Security Analysis and Portfolio Management
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Lecture 48
Mean Variance Portfolio Optimization VIII

Welcome back, so let us continue from where we left off. We were talking about the linear combinations of two efficient portfolios being efficient and as I mentioned if I take two portfolios P and Q and solve the optimization equations and they would be satisfying the optimized equations, because and they are efficient portfolios.

Now if I take a linear combination of P and Q, I find that the new portfolio that is alpha P plus beta Q will also satisfy the optimization equations with a different risk free rate. So the outcome is that if we form linear combinations of efficient portfolios we get another efficient portfolio. How does this help us?

It helps us in the sense that the entire efficient frontier can now be defined as the locus of linear combinations of two efficient portfolios, I repeat the entire efficient frontier is now the locus of linear combinations of two efficient portfolios. Hence, if we know the coordinates of any two points on the efficient frontier that is P and Q we can trace the entire efficient frontier.

Because every other point will be a linear combination of these two points, I repeat again if we know the coordinates of any two points on the efficient frontiers P and Q we can trace the entire efficient frontier because every point will be a linear combination of these two points, thus the problem of tracing the efficient frontier turns out to be a two security problem it resolves down to two security problem.

Because you see we are all having two portfolios P and Q which are efficient and we want to work out combinations of these two portfolios, now portfolios and securities and so far as portfolio management is concerned are identical in the sense that they represent points in risk return space.

So, whether we are talking about portfolios or we are talking about securities that amounts to the same thing in so far as the risk return space is concerned, so they represent points in the risk return space whether it is a single isolated security or whether it is a combination of separate securities representing a portfolio.

So the bottom line is that we want to work out linear combinations of two securities P and Q which may be portfolios but we take them as securities but because it does not really matter they are they are identified by a pair of numbers in risk return space.

That being the case we have P and Q we if we are able to identify them in risk return space they and then we form linear combinations of P and Q just as we form linear combinations of two securities A and B and then we arrived at the hyperbola that contained all the portfolio possibilities curve.

Then out of that we identified the efficient frontier, so the same process the same exercise can now hold for these portfolios P and Q but the result that we are going to get now is the set of efficient portfolios comprising of the end security that we started with. We have now resolved this n security problem to a two security problem we identify two efficient portfolios and we take linear combinations of these two efficient portfolios in the same manner as we had linear combination of two securities A and B.

And using the same process same methodology parallel absolutely we end up with the efficient frontier of P and Q which is essentially the efficient frontier of the n securities for A B C D…n up to n. I repeat let me repeat it is worth repeating we have got two efficient portfolios comprising of the n securities A B C D…n up to n portfolio corresponding to the risk free rate of 5 percent and Q portfolio corresponding to the risk free rate of 2 percent.

We have these two portfolios which are efficient, let us forget the name portfolio let us call them securities. So we have got two securities P and Q and we know that our efficient frontier that we are desperately looking for is a linear combination of P and Q. We also know that given any two securities A and B how to form linear combinations and the outcome is the hyperbola and the arc of the hyperbola forms the efficient frontier beyond the point of minimum variance.

In other words that methodology completely devolves down to the current problem in terms of the portfolio of securities P and Q and the efficient frontier comprising of P and Q is also the efficient frontier comprising of the securities A B C up to n. Therefore, the problem that we started with of n securities has now been resolved to a two security problem, so let us continue.

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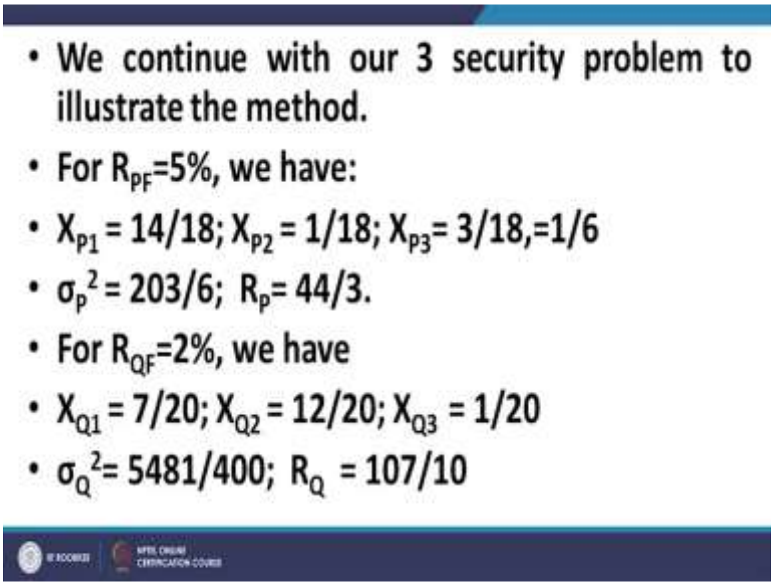

We continue with our three security problem to illustrate this method for RFP equal to 5 percent we had the portfolio P as the point of contact of the tangent from F to the arc of the hyperbola in the plane z is equal to 1 by 6 and we had this composition XP1 is equal to 14 by 18, XP2 is equal to 1 by 18, XP3 is equal to 1 by 6, sigma P is equal to under root 203 by 6 and RP is equal to 44 by 3.

This was the situation for the portfolio P where FP was tangent and which corresponding to F equal to or RF equal to 5 percent. Then we did the same exercise with RF equal to 2 percent, we found a new optimal hyperbola we found a new plane which was given by z is equal to 1 by 20 and the essentials or the cardinals of this new point Q which lies on the hyperbola and which is the tangent where F is the 2 percent one, so FQ is the tangent to this new hyperbola which lies on the plane z is equal to 1 bu 20 where F corresponds to rsk free rate of 2 percent.

The composition of this point Q is XQ1 is equal to 7 by 20, XQ2 is equal to 12 by 20, XQ3 is equal to 1 by 20 and when we work out the cardinals the standard deviation and the expected return of Q, we find that sigma Q is equal to under root 5481 divided by 400 and sigma RQ is equal to 107 upon 10.

So again we have full information about both the portfolios P corresponding to the risk free rate of 5 percent and the portfolio Q corresponding to the risk free rate of 2 percent and reiterate that both P and Q are efficient because they have been obtained by solving the fundamental optimization equations, so they naturally have to be efficient .

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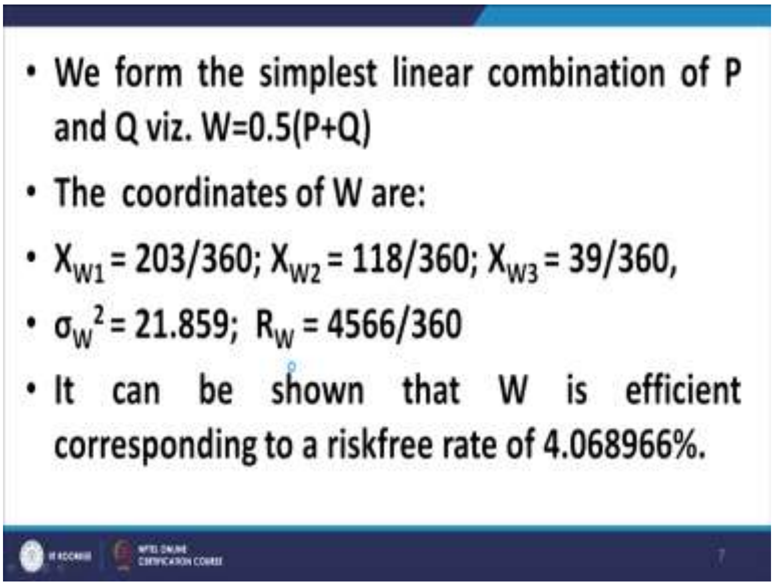

We know we formed a linear combination of P and Q as I discussed just now we form a linear combination of P and Q for simplicity for minimization of calculations we form the simplest possible linear combination that is half of P and half of Q, we mix half of P and half of Q in terms of the amounts of course.

The coordinates of W, this new portfolio we call W, in other words we constitute a portfolio W which comprises of 50 percent of Q and 50 percent of P. Combining these two, 50 percent of P, 50 percent of Q we form a new portfolio W, the cardinals of the portfolio W can be easily derived because it is a weighted portfolio of P and Q.

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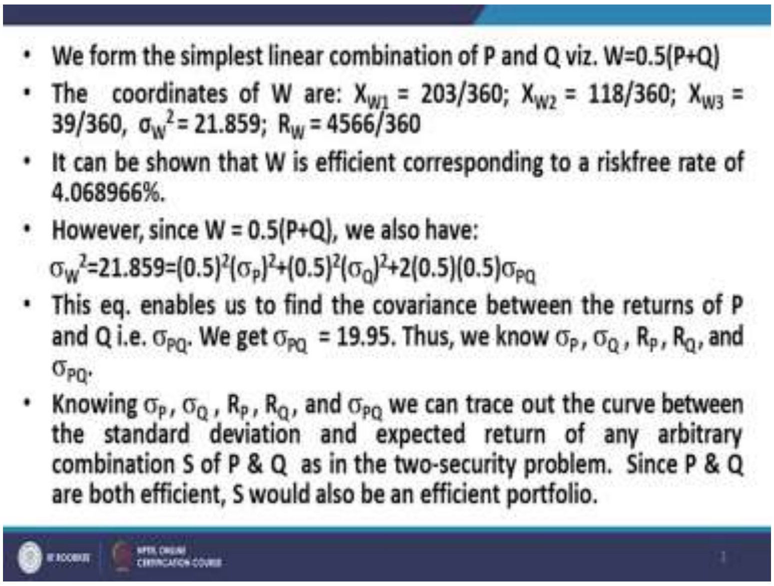

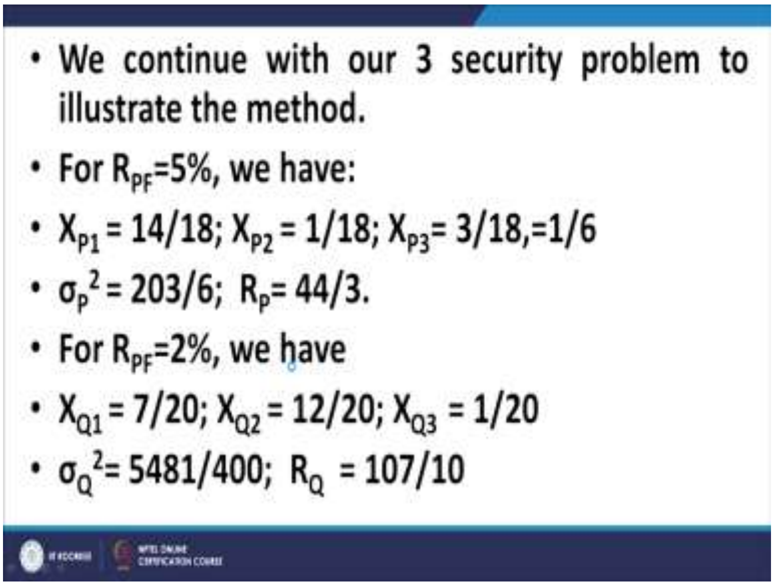

We have XQ1 is equal to 203 divided by 360, XW2 is equal to 118 divided by 360 and XW3 is equal to 39 divided by 360 and sigma W is equal to under root 21.859. In other words sigma W square, the variance of W is equal to 21.859 and the expected return of W is equal to 4566 divided by 360.

It can be shown if you use this process, if you use the reverse process of the optimization equations it can be shown that W is efficient it will solve the optimization equations but with the risk free rate of 4.06896 percent. So I repeat this portfolio W that we have formed of half of P and half of Q can be explicitly shown to satisfy the fundamental optimization equations and hence it is an efficient portfolio corresponding to a risk free rate of 4.068966 percent.

Because W, now is a very important point this is a very important slide, now W is equal to 0.5 into P and Q. We know the cardinals of P and we know the cardinals of Q in essence we know the variance and the co-variance and the expected return of P and Q and the variance and expected return of Q.

For the moment let us focus on the variance we know the variance of P, we know the variance of Q, we know the weight of P, we know the weight of Q, what do we not know? What we do not know is the covariance between P and Q, so if we can find out the covariance of P and Q we are done our problem is solved.

We have a two security portfolio where all the parameters are given we have the expected return of P and Q we have the variance of P and Q and if we can find out the covariance between P and Q, we can simply plot the portfolio possibilities curve the hyperbola and the efficient frontier can be identified therefrom.

So our problem boils down to finding the covariance of P and Q. To find the covariance of P and Q we use an ingenious trick. We know that W comprises of 50 percent of P and 50 percent of Q, therefore the variance of W will consist of which is already given is which we have already derived from where have we derived, we have derived it from the composition vector of W using the composition vector XW1, XW2, XW3 and the variance covariance matrix of A B and C we are able to derive the covariance of W.

We find it to be 21.859, we have already done that exercise in the previous slide, but W consists of half of P and half of Q, therefore the variance of W which we have derived as 21.859 must be also equal to what? 1 by 2 square sigma P square plus 1 by 2 square sigma Q square plus 2 into covariance of P and Q into 1 by 2 into 1 by 2, this is the standard formula for the variance of a two security portfolio.

Now we are using where P and Q are the two securities as I mentioned it does not matter if we are having a security or a collection of securities called a portfolio is simply a nomenclature. We now have sigma W square which is already derived as 21.859 as given by this expression, we all we also know sigma P square, what is sigma P square?

Sigma P square is 203 upon 6 and what is sigma Q square let me check sigma Q square is equal to 5481 upon 400 I repeat sigma P square is equal to 203 upon 6, sigma Q square is equal to 5481 upon 400. So we know sigma P square we know sigma Q square we know XQ we know XP.

So, what else is left we the only thing that is left is the only unknown in this equation is sigma PQ the covariance between P and Q if we solve this equation for sigma PQ we end up with getting sigma PQ is equal to 19.95, I repeat sigma PQ is 19.95. Now as far as portfolio P and Q are concerned what information do we have we have sigma P we have sigma Q we have sigma PQ, we have expected return P, we have expected return Q.

All the cardinals that are required for working out the efficient frontier or the portfolio possibilities curve of this pair of risky securities P and Q is available with us, we simply do that exercise as we did in the two security risky problem to obtain that hyperbola and the arc of the hyperbola representing the efficient frontier

We do that exercise, when we do that exercise what we get is x square is equal to 0.4853 y square minus 7.2353 y plus 35.5588. So what I have done let me reiterate I have simply use this parameters of the information about the variance covariance of P and Q and their respective returns I have used the feasibility equations and eliminated XP and XQ from them and the result is the hyperbola that we did in the two security risky problem which is represented by this equation x square is equal to 0.4853 y square minus 7.2353 y plus 35.5588.

This is the hyperbola that represents all feasible combinations that is the portfolio possibilities curve comprising of P and Q and please note this P and Q are themselves combinations of securities A B C up to n and in various proportions. We have got now is the portfolio possibilities curve comprising of P and Q the equation for that we have already got is the equation and that is represented here.

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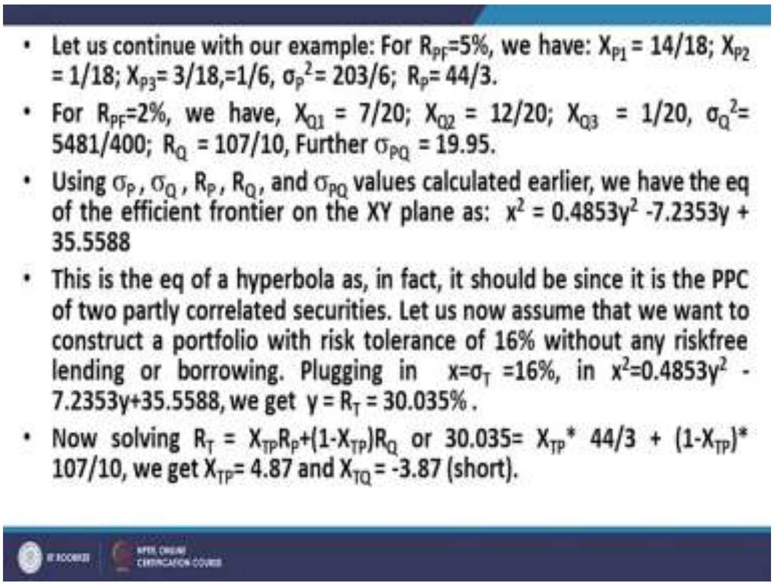

Now suppose we want to work out a portfolio with the risk tolerance of say given as 16 percent without any risk free lending and borrowing, what do we do? We plug in x equal to sigma T where T is the name of the portfolio that we are arriving at XT equal to 16 percent in this expression in this portfolio possibilities curve, x square is equal to 0.4853 y square minus 7.2353y plus 35.5588.

We get y is equal to RT is equal to 30.035 percent and now we can solve this expression for the expected return in terms of the composition vector we get RT is equal to XTP RP plus 1 minus XTP RQ, we know RP, we know RQ and we know RT as well, we can find out the content of XTP that is the composition of P in T and the full composition of T in terms of A B and C is now obtained as XTi is equal to 4.87 XPi minus 3.87 XQi where i represents A B and C.

Let me recap what we have done, I have spoken in a bit of pace so let us go back and let us recap, what we have done after arriving at the portfolios P and Q. We know sigma P, sigma Q, RP, RQ, and sigma PQ that is all that is required for determining the portfolio possibilities curve comprising of P and Q and we can trace out the curve between the standard deviation and expected return for any arbitrary combination S of P and Q as in the two security problem. So that gives us the portfolio possibilities curve from which we can identify the efficient frontier.

Now construction of the efficient portfolio if we want to construct a portfolio efficient portfolio corresponding to a given parameter, let us say we want to construct a portfolio T corresponding to a given level of risk. We simply plug in the value of sigma T in the portfolio possibilities curve and that we have obtained and we arrive at the corresponding value of RT.

I repeat we have got the equation of the hyperbola which is a combination of P and Q, a linear combination of P and Q, for choosing an efficient portfolio or for working out an efficient portfolio corresponding to a given level of risk what we do is let us say sigma T is the given level of risk corresponding to a portfolio T which we want to determine the composition of in terms of A B and C. We are given the composition of…we are given the risk I am sorry of sigma T.

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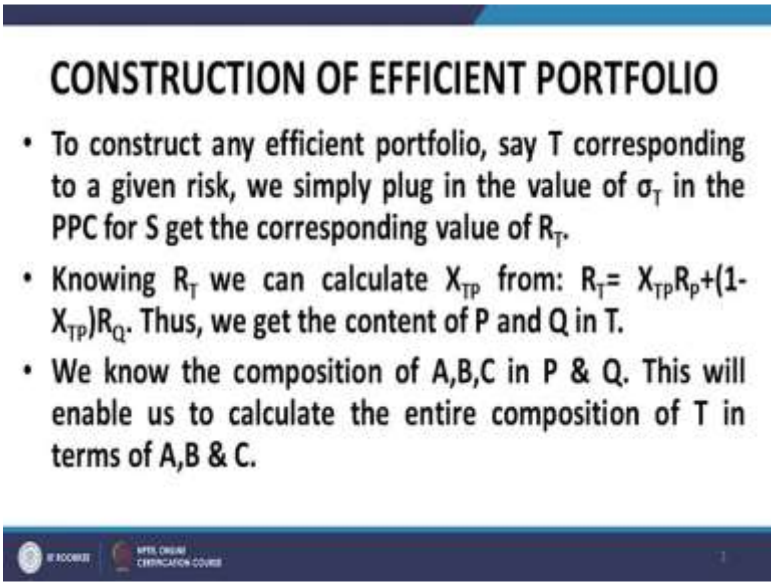

We solve this portfolio possibilities curve equation and we given sigma T we can find out RT. Now we have RT is equal to XTP RP plus 1 minus XTP RQ, so this is the second step. Now in this equation if you look at it carefully we know RT we have found out RT how? We have found out RT by substituting sigma T which is a given parameter by the investor and substituting in the portfolio possibilities curve, we get RT. Using that RT we know RP and RQ as well, they are already derived.

Let me repeat RP is equal to 44 by 3, RQ is equal to 107 by 10, so we know everything in this equation except XTP. From this equation we can find out XTP. If we know XTP we can find out XTQ, so we know the composition of P and Q in T but we also know the composition of A B C in P and A B C in Q. We can find out the composition of A B C in T, so that is the step methodology that we follow if we are given a level of risk tolerance then we can determine the efficient frontier comprising of n securities given to us.

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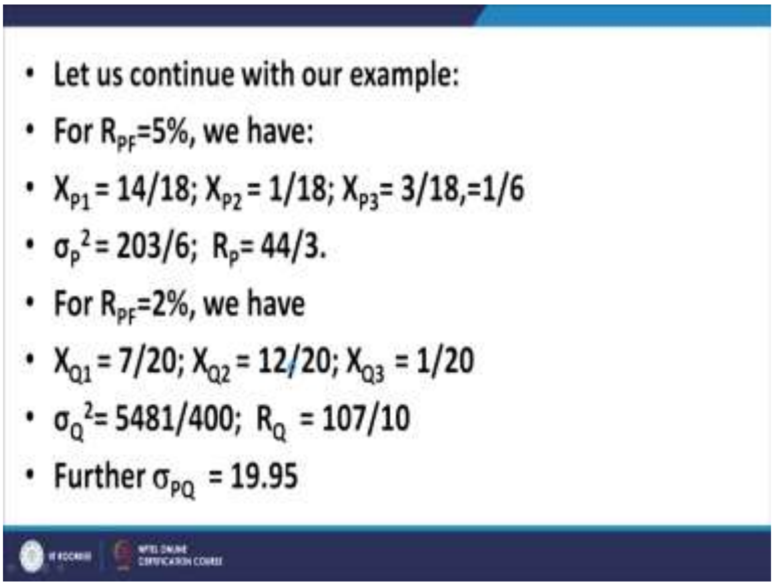

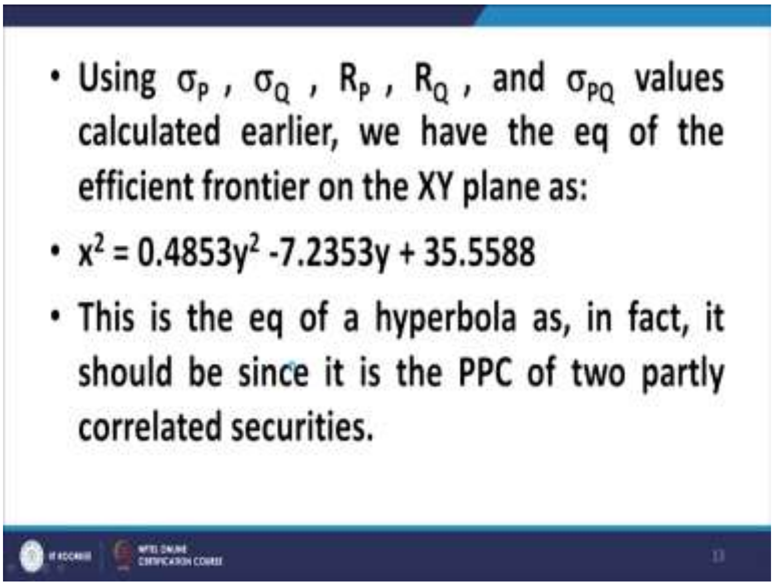

Then I discuss this example if RP is equal to 5 percent we have this result, if RP is equal to 2 percent we have this result, both of the results we have already derived and which are here in this which are summarized here. Once more XP1 is equal to 14 by 18, XP2 is equal to 1 by 18, XP3 is equal to 1 by 6, sigma P square is equal to 203 by 6, RP is equal to 44 by 3.

For RF is equal to 2 percent, we get XQ1 is equal to 7 by 20, XQ2 is equal to 12 by 20, XQ3 is equal to 1 by 20, sigma Q square is equal to 5481 upon 400, RQ is equal to 107 by 10 and we have also found out that sigma PQ is equal to 19.95. We can get the equation of the hyperbola as x square is equal to 0.4853 y square minus 7.2353y plus 35.5588; this is the equation of the hyperbola as it should be because it is the portfolio possibilities curve of a two security portfolio P and Q.

Now let us take this example let us carry forward this example suppose we want to construct a portfolio T having a risk tolerance of 16 percent we plug in sigma T equal to 16 percent is equal to x and we find the value of y which represents the expected return of T that comes to 30.035 percent.

Using this return of T which is now given, which is now obtained as 30.035 we can find out the content of P in T, and then we can find out the content of Q in T using the content of P in T, we can find the composition or the content of A B and C in T.

So this is how the portfolio optimization problem in the mean variance portfolio optimization can be solved for n securities, the procedure can be directly extended to n securities where instead of 1, 2, 3 we can have 1, 2, 3,….n, so the three security problem completely epitomizes, completely captures the features of the n security problem.

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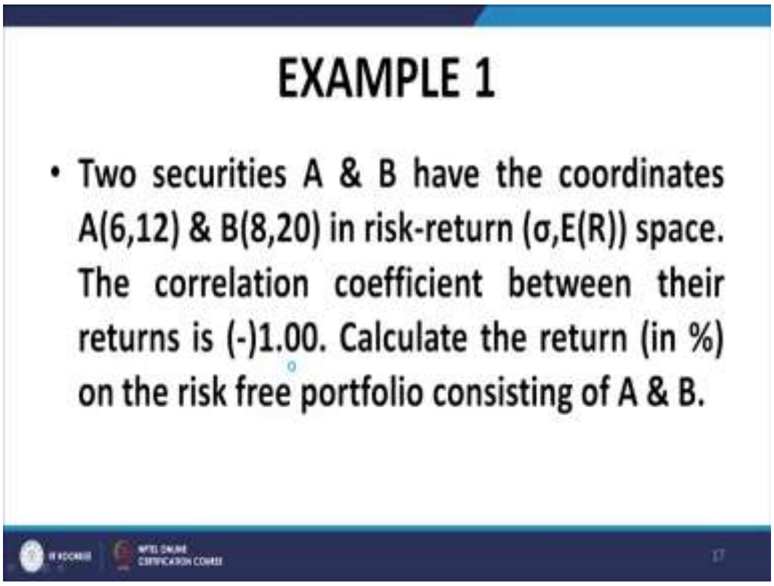

STD DEVIATION	6	8
EXPECTED RETURN	12	20
CORRELATION	-1	
X(A)=σ(B)/[σ(B)+σ(A)]	0.57142857	
X(B)=1-X(A)	0.42857143	
SD(P)	0	
R(P)	15.4285714	

Let us continue let us start with some examples two securities A and B have coordinates of A 6 by 12 and B 8 comma 20. Let me repeat two securities A and B have the coordinates A 6 comma 12 and B 8 comma 20 in risk return that is sigma expected return space the correlation coefficient between the returns is minus 1. Calculate the return in percentage on the risk free portfolio comprising of A and B.

The solution is quite simple, we know that if the two securities are perfectly anti correlated then the risk free return is given by sigma 1 R2 plus sigma 2 R1 divided by sigma 1 plus sigma 2. You have got the values of all the contents or the constituents of this formula you have got sigma 1, you have got sigma 2, you got R1 you got R2. Simply plug in the values you get the expression for the risk free rate which is 15.4285 percent.

Alternatively you can go through a slightly longer way, you can find out the content of A in the risk free portfolio that is given by sigma B divided by sigma B plus sigma A, the content of B would be 1 minus XA. You know the content of A and B; you know that returns on A and B, you can work out the risk free rate. Either way you arrive at the answer of 54.4285 percent. Let us look at a slightly more interesting example.

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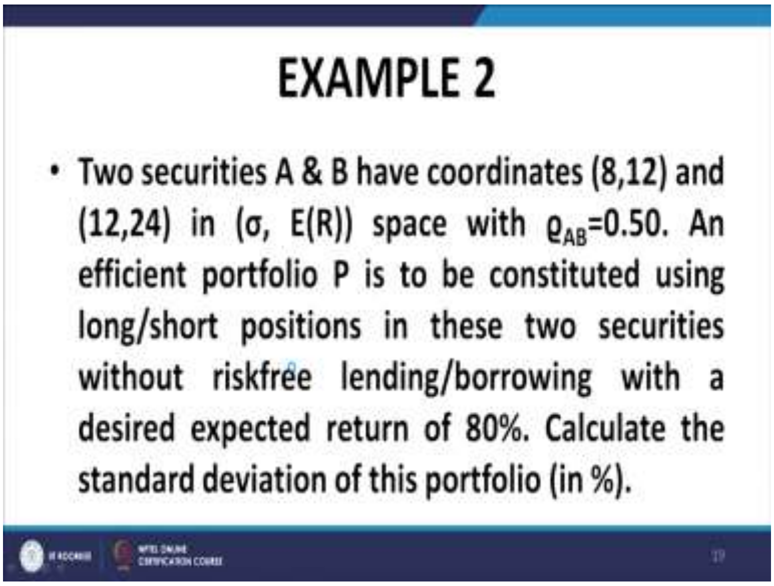

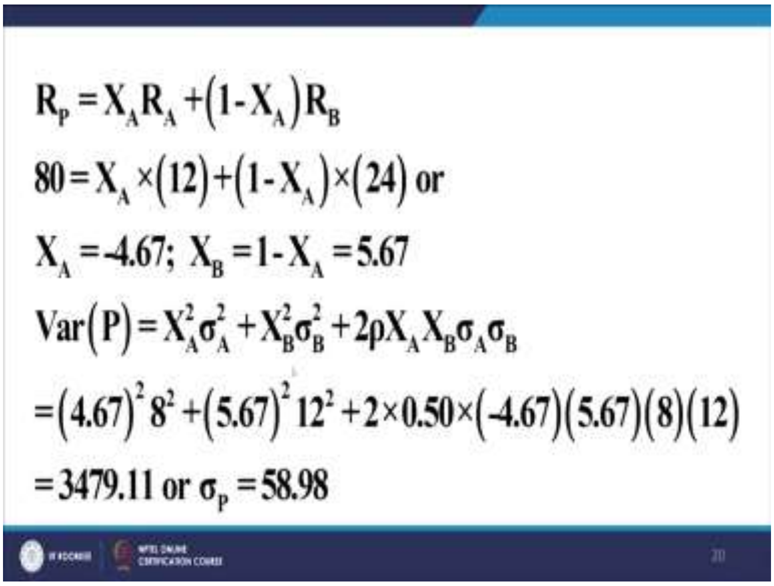

Two securities A and B have coordinates A 8 comma 12 and 12 comma 24 in sigma expected return space. The correlation coefficient is 0.50 and efficient portfolio P is to be constituted using long oblique short positions in these two securities without risk free borrowing or lending with a desired expected return of 80 percent. Please do not get disturbed by the 80 percent, I have put in this figure especially to highlight certain things this exorbitant rate of return but this conveys a message which I should tell you.

Calculate the standard deviation of this portfolio in percentage. What are we given we are given sigma A RA or sigma R1, we are given sigma 2 2R and we are given rho 1 2 or rho A B as the case may be and we are also given the expected return on the target portfolio that is given as 80 percent.

Using the first equation that we have on the slide RP is equal to or the expected return on the portfolio is the weighted average of the expected return of its constituents, I can work out the composition or the content of the target portfolio which requires expected return of 80 and we find that XA is equal to minus 4.67 and XB is equal to 5.67.

Now it is very simple to find out the variance y, because we know XA, we know XB, we know sigma A, we know sigma B, we are also given rho A B and remember the covariance A B is equal to rho A B sigma A sigma B. So we know everything in this formula and we simply have to plug in the values and what we get is the variance is equal to 3479.11 percent squared and therefore the standard deviation is 58.98 percent.

Another problem two securities A and B have coordinates 8 comma 12 and 12 comma 24 in sigma expected return space with rho A B equal to 0.50 and efficient portfolio please note this word efficient, it is very important, an efficient portfolio is to be constituted using long oblique short positions in these two securities together with risk free lending and borrowing.

So we have freedom as far as long short positions in the securities as well as risk free lending and borrowing with an expected return of 80 percent calculate the standard deviation of this portfolio. Please note again the expected return is 80 percent and as far as the other parameters of A and B are concerned they are similar to what we had in the immediately preceding example, rho A B is equal to 1 by 2 and the coordinates of A and B are 8 comma 12 and 12 comma 24 respectively.

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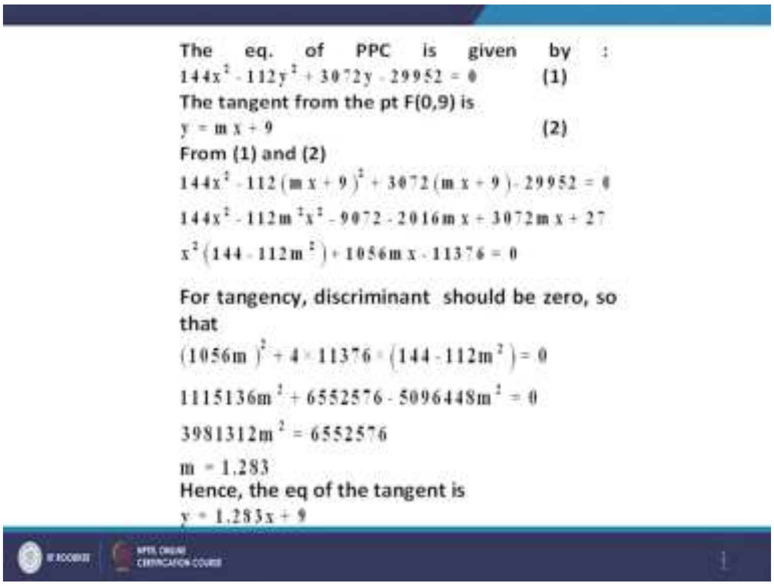

So the first step is to calculate the equation of the portfolio possibilities curve which we can do in the standard procedure using the feasibility conditions and we arrive at the portfolio possibilities curve which is given by 144x square minus 112y square plus 3072y minus 29952 is equal to 0, equation number 1 on the slide. I repeat this is simply obtained by eliminating X1 and X2 from the expressions for the expected return and the variance of the portfolio which is targeted as to have a return of 80 percent.

Assuming that the targeted portfolio has the standard deviation x and has the expected return y and we solve the feasibility conditions eliminate X1 and X2 from them and what we get is this equation of the tangent. Now the tangent, now we are given that the risk free rate is 9 percent, therefore the efficient frontier is the straight line joining the point F which has the coordinate 0 comma 9 with the arc of the hyperbola which is represented by equation number 1.

So to obtain the equation of this tangent which will be the efficient frontier what we do is we assume that y is equal to m x plus 9 because the y intercept is 9, so we assume that y is equal to m x plus 9 is tangent to this particular hyperbola, let us say this is equation number 2 Substituting equation number 2 in equation number 1, what we get is the equation right at the bottom of your slide.

Now because we want that equation number 2 should be tangent to equation number 1 it is necessary that the roots of this equation must be…the discriminant of this equation must be 0, the roots must coincide, the discriminant must be 0 and when we solve for the discriminant of this equation equal to 0 we can get the value of m, the value of m turns out to be 1.283.

Please note we are neglecting the other value of m because that would give us the other tangent to the hyperbolic arc which is redundant for our purposes, because we are considering or we are concerned with the efficient frontier only. Hence, we are concerned with the upper tangent, the tangent with the positive slope and the upper tangent which lies on the upper side of the two tangents, because that would be the efficient frontier.

So we have m is equal to 1.283 and we know the equation of the tangent the equation of the tangent is given by y is equal 1.283x plus 9. We are given that the standard deviation, I am sorry the expected return of the target portfolio is 80. So substituting y is equal to 80 in this equation we get the value of x which is 55.34, so this is the solution of this problem.

The interesting point that you would observe in examples 2 and 3 is that while the expected return in both cases is 80, the standard deviation in the case where short sales where restless lending or borrowing is not allowed was slightly higher, it was slightly above 58 percent. However when you allow risk free lending on borrowing you can construct a relatively better portfolio a relatively superior portfolio a portfolio which is even more efficient, because it has a lower standard deviation corresponding to the same level of expected return.

In other words what I simply want to convey by this pair of examples is that if you add risk free lending and borrowing to your feasibility structure you are able to enhance value of your portfolio optimization problem in the sense that you can construct a portfolio which is better than the portfolio which is superior to the portfolio that you would have constructed if risk free lending and borrowing is not allowed. I shall continue from here in the next class with the single index model. Thank you.