Security Analysis and Portfolio Management
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Lecture 46
Mean Variance Portfolio Optimization VI

Welcome back, so let us continue. Now we move on to the case of 3 risky securities. This is the real representation of the general case or the end security case. The 2 security case had certain singularities because we could develop the entire theory on a 2 dimensional representation. However, in this case the situation gets more intriguing, more interesting, and it is here that we encounter the aesthetic beauty of this particular theory. So let us get into it quickly.

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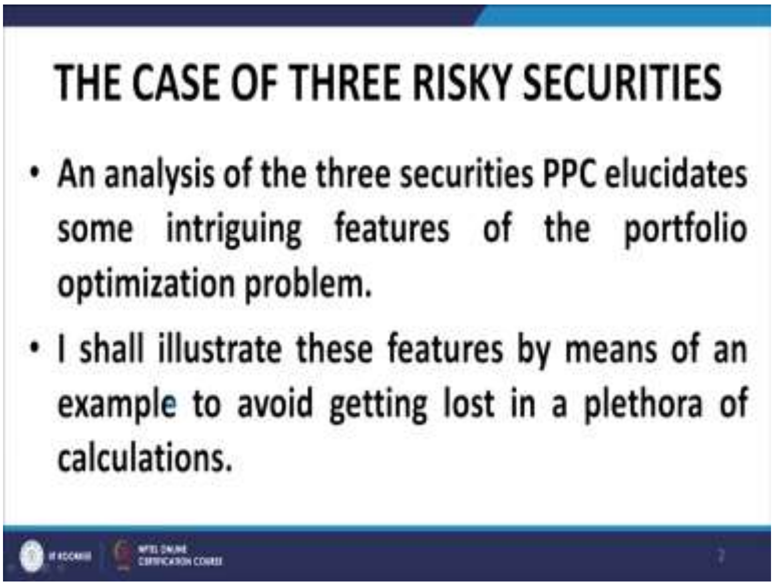

An analysis of the 3 securities elucidates some intriguing features of the portfolio optimization problem. I shall illustrate this by virtue of an example, because what will happen is the calculations and the expressions would be so cumbersome that the real nuances would be lost in the plethora of calculations. I will take this up in the form of an example, and illustrate each and every step which goes into this theory for 3 securities or more than 3 securities. Let us start with the 3 securities.

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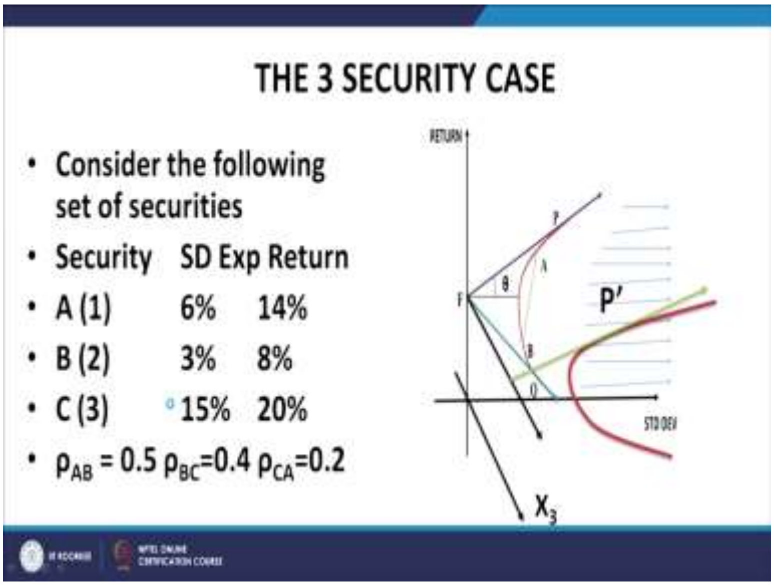

Let us consider 3 securities A, B, and C. The standard deviations and the expected returns of the 3 securities are given on the slide, 6 percent, 3 percent and 15 percent are the standard deviations, and the expected returns are 14 percent, 8 percent, 20 percent for the 3 securities respectively. We are also given the correlation coefficient of between the pair of securities, rho AB is equal to 0.5, rho BC is equal to 0.4, and rho CA is equal to 0.2.

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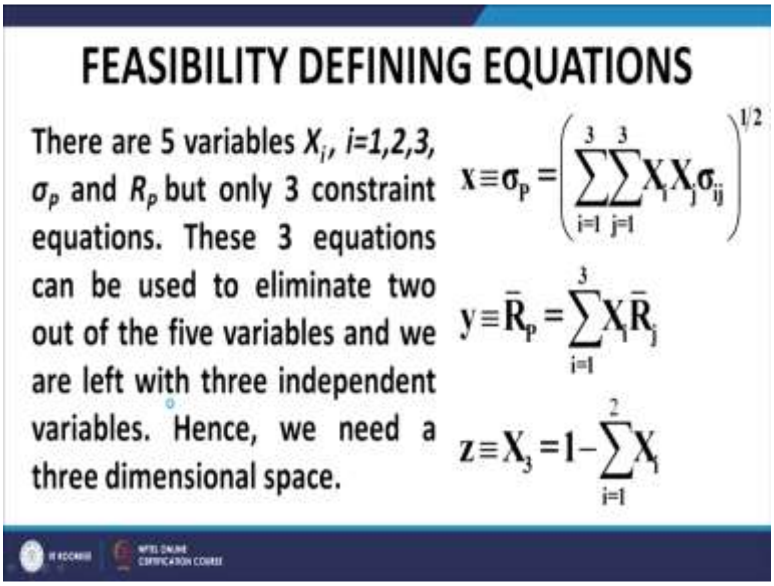

And what are the feasibility defining equations, these are standard, I believe, the learner would have become familiar by now. We have the expression for sigma p of the portfolio which is given here on the right hand side the first equation and the right hand panel. Then the expression for the expected return on the portfolio which is the second equation, then we have the aggregate of the weights must be equal to 1.

I am representing sigma P by x Rp bar that is the expected return by y and X3 by 1 minus X1 minus X2 as I said. Now, the important thing is, please note here we have got 5 variables Xi that is the composition vector, which are 3 components X1, X2, X3. Plus we have sigma P, and we have the expected return of the portfolio Rp. But we have only 3 equations which are shown on the panel on the right hand side.

And therefore, we can eliminate 2 unknowns, but we are left with 3 degrees of freedom. To represent the 3 degrees of freedom I choose the 3 coordinate axis or 3 dimensional space with x representing the standard deviation, y represented the expected return of the portfolio object represent the composition of the third security, security C, which we have so, Z is representing X3.

Now, even we simplify the expressions when we eliminate X1 and X2 and within this panel of equations, within the set of equations, if we eliminate X1 and X2, we get an expression from between or a quadratic between X, Y and Z which is given in this particular slide.

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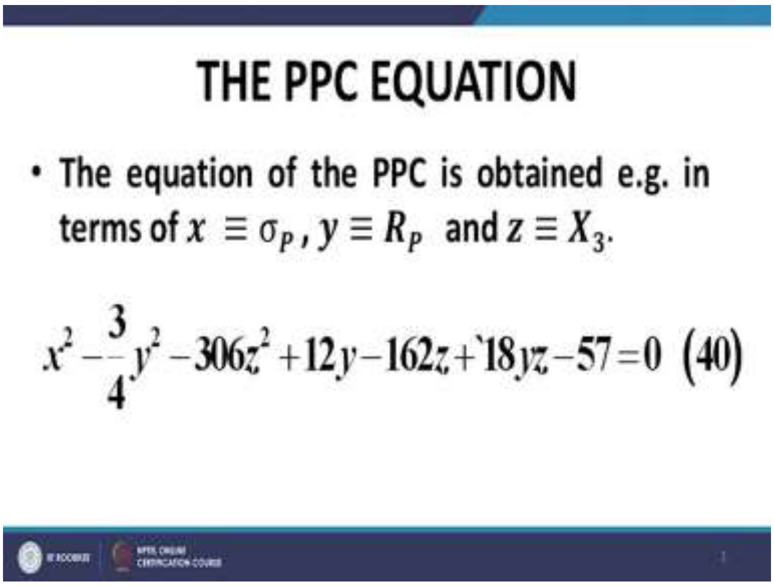

x squared minus 3 by 4 y squared minus 306 z square plus 12 y minus 162 z plus 18 y is at minus 57 is equal to zero. Now, let me repeat this is obtained by eliminating X1 and X2 between the 3 equations that were given in the previous slide if you eliminate X1 or X2, substitute sigma P equal to x RP equal to y and X3 equal to z, what we get is this expression which is represented by equation number 40.

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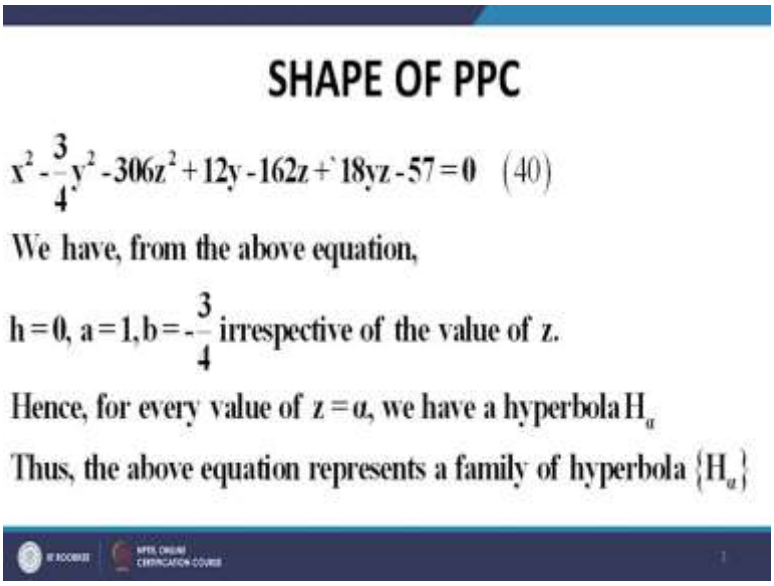

If you look carefully at the shape of this particular equation or the representation of this particular equation in the 3 dimensional space, what we find is that irrespective of the value of z, what we find is that the quadratic or the second degree equation represents a hyperbola. In other words, whatever value of z you take you end up with a hyperbola.

So, what does this mean? This means that given this equation in 3 dimensional space, if I pick up any value of z, say z equal to 1 by 2, then I get a hyperbola by plotting different values of x and y on the plane, which is which has the equation z equal to 1 by 2. In other words, the plane which is parallel to the xy plane, and which has a z intercept of 1 by 2.

For every value of z now, if short sales are allowed, then the value of set can range from minus infinity to plus infinity. For every value of z, let me repeat this is fundamental, this is very important, for every value of z along the z axis, if you pick up any value of said you get a hyperbola with the points on the hyperbola being confined to the plane that is represented by z equal to the given values let us call it alpha.

If you pick up any value of z is equal to alpha then the plotting of x and y with z equal to alpha you pick up that equal to alpha, you fix that equal to alpha and you plot x and y, what you get is a section of the hyperbola just in like in the 2 security scale.

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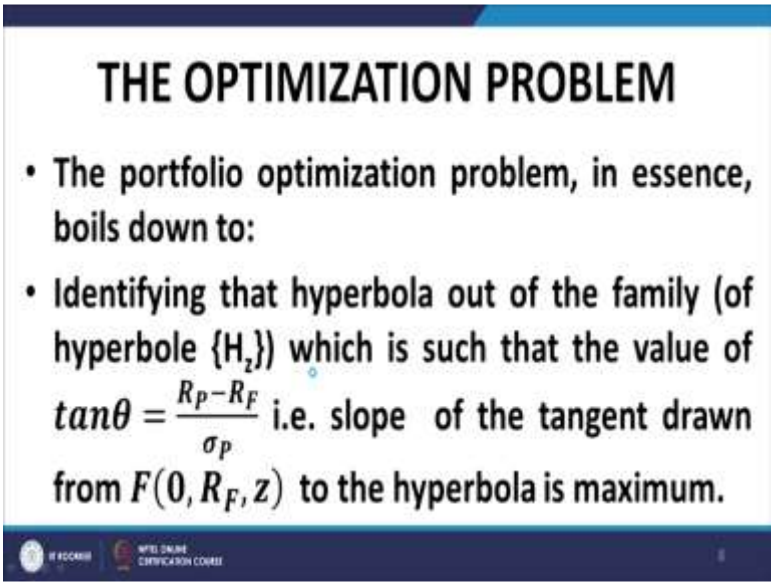

But this section of the hyperbola will lie on the plane that is given by z equal to alpha or a plane which is parallel to the xy plane, but we are at a distance of alpha from it. So that is the shape of this second degree equation. It comprises of a family of hyperola, which can be identified with the value of z because that hyperbola will lie on the plane, which is represented by z equal to alpha parallel to the xy axis at a distance equal to alpha from the xy plane I am sorry, not xy axis xy plane.

So, this will show you, I have done my best to present this situation which I just explained in the form of a diagram, we have the z axis or the X3 coming out from the plane of this particular tablet, and we have the expected return upwards and the standard deviation towards the right, so this is the 3 dimensional framework.

If you pick up any point on the z axis, and you have a plane, which is parallel to the xy plane cutting the z axis at that point, which is to say, let us call it z equal to alpha, then if you plot x and y with z equal to alpha being fixed, you get a hyperbola for example, which is shown here in this diagram as containing the point v dash.

This is how the portfolio possibilities region would look like. Please note, we are examining the case where both were short scales and all the 3 securities are allowed and it is less borrowing and lending is allowed. So, what we do now is we draw tangents from the point zero RF comma z equal to alpha, let us say we identify this plane by z equal to alpha.

So, this particular point, let me illustrate it this point it has the coordinates what are the coordinates of this point? Let me call it if suppose they pick up this point RF, which is lying on the plane z equal to alpha, what will be the coordinates of this point RF the coordinates of this point RF will be x coordinate would be zero.

Because this is representing zero as sigma, sigma P is equal to zero and this is the zero risk point, the y coordinate will be equal to the risk free rate and this z coordinate will be equal to alpha, because it is lying on the plane is equal to alpha parallel to the xy plane.

The coordinates of this point will be zero comma RF comma alpha, what we do is from this point zero comma RF comma alpha, we draw a tangent to the hyperbola that is lying on this plane z equal to alpha by plotting various combinations of x and y. We get the hyperbola let us say RFP dash I am sorry, we get the tangent, not the hyperbola.

We get the tangent RFP. dash which is tangent to the hyperbola that lies on the plane is equal to alpha and which is obtained, the hyperbola is obtained by plotting various combinations of x and y with z equal to alpha being fixed. Similarly, for each value of alpha, we will have a similar situation for alpha equal to say alpha 1 alpha 2 alpha whatever.

So, basically the portfolio possibilities region would be a combination or a family rather, family of hyperbola and get together with these kind of tangents depending on what we choose for the position or the scenario that we choose in, far as short selling of ABC is allowed or not, and whether risk free borrowing is allowed or not.

This would be the hyperbola and this would be the tangent. So, then you can apply the theory that we have discussed earlier for the 2 securities case. So, this is how it would evolve this is how the situation would evolve. Now, our problem is to obtain the optimization or the efficient frontier, how to obtain the efficient frontier let us look at the case where we have short sales in all the most general case, short sales in all the securities is allowed a risk free lending and borrowing is allowed.

Then what did we do in the previous case, when we discuss it to securities case, we try to maximize this slope of this particular line RFP dash which would give us a tangent. In this case, what we'll do what we will do is corresponding to each is equal to alpha we will try to maximize this particular this slope RFP dash.

For every value of alpha we have this kind of hyperbola we have this kind of situation and on every value of hyperbola we try to maximize this slope and we will get different points like P dash on each of the hyperbola corresponding to each value of alpha. Out of this family of hyperbola and these tangents to each of these hyperbola we pick that particular hyperbola, we pick out that particular hyperbola listen carefully, which for which the slope is the maximum.

The first step is that we try to find out the slopes of all these hyperbola which fly on different planes parallel to the xy plane at different points along the z axis. For each of these planes, we will get a particular tangent which has a maximum slope because the tangent is obtained by maximizing the slope.

Then we out of this for maximum slopes, we pick out that slope which is which has the maximum value that gives us the optimal hyperbola, the hyperbola that we now need to focus on and the corresponding intercept or the plane on which that optimal hyperbola lies will be represented obviously, again it could be a plane parallel to the x axis and that would be represented by let us set some value of z, let us say z equal to alpha.

Then z equal to alpha plane which has now been selected as the plane which contains the hyperbola for which the tangent has the maximum slope that is our optimal hyperbola. And then the next step is… this is the first step of the optimization process, maximize the slope of the hyperbola. The second step is and then pick out the hyperbola with the maximum of those maximize slopes.

The second step is that we know try to work out the coordinates of the point of contact of this particular special hyperbola which corresponds to the maximum slope with the tangent. In other words at point P dash which we are represented in this diagram, we try to work out the coordinates of this P dash where this P dash is the hyperbola, which is the optimal hyperbola, which is the hyperbola having the slope of the tangent.

So, this point P dash coordinates we can work out just as in the case of 2 security problem, x and y coordinates can be worked out, and the z coordinate will obviously be z equal to alpha because this entire hyperbola as well as the tangent there to rely on the plane, z is equal to alpha which is parallel to the x y axis. Let me recap all the steps and then we will move forward.

The portfolio optimization problem, in essence, boils down to number 1, identifying that hyperbola out of the family of hyperbolas represented by Hz that is the point is the point on this z axis, an arbitrary point on this z axis, which is such that the value of tan theta RP minus RF upon sigma P is that is the slope of the tangent drawn from F comma, RF commerce z to the hyperbola is maximum.

For each of these hyperbolas, we try to draw the tangent, and then out of those, our entire set of hyperbolas with their tangents, respective tangents from these points, zero RF and z, we find out that hyperbola for with the slope is the maximum that is the special hyperbola which has the optimal hyperbola, let this hyperbola H alpha, which has the maximum slope.

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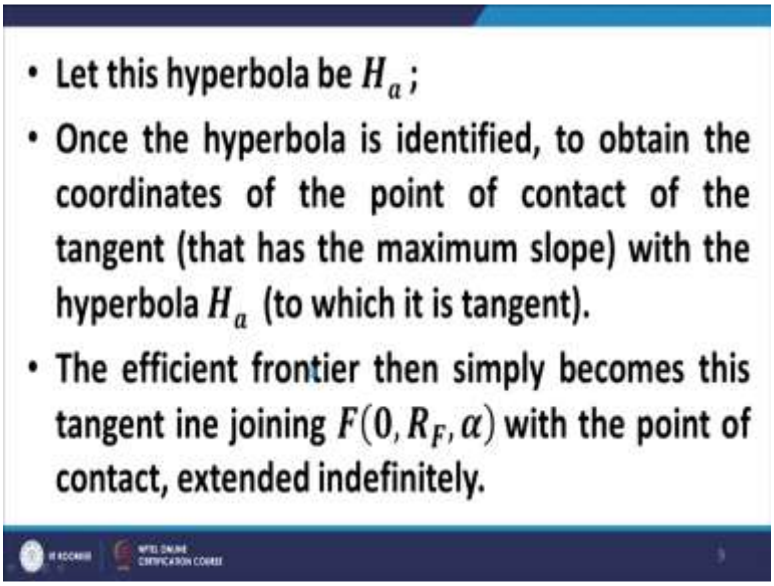

Once the hyperbola is identified to obtain the coordinates of the point of contact of the tangent that has the maximum slope with the hyperbola H alpha to which it is tangent that is the next step. First of all, we identify the hyperbola H alpha, which is such that the slope from the point zero RF comma z or in this case, it would be z equal to alpha to the hyperbola is the maximum of all those hyperbolas and all those tangents to those hyperbolas, then we find the point of contact of this tangent with this optimal hyperbola.

Now, the efficient frontier in this case, where short sales and all the securities is allowed. And the risk free lending and borrowing are allowed, simply the tangent line joining the point zero RF alpha and the point of contract was extended indefinitely, this is the point that we are talking about is a point of contact of the tangent with this optimal hyperbola. That is the situation, that is the geometry of this particular 3 security case that becomes very intriguing, very interesting. Now we do the math.

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For optimality, Maximize $\tan\theta = \dfrac{\bar{R}_P - R_F}{\sigma_P}$

$$\tan\theta = \frac{\sum_{i=1}^{N} X_i(\bar{R}_i - R_F)}{\left[\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n} X_i X_j \sigma_{ij}\right]^{1/2}} \qquad (41)$$

Obviously, the first step is to find the hyperbola with the maximum slope for which or the tangent with the with the maximum slope for which you maximize the function can take as your call to RP minus RF upon sigma p, we substitute RP as a sigma Xi Ri, and then we use the packet sigma Xi is equal to 1 so that we can write the numerator in the form that is given in equation 41. The denominator is written in the usual way.

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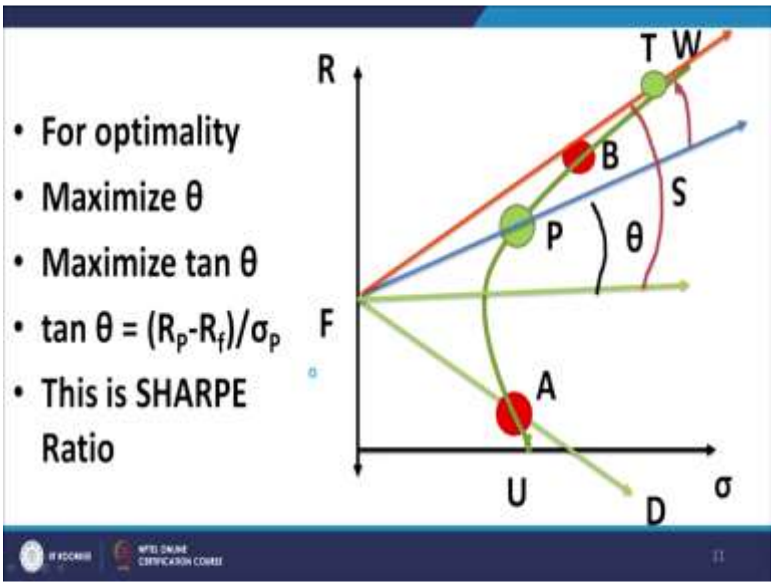

For optimality as you can see, we maximize theta we have discussed this in a lot of detail earlier, this is called the Sharpe ratio.

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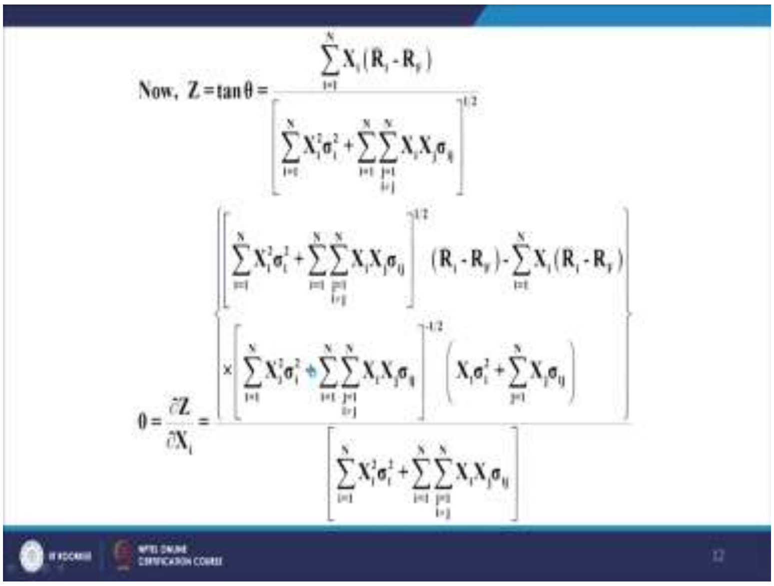

And when we do the maximization exercise, we differentiate with respect to each of these, what are the free variables; the free variables are the composition vectors or the components of the composition vectors. Capital X 1, capital X 2 capital X 3, and so on, if it is an insecurity portfolio, up to capital Xn, and in our case, we are having a 3 security portfolio. So our independent or the free variables are X1 X 2 and X3.

So we differentiate this expression for tan theta or the Sharpe ratio with respect to X1, X2, and X3 partially and then we do some algebraic simplifications. And after doing this algebraic simplification, we get this set of equations, which are represented right at the bottom of your slide, which is called the fundamental form of the optimization equations.

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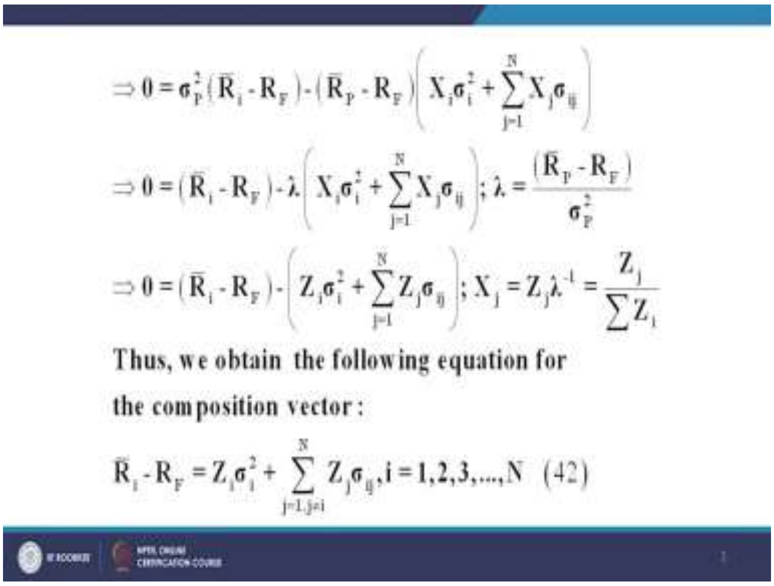

So this is equation number 42, which is the fundamental form. Now please note, these are a set of n equations, we have no unknowns Z 1, Z 2, Z 3, and so on up to Z n. So this should be sufficient to give us a unique set of Z's, Z1, Z2, Z3, and from which we can extract out the X1, X2, X3 by using this expression right at the right hand corner of your slide.

Knowing all the Z's, we can work out the X's as well, straight away. So here we have these n equations in Z's, and we have n unknowns, we can get a unique set of Z's and from there, we can obtain the complete set of components of the composition vector.

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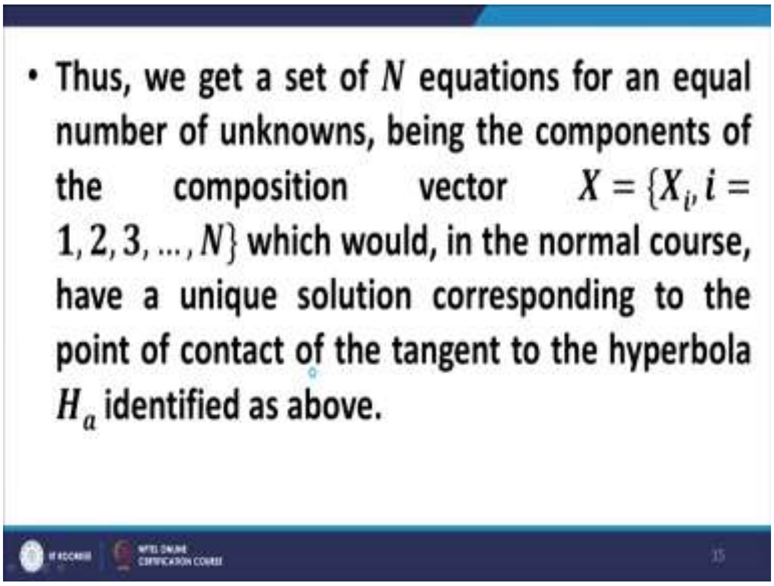

So thus we get a set of n equations for an equal number of unknowns being the components of the composition lecture which would in the normal course have a unique solution corresponding to the point of contact of the tangent to the hyperbola H alpha identified as above.

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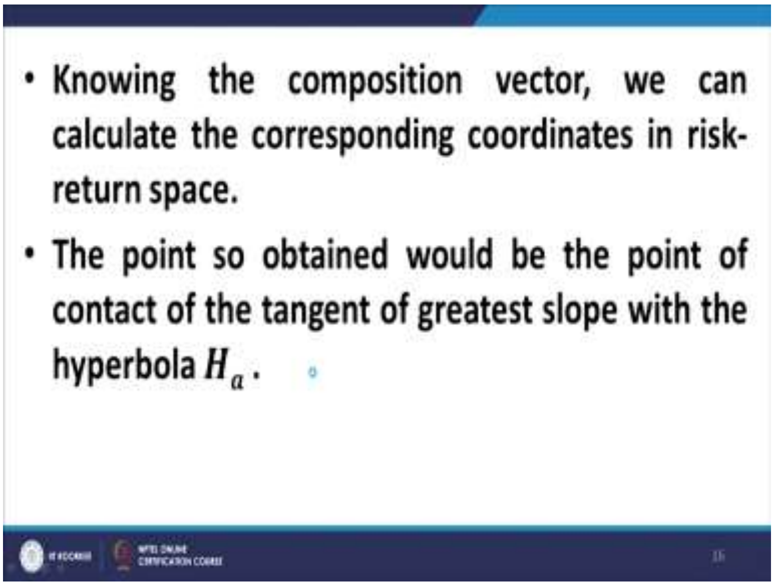

Knowing the composition vector, we can calculate the corresponding coordinates and risk return space of that point of contact and let us call that point P dash. And then we can because we know that coordinates of P dash, we know the coordinates of the risk free point that is zero RF and alpha and therefore, we can find out the equation of the straight line joining the point RF and P which represents the efficient frontier.

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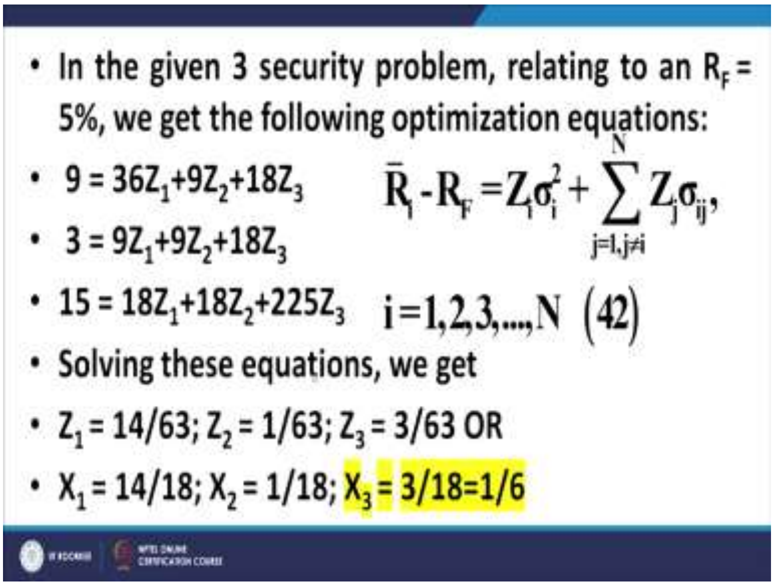

So, now, let us continue with this problem. So, we now continue with the 3 security problem and we assume the risk free rate at 5 percent we are making this assumption that the risk free rate is 5 percent then we get, if we use the equations number 42, which are given here on the right hand panel of your slide and we substitute the expressions for sigma 1, sigma 2 sigma 3, R1 R2 and R3 and rho 1 2, rho 2 3 and rho 3 1 what we end up with is the equation that are given on the left hand panel of the slide, 9 is equal to 36 Z1 plus 9 Z2 plus 18 Z3, 3 is equal to 9 Z1 plus 9 Z2 plus 18 Z3 then 15 is equal to 18 Z1 plus 18 Z2 plus 225 Z3.

If you solve this equation, these are 3 equations and 3 unknowns. So, the solution is quite straightforward and it can be done by any method that applies to simultaneous linear equations, we can Z1 is equal to 14 upon 63, Z2 is equal to 1 upon 63 and Z 3 is equal to 3 upon 63, the corresponding values of X1, X2 and X3 are 14 by 18, 1 by 18 and 1 by 6 and the standard deviation we have got X1 X2 and X3 we know sigma 1 sigma 2 and sigma 3, we know rho 1 2, rho 2 3 rho 3 1.

So, we have got all the inputs that are required for calculating the standard deviation of the portfolio comprising of these securities in these proportions and we find that the standard deviation is equal to under root 203 upon 6 and similarly, we can work out the expected return on this portfolio and we find that expected return is equal to 44 upon 3.

Hence what are the coordinates now, let us continue with that what are the coordinates of P dash, the coordinates are P dash are sigma, sorry, x is equal to under root 203 upon 6 y is equal to 44 by 3 and what is z? z is the composition of X3 or the content of X3 which we have already found it is 1 by 6.

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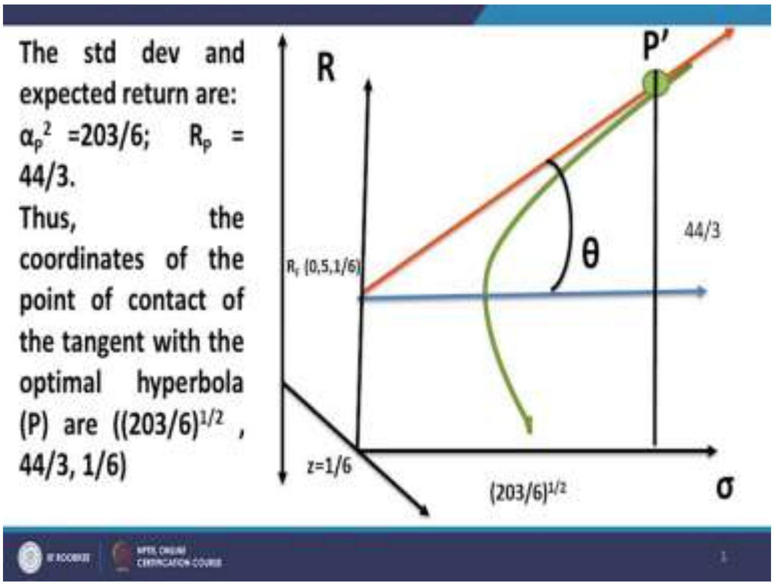

So, the coordinates of the point P dash are under root 203 by 6, 44 by 3 and 1 by 6. So, the efficient frontier is the straight line joining the 2 points RF and P dash. What are the correlates of RF, let me repeat zero comma it was the 5, years 5 percent was the risk free rate and Z3 I am sorry X3 which is equal to 1 by 6.

So, the coordinates of RF are 0, 5 and1 by 6 and the coordinates of P dash we have already worked it out under 203 upon 6 comma 44 by 3 comma 1 by 6. Please note both these points are coplanar both these points lie on the plane z is equal to 1 by 6 which is parallel to the xy plane and at least at a distance of a 1 by 6 along the z axis.

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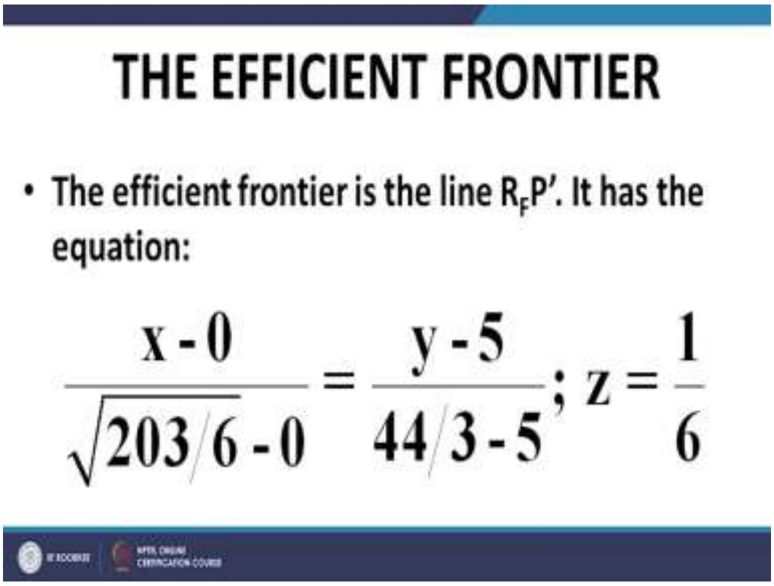

What is the equation of the straight line joining RF and P dash that is given by x minus zero divided by under 203 upon 6 minus zero is equal to y minus 5 divided by 44 by 3 minus 5 and z is equal to 1 by 6 because the line lies on the plane that is equal to 1 by 6. So, this is the equation of the efficient frontier and any point on this line could be an efficient portfolio. It would comprise of the risk free asset lending or borrowing on depending on whether the point lies beyond P dash or within P dahs.

If it lies within P dash and both F and P dash or long term if it lies beyond P dash, then F is short that means the risk is borrowing plus P dash which is long. Now the final question is what happens when neither risk free borrowing nor lending is permitted only short sales of risky assets are allowed. So far I have tackled the case where short sales in 3 securities, A, B and C was allowed and risk free lending and borrowing is allowed.

To be able to calculate the entire efficient frontier, we need also to address the situation where risk free lending and borrowing is not allowed. So, that is the case that we take up now.I will illustrate the procedure, it is again, somewhat intriguing, but I will illustrate this procedure in detail step by step. Step 1, we take a risk free rate, just like we have already done; we have taken this 3 rate as 5 percent resolve the optimization equations.

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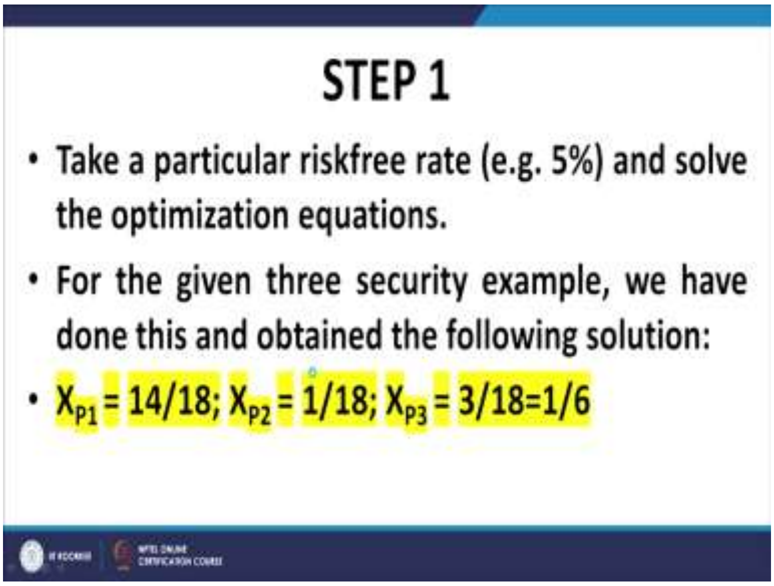

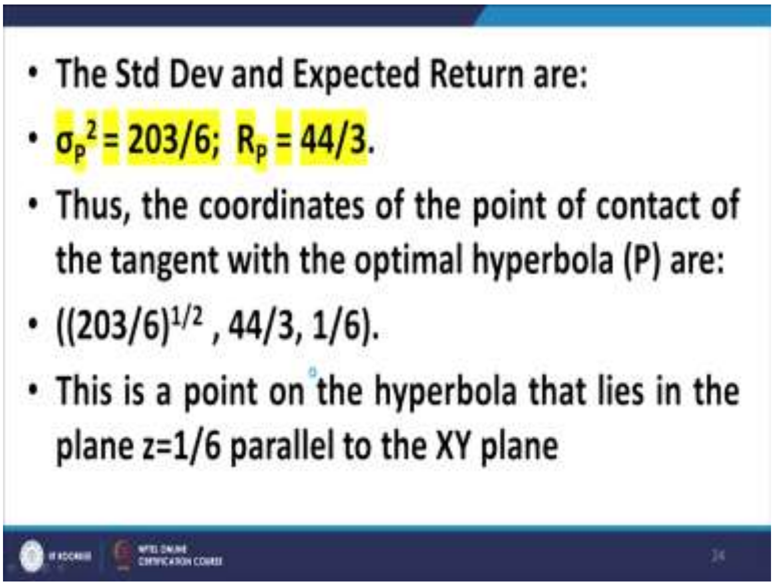

Now, for the given set 3 security problem A, B, and C with risk free rate equal to 5 percent. We arrived at the composition of the point P dash as on let us call it point P. Let us simplify the notation let us use brevity, let us call this point P. So this point P, where the efficient frontier is the let us call RF the point i, so the FP is my efficient frontier and the composition of the point P is given we have already worked it out, it is 14 by 18, 1 by 18.

And 1 by 6, this is the composition of the point P and the coordinates of the point P are under root 203 by 6 comma 44 by 3 and 1 by 6 and the coordinates of F zero comma 5, comma 1 by 6. So this is the situation with all these figures, we have already worked it out or are available in the problem.

So the standard deviation and the expected returns are under root 203 by 6 and 44 by 3 and the coordinates of the point of contact I have already mentioned coordinates of P are under root 203 by 6, 44 by 3 and 1 by 6 and this is a point on the hyperbola that lies on the plane set is equal to 1 by 6 parallel to the xy plane.

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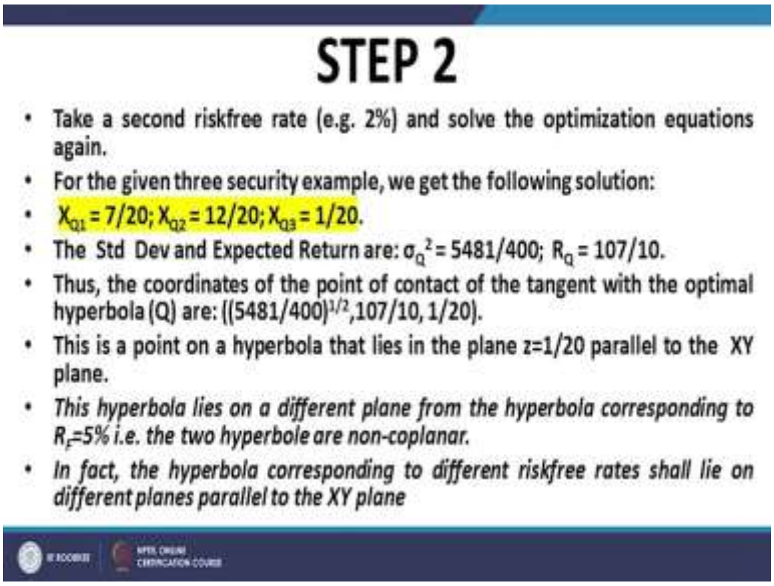

Step 2; step 2 is you take a second risk free rate take a different risk free rate, let us say we take a risk free rate of 2 percent and again solve the optimization problem. The procedure is exactly same as what we had done earlier. We simply have to substitute RF as 2 percent instead of 5 percent sigma 1, sigma 2 sigma 3, R1 R2 R3 and rho 1 2 rho 2 3, rho 3 1 will all remain the same. The only change that would occur is that RF would change from 5 percent to 2 percent.

And when we solve this set of equations, let us call this new point Q and solve the set of equations we arrive at the composition of Q as XQ 1 is equal to 7 by 20, XQ 2 that is the composition of security A in portfolio Q is 7 by 20, composition of security B in portfolio Q is 12 by 20, and composition of security C in portfolio Q is 1 by 20. The standard deviation and the expected returns can again be worked out in the same way.

We know X1, X2, X3, we know sigma 1, sigma 2 sigma 3, we know rho 1 2, rho 2 3, and rho 3 1. So we can calculate sigma Q that works out to under root 5481 upon 400. And we can also work out the expected return because we know R1 R2 R3; we know X1 X2 X3, so our Q works out to 107 divided by 10.

The coordinates have Q that is the coordinates of the point of contact of this new tangent, new hyperbola new tangent, this is different from the hyperbola that we had for RF equal to 5 that was lying on the plane z equal to 1 by 6. This is lying on the plane z equal to 1 by 20. So they are not cooperated. Please notice the hyperbola corresponding to RF equal to 5 percent was lying on the plane z equal to 1 by 6.

The hyperbola corresponding to z equal to I am sorry, RF equal to 2 percent is lying on the hyperbola is z equal to 1 by 20. So they are lying on different planes but obviously, both the planes are parallel to the xy plane, at different perpendicular distances from them. The coordinates of this point of contact Q, which is similar to the point P in the case of the earlier hyperbola which corresponded to RF equal to 5 percent, we now have the point Q, which is the coordinate 5481 upon 400 square root comma 107 by 10 comma 1 by 20.

So, let me reiterate this point we have this hyperbola corresponding to RF equal to 2 percent lies on a different plane from the hyperbola corresponding to RF equal to 5 percent that is the two hyperbola are non coplanar therefore, and in fact, the hyperbola corresponding to different risk free rates will lie on different planes, but each plane would be parallel to the xy plane and would be at different perpendicular distances from the xy plane.

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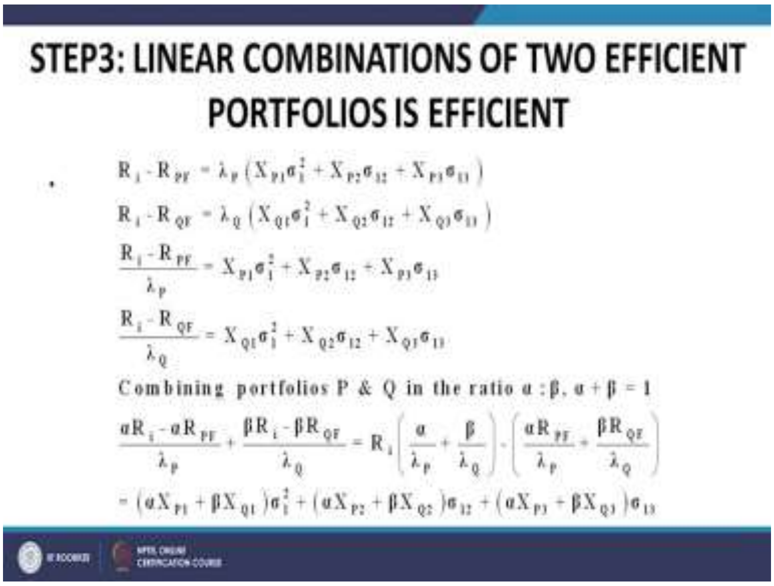

Now, we need to prove a simple theorem. This theorem is that linear combinations of two efficient portfolios is efficient, the linear combination of two efficient portfolios is efficient, the proof is rather straightforward, I will not spend time on it, you simply write out the optimization equations for the two portfolios, two given portfolios let us say we acquired the portfolio P and the portfolio Q which we have for example, which we are just calculated, let us use these two portfolios both are efficient because we arrived at those portfolios by solving the optimization equations. So, they are optimal portfolios.

We use these two portfolios for illustration and we like the optimization equations and then we form a combination a linear combination of the two portfolios in the ratio alpha is to beta and alpha plus beta is equal to 1 and when we simplify this, we find that the new portfolio that is constructed that is a portfolio comprising of P and Q in the ratio of alpha is to beta also satisfies the optimization equations, but with a different risk free rate.

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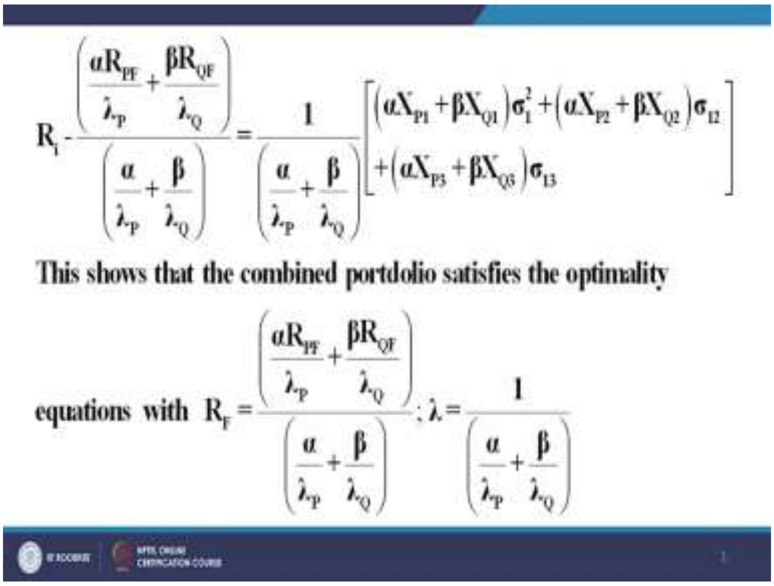

So, this is shown in this particular slide, you can see here that the combined portfolio the portfolio comprising of P and Q in a arbitrary ratio alpha is to beta where alpha plus beta is equal to 1, results in a portfolio which satisfies the optimization equations given by or with respect to a new risk free rate and a new lambda which are shown in the in this particular slide right at the bottom.

So, we have established that a linear combination of two efficient portfolios is also efficient, let me repeat a linear combination of two efficient portfolios is also efficient. Therefore, it follows that every point on the efficient frontier is a linear combination of two other points on the frontier, corresponding to different risk free rates.

Hence if we know the coordinates of any two points on the efficient frontier, which will represent two efficient portfolios, we can trace the entire efficient frontier because every other point will be a linear combination of these two points. Thus, now, if you look at the beauty of this problem, here is according to me or as per my perception, the most interesting step in this whole analysis,

What we find is that a linear combination of two efficient portfolios is efficient. Therefore, every portfolio, every efficient portfolio that can be formed, can be expressed as a linear combination of these two efficient portfolios and that means, what that means, we simply need to construct the poor the portfolio possibilities curve or the efficient frontier of this portfolios P and Q and that will give us the entire efficient frontier that we are looking for.

So, the bottom line is what I am trying to say is that the entire problem has now been condensed to a two security problem from the insecurity problem, we have been able to resolve this into a two security problem, the theory of which you already know with a lot of detail, which we have already discussed. So, let me repeat, we have established that the linear combination of two efficient portfolios is efficient, that means what?

That means is any portfolio that would be efficient can be expressed as a linear combination of two efficient portfolios. And that being the case, now, any portfolio means an arbitrary term that means we are talking about the efficient frontier in totality, because any portfolio can be any arbitrary portfolio on the efficient frontier.

In other words, we are talking about the entire efficient frontier and the entire efficient frontier would be represented by a linear combination of P and Q. But we know how to do how to work out the efficient frontier of a two risky portfolio problem or two risky security problems. That is the hyperbola arc that we have discussed in a lot of detail.

So the bottom line is, if we know the leaking the cardinals of P, the cardinals of Q, what are the cardinals of P and Q that we need to know, we need to know the standard deviation of P, we need to know the standard deviation of Q, we need to know the covariance between P and Q or the correlation coefficient between P and Q.

And we need to know the expected returns of P and Q. If we know all these five figures, then we can work out the efficient frontier comprising of P and Q, which would be the efficient frontier corresponding to all portfolios, which we are looking for. So from here I will take up in the next lecture. Thank you.