method. However, the problem is that the approximate method does not yield us the consistency, we cannot do a consistency check as we have seen in our RStudio exercise.

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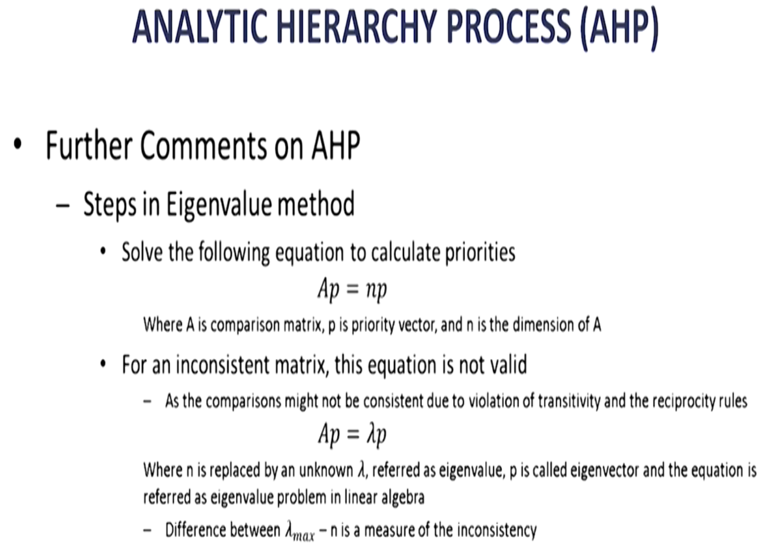

If we look at the eigenvalue method, so this is the equation that is Ap = np where A is the comparison matrix and p is priority vector and n is the dimension of A. So this equation has to be solved to calculate priorities. If the matrix is inconsistent, then this equation is not going to be valid and we use another unknown variable called lambda. So the equation becomes now Ap = lambda p, in this lambda is actually referred as eigenvalue and the p is actually the associated eigenvector.

This particular kind of problem formulation the equation formulation is referred as eigenvalue problem in linear algebra. Now in this here you would also see that this lambda value the maximum value that is there that we can calculate satisfying the equation Ap lambda p. So that lambda max - n, this difference is actually becomes the major of inconsistency.

So this is the value lambda x - n if you remember we have used in the consistency ratio, so we go back. This is the formula for consistency index and you can see in the numerator we have lambda x - n.

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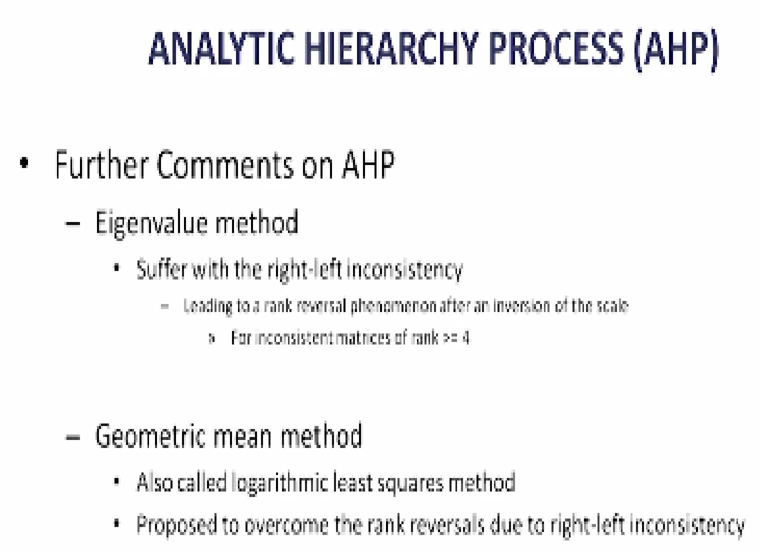

Now few more points about eigenvalue method. So there are some issues with even this method. So for example approximate method, we had the problem that we could not compute consistency. Now here, this method might suffer from right-left inconsistency. So this right-left inconsistency will actually lead to rank reversal phenomenon if the scale is inverted. So what typically happens is if you are using certain scale and if the values are reverse, then what happens is that the scores that we get and the final ranking that we get that might get changed.

However, this does not happen for consistent matrix and it only happens for inconsistent matrices of rank greater than r = 4. Now let us talk about the next method that is geometric mean method. This method is also referred as logarithmic least squares method. This method is actually what is proposed to overcome the problem that we have in eigenvalue method, this right-left inconsistency problem that we have. So to overcome this rank reversal due to right-left inconsistency that we have, this method was proposed.

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ANALYTIC HIERARCHY PROCESS (AHP)

- Further Comments on AHP
  - Steps in Geometric mean method
    - An error term is incorporated in the comparisons as given below

$$a_{ij} = \frac{p_i}{p_j} e_{ij}$$

    - This multiplicative error is
      - To be minimized
      - Assumed to be log-normally distributed
        - » Just like additive error would be assumed to be normally distributed

Under this method, an error term was also incorporated in this equation. So previously, we talked about if we have the perfectly consistent matrix, then aij would be pi divided by pj. However, if that is not the case, this is being proposed where error component is also part of this. So now this error is to be minimized and you can see this is multiplicative error, therefore we will have to do the log transformation we can take ln on both sides and therefore we can derive the equations for minimization of some of errors.

Now, this error is assumed to be log-normally distributed just like we do in linear equation because there it is typically additive error. So if you remember the regression equations are like y = beta 0 beta 1 x1, beta 2 x2 and so on and plus error terms. So that error term is an additive term that is there. So that is typically assumed to be normally distributed. Here the error term is the multiplicative, so therefore this we have to take log transformation and therefore this is assumed to be log-normally distributed.

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ANALYTIC HIERARCHY PROCESS (AHP)

- Further Comments on AHP
  - Steps in Geometric mean method
    - Priority p_i is calculated as below

$$p_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}}$$

    - By minimizing the following error expression

$$min \sum_{i=1}^{n} \sum_{j=1}^{n} (\ln(a_{ij}) - \ln(\frac{p_i}{p_j}))^2$$

So how do we compute priorities under this method. So priority pi can be computed following this equation. So you can see pi at the root of this product all aij multiplied with each other along the row. So as I talked about in the previous equation if we take the ln of it we can derive this equation that I have written here. So these priorities are to be calculated by minimizing this error expression. So if you use this lnai the main component of this expression lnaij - ln pi/pj and the square of it.

This is actually coming directly from this equation aij = pi/pij multiplied with eij, so it is directly coming from there if you take ln and compute the error part.

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ANALYTIC HIERARCHY PROCESS (AHP)

- Further Comments on AHP
  - Aggregation
    - Additive aggregation

$$P_i = \sum_{j} w_j . p_{ij}$$

Where P_i is the global priority of alternative i,

w_j is the priority of criterion j being used as weight, and

p_{ij} is the priority of alternative i w.r.t criterion j

So now, we will come to the next part that is aggregation. So once the priorities, criteria priorities and alternative priorities have been computed, then we can go ahead and compute global priorities. So that is here referred as aggregation and that is why this method also becomes part of aggregation method, AHP is also considered part of aggregation methods.

So you can see here a first and more common approach under aggregation is additive aggregation and you can see the expression here. So priorities the global priorities Pi this is computed using this summation with wj multiplied with pij where wj is the priority of criterion j being used as weight so this we have already computed if we have the priorities for criterion and alternatives and then the small pij, this is priority of alternative i with respect to criterion j.

So you can see this weighted sum is being computed here and that becomes the global priority. Now what typically happens in additive aggregation is that we do a distributive mode normalization wherein the normalization of the sum of local priorities to unity is done that means when we divide it by the summation of all the values for normalization step, then that actually sum of those of priorities becomes unity. So the common denominator is going to be part of this calculation.

So what happens is if there is introduction or removal of certain alternatives, so again this common denominator is going to change and because of the change, all other values are going to be changed and therefore the final ranking that we produce it is also going to change. So therefore it will lead to rank reversal. So this is one problem with the additive aggregation. So the solution has been recommended.

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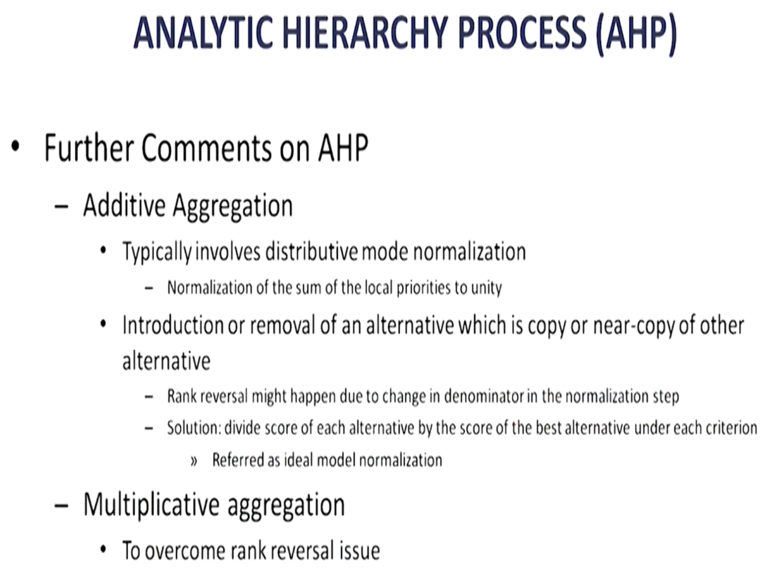

So in the normalization step instead of dividing it by the sum of all the values, what we can do is we can divide it by the score of best alternative. So in that sense, this problem can be sorted out, rank reversal problem can be sorted out. So this particular mode is referred as ideal mode normalization where instead of the sum of all the priorities, we can actually divide the score by the best alternative.

Now there are other methods for aggregation which have been proposed, for example multiplicative aggregation. So the rank reversal issue that we talked about that is part of additive aggregation. To overcome that particular issue, the multiplicative aggregation has also been proposed. So these are two methods under aggregation, additive and multiplicative and two modes we talked about, the distributive mode normalization and the ideal mode normalization. So this brings us to the end of this particular technique AHP.

So under AHP, we have talked about different methods. We have also done an exercise in RStudio using R script using R code and then we have talked about few more aspects of AHP which are important depending on the kind of decision problem that we are going. So some of these points that we have discussed will become important consideration for our problem solving. So with this, we conclude this particular lecture. Thank you.