Decision Making With Spreadsheet
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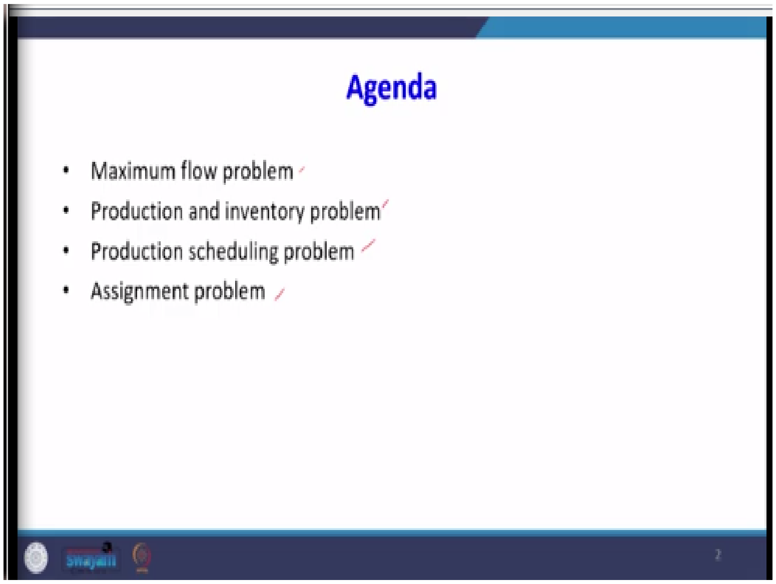
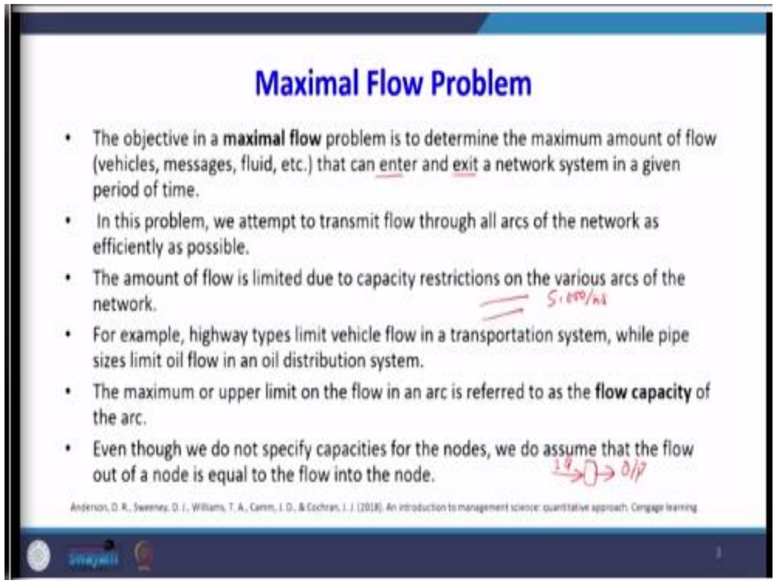
Lecture-21
Maximal Flow, Production and Inventory Application, Scheduling and Assignment Problems

Dear students, in this lecture, I am going to discuss the maximum flow problem. After that, by using the concept of network models, I am going to solve a production inventory problem, a production scheduling problem, and an assignment problem. The point which you have to notice about all this problem is that I will use the concept of network models for solving this kind of problem. The advantage of a network model is that it can be pictorially easily represented, and the visualization is so easy that you can easily understand how to formulate the problem.



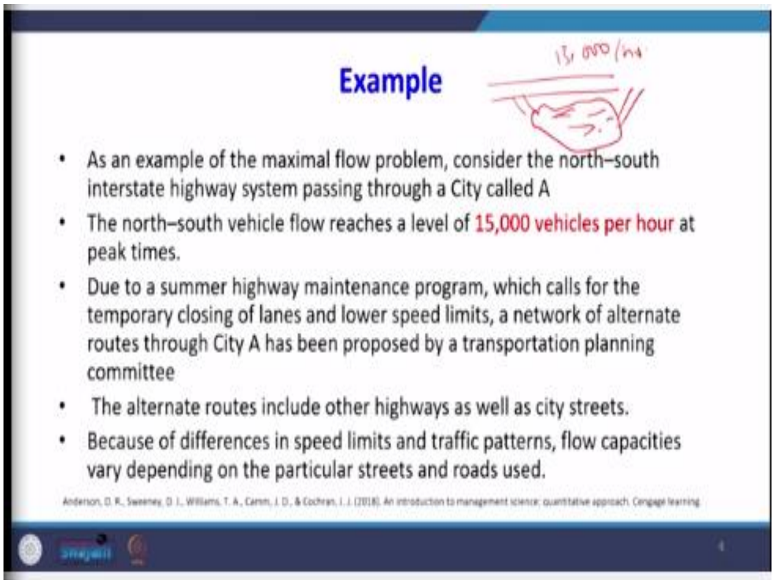
So, the agenda for this lecture is, first, will we solve the maximum flow problem? After that, I will discuss production and inventory problems. Then, I will use the network concept to solve your scheduling problem; at the end, I will take one assignment problem that I also solve with the help of network models.



First, I will explain what this maximum flow problem is. So, the objective in a maximum flow problem is to determine the maximum amount of flow; when I say flow, it can be several vehicles, it can be a number of messages, it can be the quantity of fluid that can enter and exit any network system in a given period of time. So, in this problem, we attempt to transmit flow through all arcs of the network as efficiently as possible.

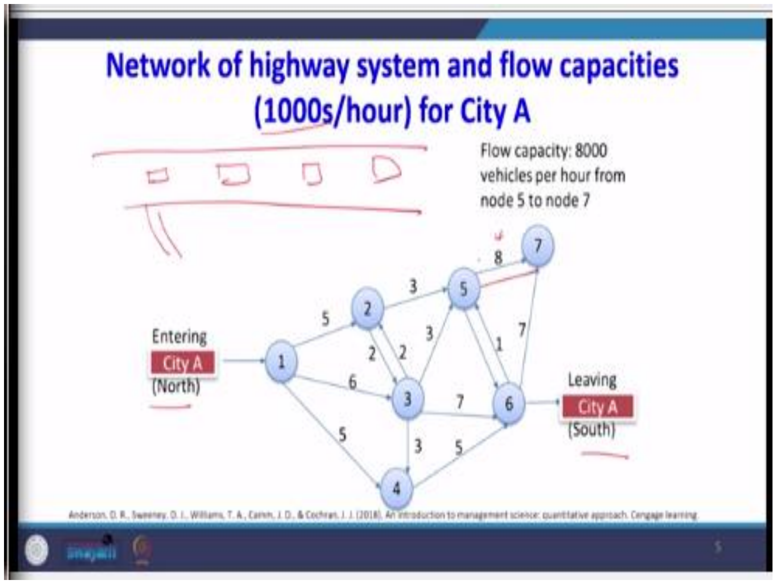
That means as maximum we try to send to the flows at the maximum capacity. Here, the flow is limited due to capacity restrictions on the various arcs of the network. For example, highway types of limit vehicle flow in a transportation system, so if there is a highway, there may be a limitation on vehicle flow. For example, maybe 5000 vehicles per hour, that may be the limitation.

While the pipe size limits oil flow in an oil distribution system, the maximum is the upper limit on the flow in an arc, which is referred to as the flow capacity of the arc. Even though we do not specify capacities for the nodes, we do assume that the flow out of a node is equal to the flow into the node. Suppose this is a node, so inflow is equal to outflow, that is the assumption.

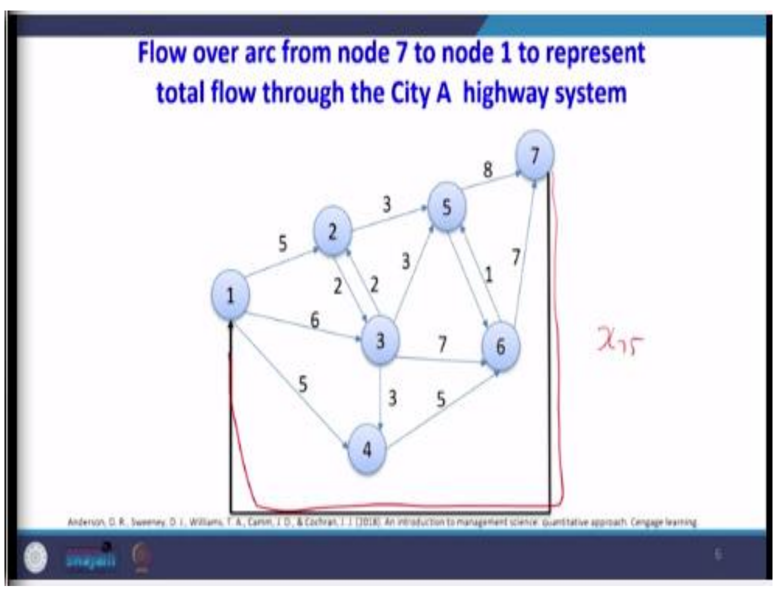


Now, we can take one example in the transportation context; then, we will use the concept of the maximum flow problem. As an example of the maximum flow problem, consider the north-south interstate highway system passing through a city called A. This problem is taken from the book by Anderson et al. So, the north-south vehicle flow reaches a level of 15000 vehicles per hour at peak times. There is a highway; the maximum flow is 15000 vehicles per hour. Due to a summer highway maintenance program that calls for the temporary closing of lanes and lower speed limits, a network of alternate routes through city A has been proposed by a transportation planning committee.

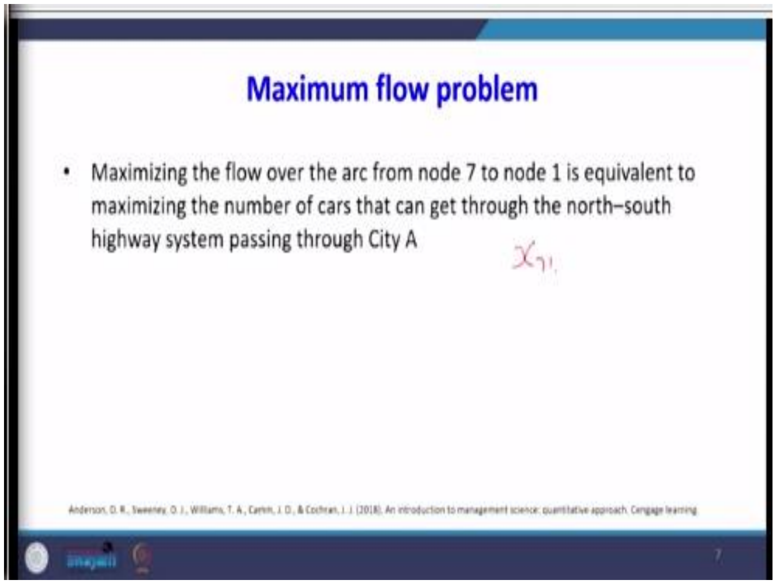
So, they need to do some maintenance, they are planning to divert this traffic inside the city, so this is a city map, they are planning to divert it. The alternate route, so inside the city, includes other highways and city streets. Because of differences in speed limits and traffic patterns, flow capacities vary depending on the particular streets and roads. So, inside the city, there may be different capacities because the size of the road may vary, and the type of road also may vary.



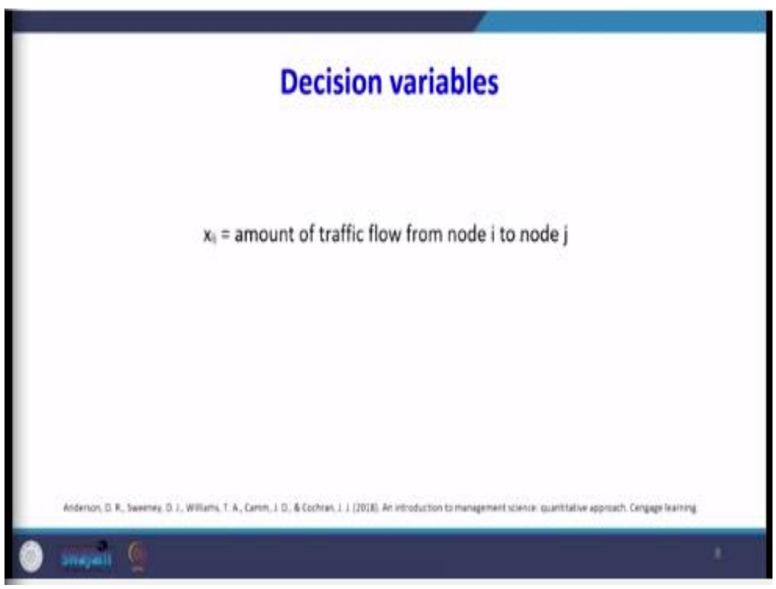
So, now the network of highway system and flow capacities all the values are in terms of 1000 vehicles per hour for city A. So, this is a diverted route, so because the initial route is like this, this is the highway, due to maintenance, they are diverting this route inside the city. So, city A north side is entering, and the same city on the south side is leaving here. The flow capacity is 8000; for example, 8 represents 8000 vehicles per hour from node 5 to 7; that is the meaning of this 8.



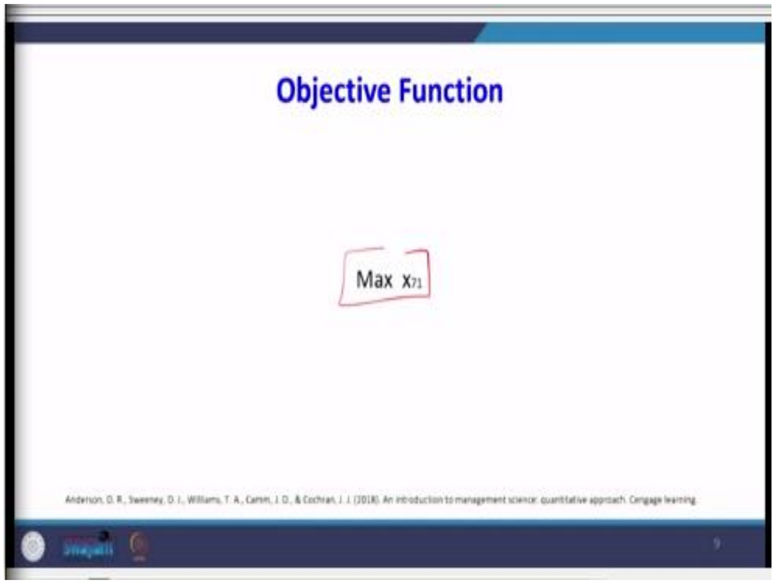
Now we are going to introduce a hypothetical network, a hypothetical arc that flows over an arc from this one node 7 to 1 to represent the total flow through city A, a highway system. So, why have we introduced this 7 to 1 network? So, maximizing this network from 7 to 1 is equivalent to maximizing the city highway system network. So, this X_{71} when you maximize this it is equivalent to maximizing the whole network.



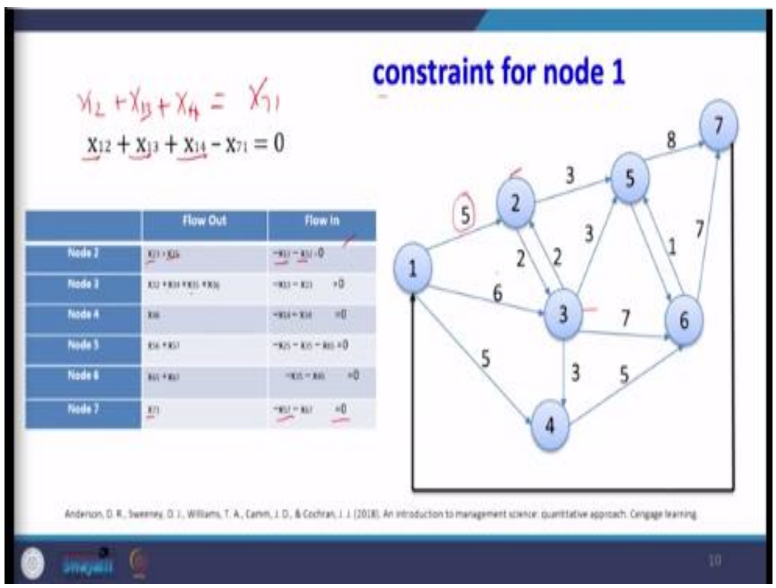
So, this is maximum flow problem, maximizing the flow over the arc from node 7 to 1 is equivalent to maximizing the number of cars that can get through the north-south highway system passing through city A, that is the meaning of this X_{71}.



So, here, the decision variables are Xij, the amount of traffic flow from node i to j, Xij.

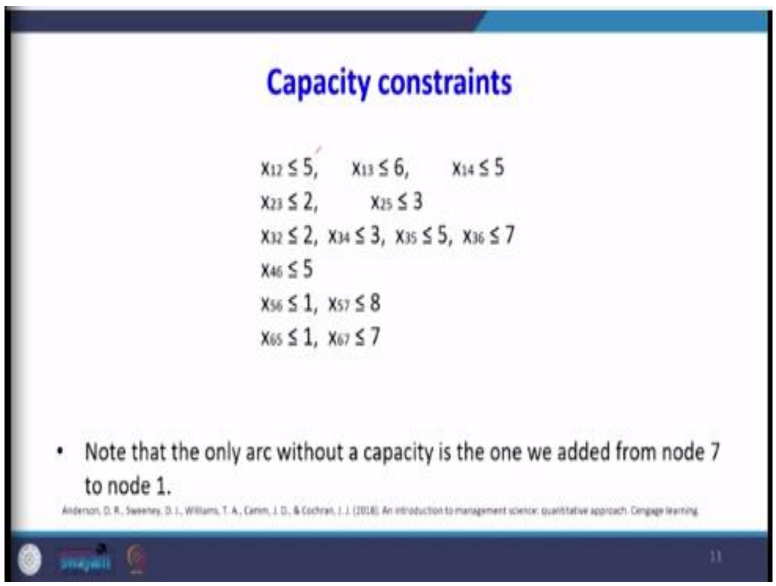


What is the objective function? We have to maximize this X_{71}, what is the X_{71}? An arc which you have hypothetically introduced.

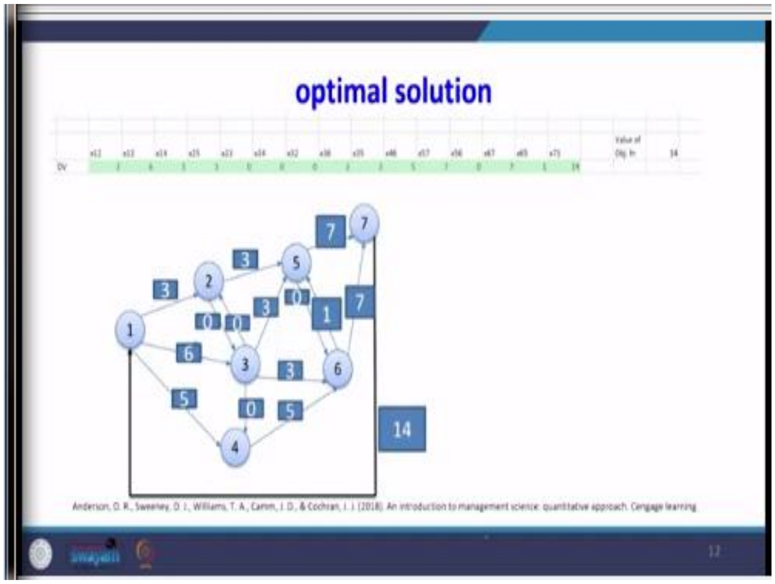


Then we are going to write the constraints. For each node we are going to write the constraint, for example constraint for node 1. We have already seen that node 1, what are the outgoing constraints? X_{12}, then X_{13}, then X_{14}. What is an incoming constraint? X_{71}. So, the number of the sum of all incoming constraints = the sum of all outgoing constraints. So, this X_7 is going from outside, so we can bring this left-hand side so that it will become $-X_{71}$; it should be this way: $X_{12} + X_{13} + X_{14} = X_{71}$, number of outgoing constraints = number of ongoing constraints, this is 14.

Similarly, we can write for flow out equal to flow in for node 2. So, when you take in node 2, what are the flow-out constraints? X_{25} is there, then X_{23} is there; what is the flow-in? 12 is there, then 32 is there, and like that, we have to write the constraint for all the seven nodes. For example node 3, what is the outgoing constraint? X_{32} is there, 34, 35, 36, incoming constraint X_{13}, X_{23}, sim. Similarly, we have to write for node 4, node 5, node, and 6. In node 7, can you see the node 7 what is the outgoing constraint? X only one outgoing constraint, that is, X_{71}. What is the incoming constraint? 57 is there, 67 is there. So, we have the objective function, we have the constraints.



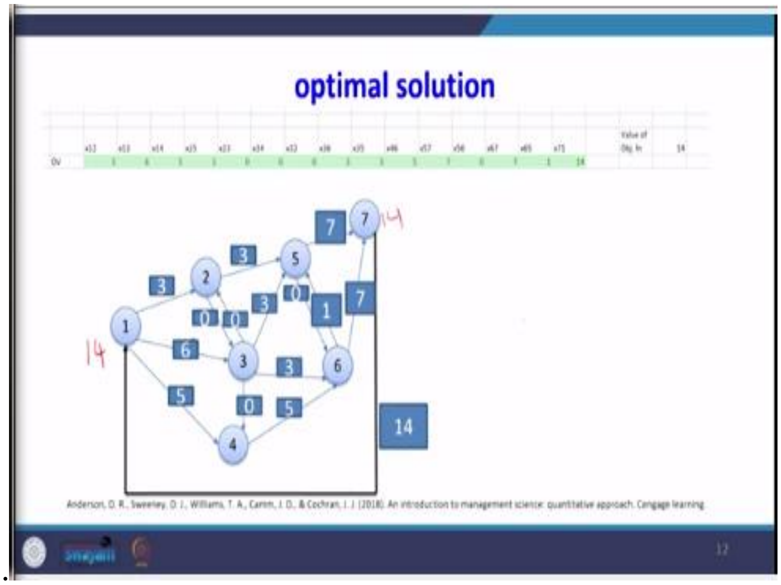
Apart from this, there is a capacity constraint. So, X_{12} is less than equal to 5; what is that? See, the value of X_{12} is the maximum of 5000 vehicles per hour that can go, so that is why the value of X_{12} cannot exceed 5; that is why X_{12} is less than or equal to 5. Similarly, I have written for all the constraints, and you see that there is no X_{71} at all. Note that the only arc without capacity is one we added from node 7 to 1. For all other constraints, there is a capacity constraint.



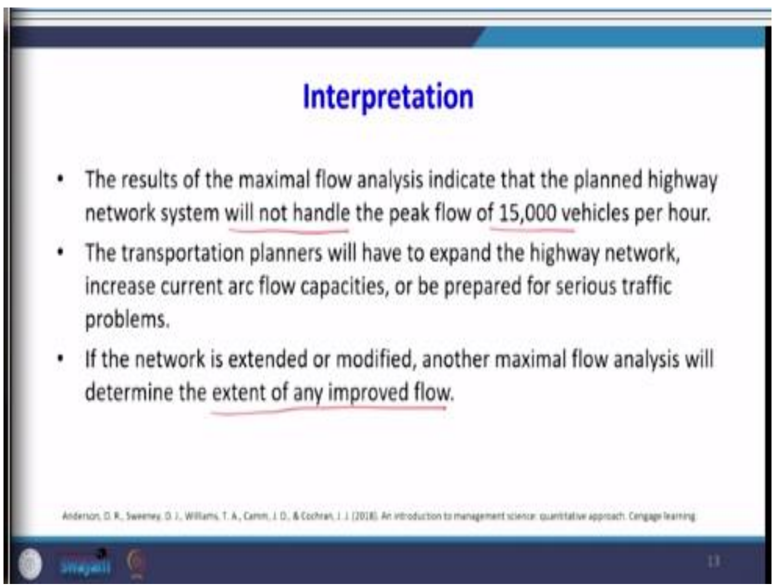
Now, I am going to solve this problem with the help of a solver; then I am going to interpret the result. Now, I am going to solve this problem with the help of Excel. So, my decision variables are X_{12}, X_{13}, and X_{14} up to X_{71}. The value of the objective function is what I have written to see if we click U6 because only the coefficient of X_{71} has to be maximized. When I click, U6 is nothing but a sum product of D7 to R7 and D8 to R8.

You can see that there is no coefficient of the objective function. So, the coefficient of the objective function is only for the X_{71} because we have to maximize only X_{71}. Then, as usual, I have written all the constraints on the left-hand side, and then I have written the right-hand side constraint. Now, when I go to Solver, I am maximizing U6. It is a maximization problem; the changing cell is D7 to R7, then I have written all the constraints, and I have included that.

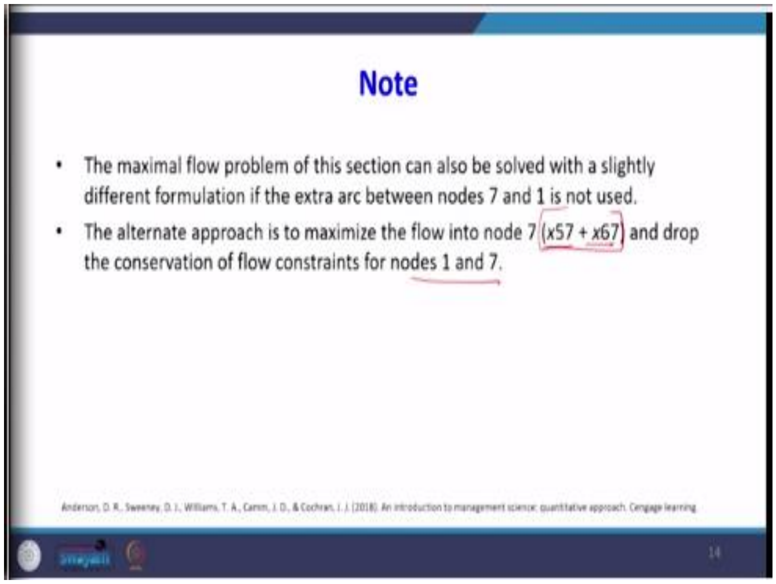
This is the linear programming problem; then I go to solve it; when I solve it, when I press ok, I see that the value of the objective function is 14. So, what did this 14 say that the new network where we have diverted the highway, after we have diverted the highway inside the city that the new network can withstand only 14000 vehicles per hour? But you see, initially, the highway can handle 15000 vehicles per hour, but when you divert that highway inside the city, it can handle only 14000 vehicles per hour. Then what is the meaning of these 14000 vehicles per hour?



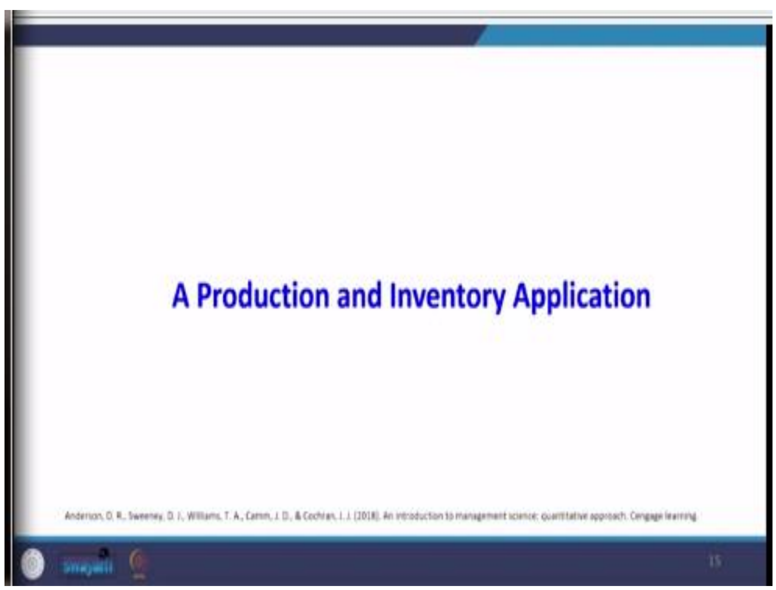
Now you look at this, so the value of objective function X_{12}, ok three and X_{13} is 6, 5, you see the total 3 + 6 = 9 it is 14 you see that how many? 5 to 6 also 7, this also 14, you see the total incoming is equal to total outgoing. So, what is the meaning of this 14?



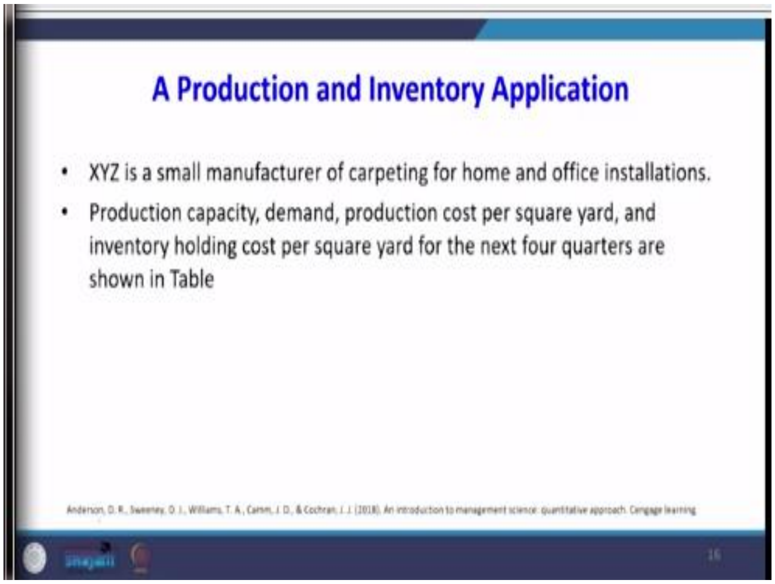
Now you see, the result of maximum flow analysis indicates that the planned highway network system will not handle the peak flow of 15000 vehicles per hour because it can handle only 14000 vehicles per hour. The transportation planners will have to expand the highway network increase current arc flow capacities, or be prepared for serious traffic problems. If the network is extended or modified, another maximum flow analysis will determine the extent of any improved flow. So, if you modify this network again, we have to resolve the problem, then after solving, we can see whether the flow capacity can be increased or not.



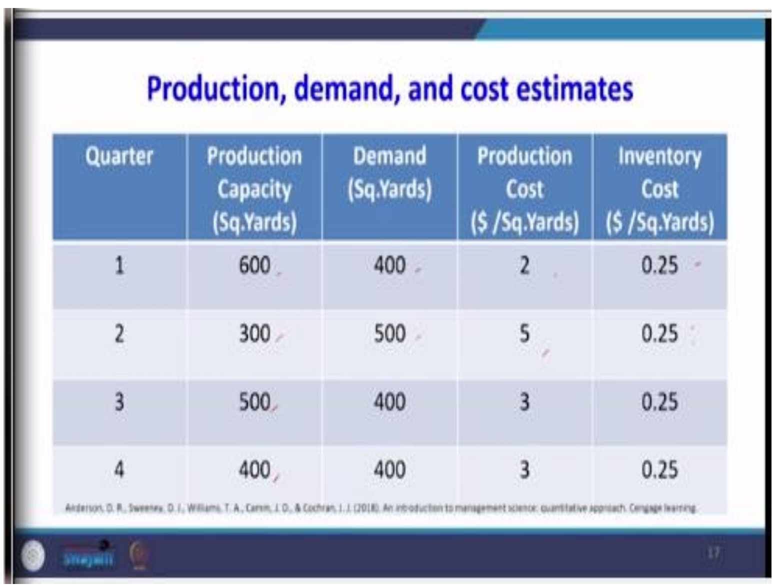
The important point that you have to remember is that the maximum flow problem of this section can also be solved with a slightly different formulation if the extra arc between nodes 7 and 1 is not used. We have introduced a hypothetical arc X_{71}; instead of this, an alternative approach is to maximize the flow into node 7. What are they? X_{57} and X_{67} and drop the conservation of flow constraint for nodes 1 to 7, so what can we do? Our new objective function maybe this one instead of X_{17}. If you maximize this again, you will get the same answer, 14.



Now I am going to solve a production and inventory problem with the help of network models.

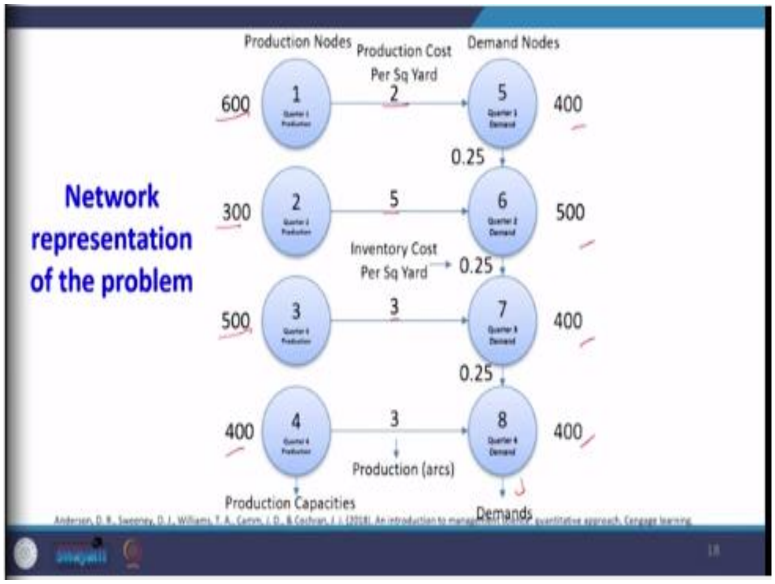


XYZ is a small manufacturer of carpeting for home and office installations. Production capacity, demand, production cost per square yard, and inventory holding cost per square yard for the next four quarters are shown in the table. This problem is taken from the book by Anderson et al.

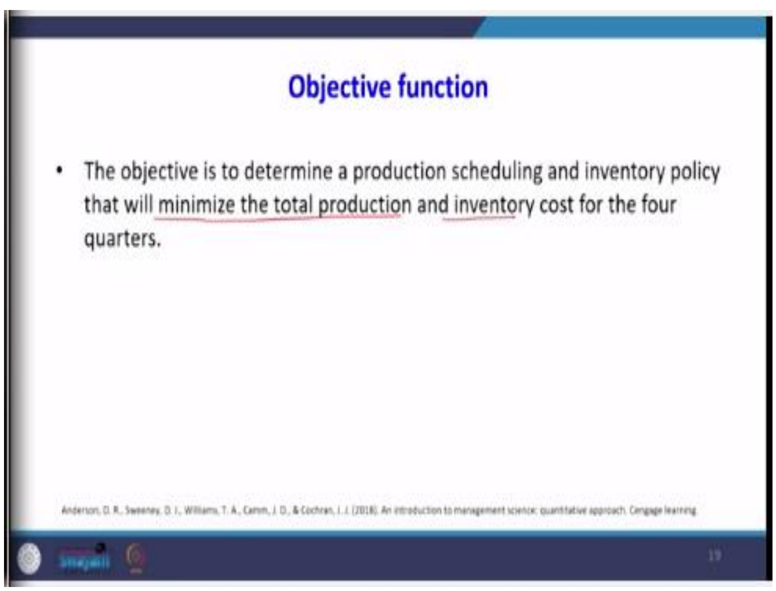


Quarter	Production Capacity (Sq.Yards)	Demand (Sq.Yards)	Production Cost ($ /Sq.Yards)	Inventory Cost ($ /Sq.Yards)
1	600	400	2	0.25
2	300	500	5	0.25
3	500	400	3	0.25
4	400	400	3	0.25

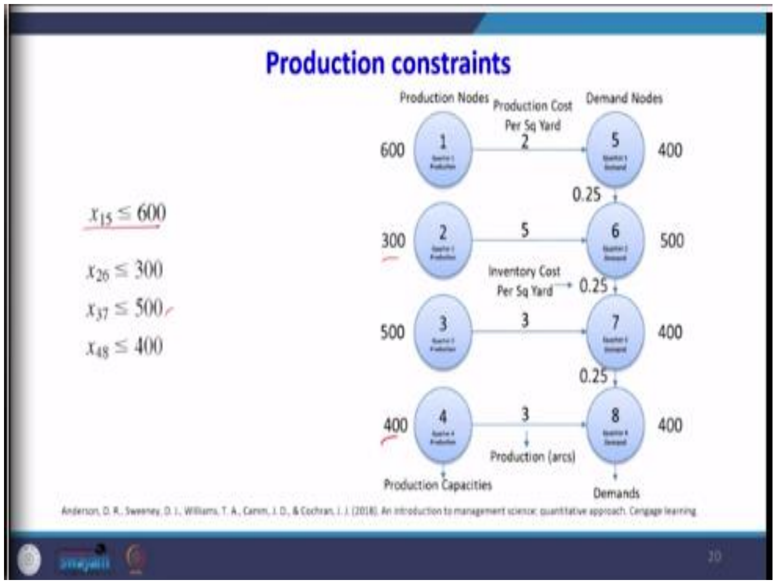
So, this table shows the production capacity for all 4 quarters: 600, 300, 500, and 400; demand is given, the production cost is given, and the inventory cost is also given.



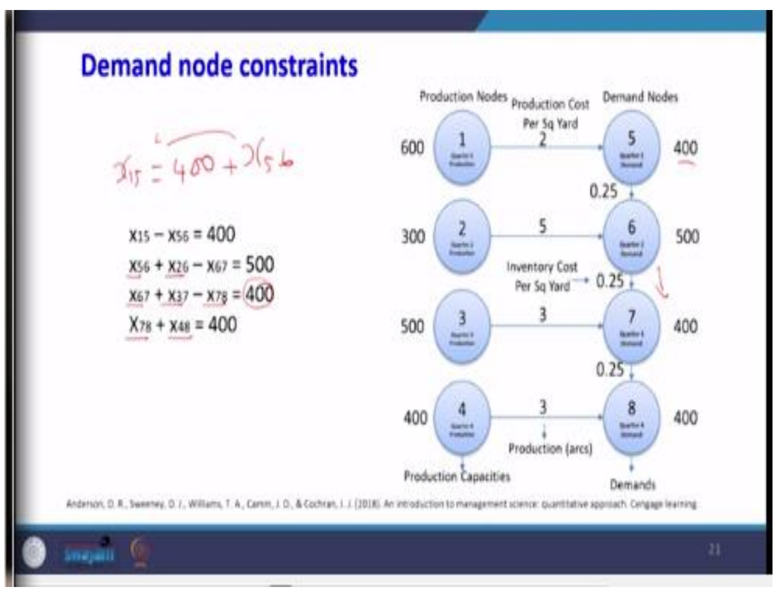
That I have represented in the form of a network, what is that? The quarter 1, you see the production capacity is 600, this is the capacity 600. Quarter 2, 300, 500, 400, and go back, you see the demand nodes, what is the demand 400, 500, so 400, 500, 500, 500, again 400 and 400. Then, if you produce in quarter 1, to satisfy the demand of quarter 1, the production cost is 2 dollars. If you stock an inventory, the production cost is 0.25 dollars; for the second quarter, the production cost is 5 dollars per square yard. So, this rises to 0.25 inventory carrying cost per square yard; similarly, in quarter 3, production cost is 3 dollars per square yard. And you see that after the fourth quarter there is no inventory, this 0.25 represents the inventory carrying cost.



So, here the objective function is to determine production scheduling an inventory policy that will minimize the total production and inventory cost for the 4 quarters. So, we must suggest how much quantity has to be produced and how much unit has to be stored in the form of inventory, so that the overall production cost is minimized.



First, we will write the production constraint. So, what notation we are using for decision variables? If I write X 15, this represents the production quantity for the quarter 1 and that is consumed in the quarter 1 itself. So, X_{15} is the number of quantities that has to be produced in the quarter 1. X_{26} is 300, X_{37} is 500, and X_{48} is 400, so these are the production constraints.

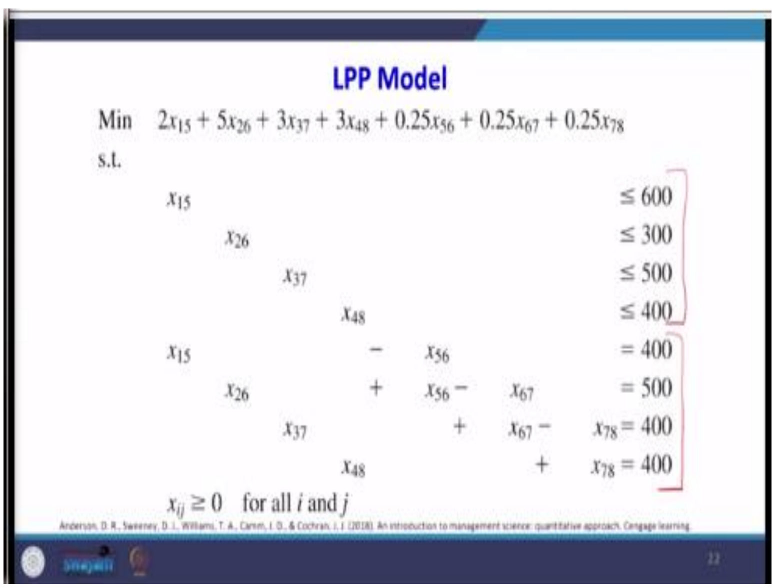


In the next one, I am going to write about the demand constraint. Here what is the demand constraint? You see that whatever you produce X_{15} should be used for satisfying these 400 units of demand; after satisfying this 400, if any extra units have to be stored, so what will happen?

So, X_{15} = 400. If there are any additional units, that will be X_{56}. So, X_{56} represents inventory which is stored at the end of quarter 1.

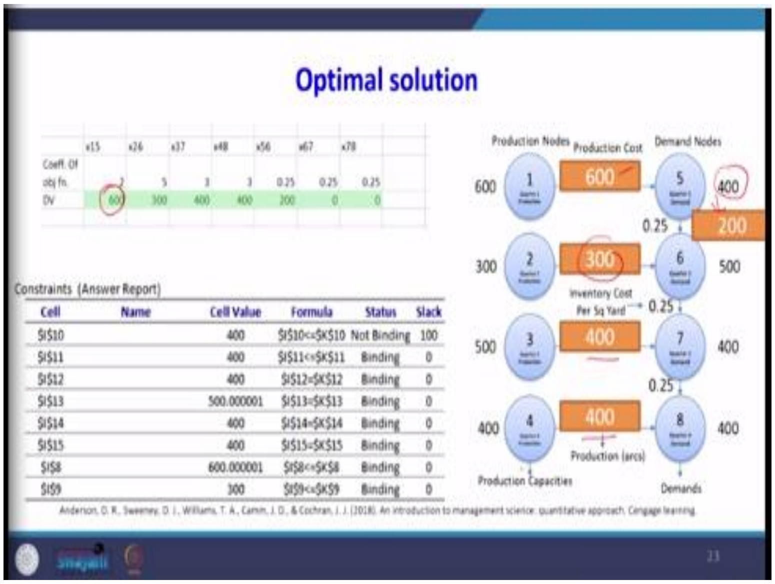
So, when I bring on this left-hand side, this will be $X_{15} - X_{56} = 400$. The second one is the production quantity plus the previous period inventory X_{26}, which should be equal to 500. Even after satisfying 500, if there is any additional unit that will be stored as the X_7, this X_7 is the quantity stored after quarter 2. Then again, this X_7 will be carried as an inventory, and X_{37} will be the production quantity for quarter 3.

Even after satisfying 400 any additional unit that will be stored as X_{78}, that is the inventory. In the fourth quarter, X_{78} will be the beginning inventory, X_{48} will be the production quantity, and there will not be any inventory term here. Because the company policy is that after quarter 4, there should not be any inventory, these are the demand constraints.

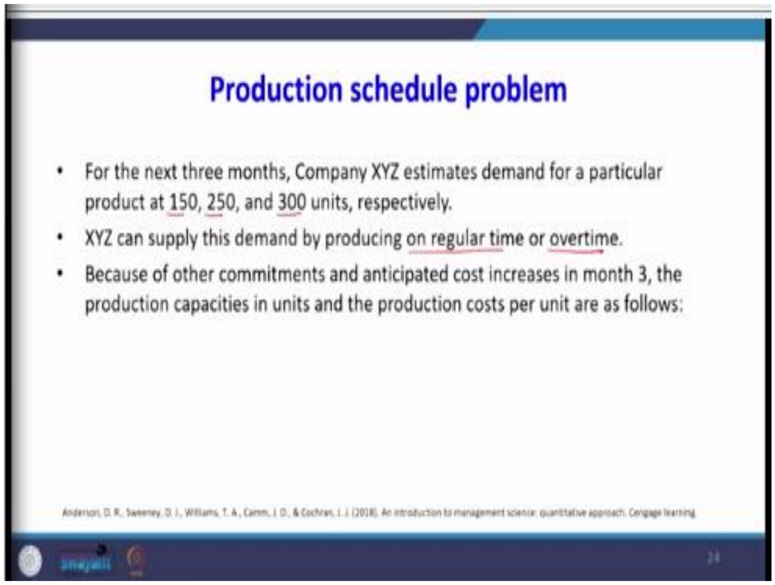


Now, this is my complete LP model. So, this is a minimization problem; we have the production constraint, and we have the demand constraint. So, I am going to solve this problem with the help of a solver. The problem that I have formulated I am going to solve with the help of a solver. So, I have written the coefficient objective function, the value of decision variables, I have written all the production constraints, and the demand constraints.

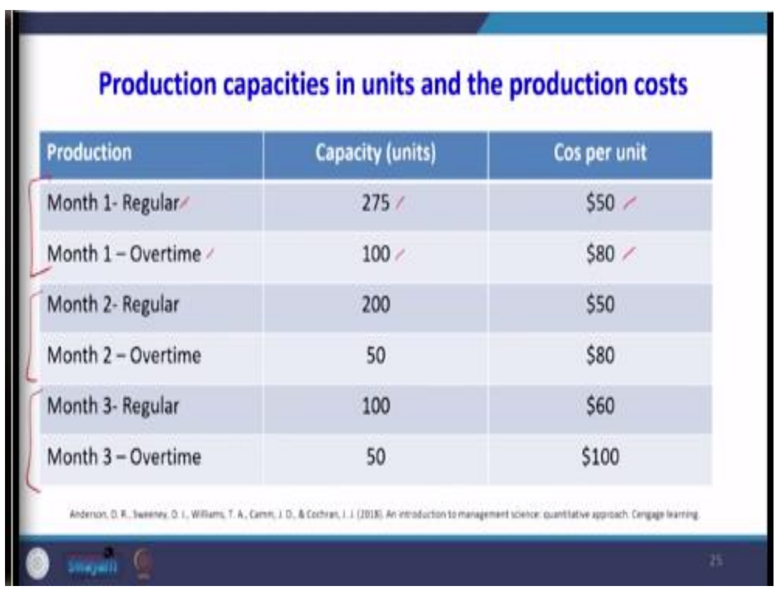
Now go to data and solver, so here the value of the objective function is D17; this is a minimization problem. So, the changing cell is B5 to H5; I have written all the constraints. Here, I am going to solve this in a linear model; I go to change it into linear; this is not nonlinear simplex LP. So, when I solve it, I press ok, and now the value of the objective function is 5150; now I got the production quantity X_{15} = 500, X_{26} = 300, X_{37} = 700, X_{48}= 400. The inventory units are only for the first month. I am going to stock 200 units. After that, I will not maintain any inventory. Now I will interpret the result in the lecture.



Now you see that I have got a value of 600, so this is the value of 600, so I have to produce 600 units. So, 400 I will consume for this month, then the remaining 200 I will stock as an inventory. In the second quarter, I will produce 300, I already have 300, but the demand is 500, so 300 + 200 will become 500. In the third quarter, I will produce exactly what is required, that is 400. Similarly, for the fourth quarter also, I will produce only 400 units.

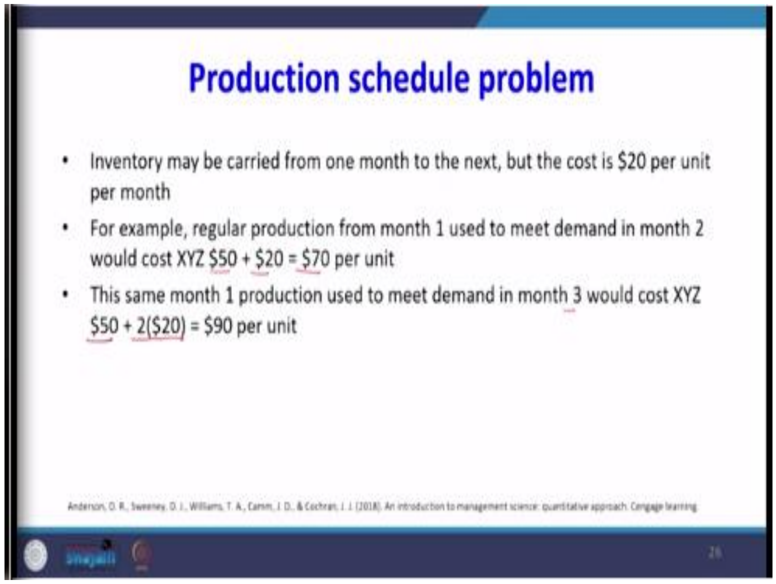


I am going to take another problem, a production scheduling problem, that I am going to solve with the help of network models. So, the problem is for the next 3 months, a company XYZ estimates demand for a particular product. What is the demand? 150, 250, and 300 for the next 3 months; XYZ can supply this demand by producing at regular time and over time. Because of other commitments and anticipated cost increases in month 3, the production capacities in units and the production cost per unit are as follows. What are they?

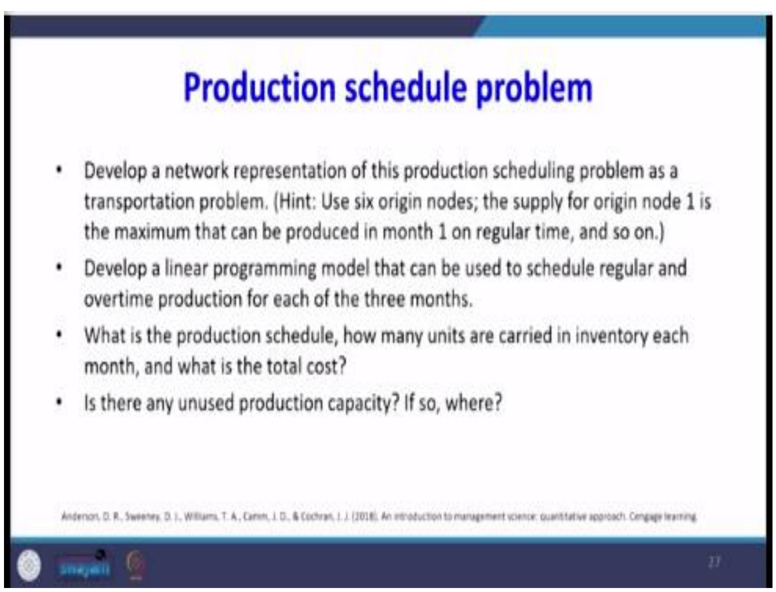


Production	Capacity (units)	Cos per unit
Month 1- Regular	275	$50
Month 1 – Overtime	100	$80
Month 2- Regular	200	$50
Month 2 – Overtime	50	$80
Month 3- Regular	100	$60
Month 3 – Overtime	50	$100

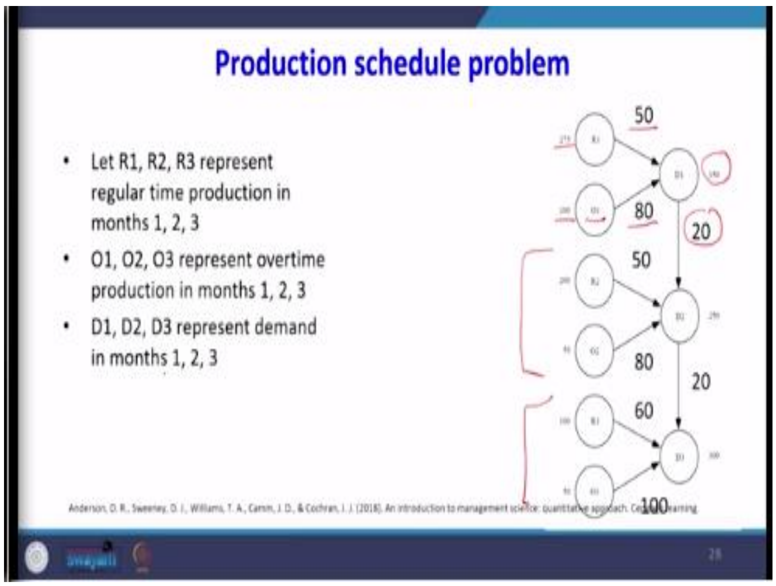
Three months they can go for regular production; maximum capacity is 275, and the cost per unit is 50. The same month, they can go for overtime also; capacity is a maximum of 100, so the cost for overtime units is 80 dollars. Similarly, for month 1, I can go for regular and overtime; in month 2, I can go for regular production, overtime production, and in month 3, I can also go for regular production, overtime production. Now the solution which you need to give is, in each period how many units have to be produced regularly, how many units should be produced by overtime?



The inventory may be carried from one month to the next, but the cost of inventory is 20 dollars per unit per month. For example, the regular production from month 1 used to meet demand in month 2 would cost 50 dollars + 20 dollars, so the total is 70 dollars per unit. In the same month, 1 production used to meet the demand in the third month would cost, so 50 normal production plus because it is stored for 2 times. So, per month, inventory cost is 20 dollars, so it will become 40, so 50 + 40 is 90 dollars per unit. So, the inventory carrying cost is 20 dollars per unit per month.

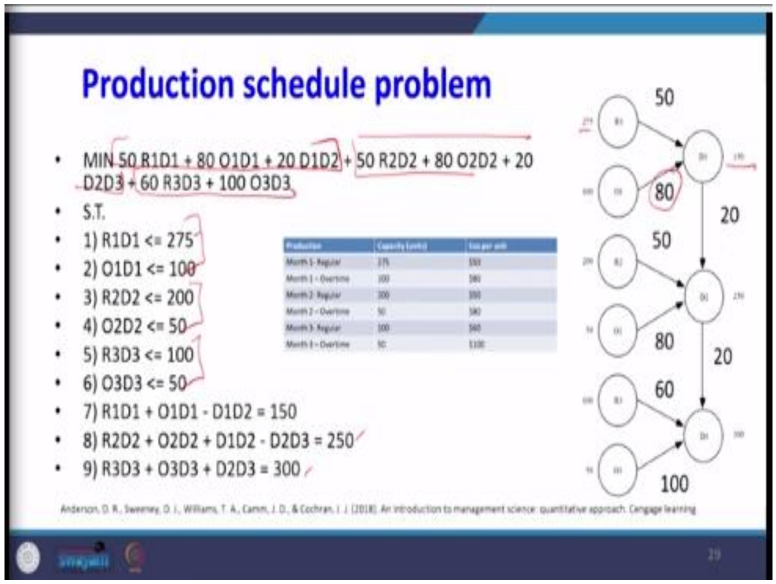


So, what do we have to do? We have to develop a network representation of this production scheduling problem as a transportation problem. What can we use? We can use 6 origin nodes; the supply from origin node 1 is the maximum that can be produced in month 1 on regular time and so on. For the next one, we have to develop an LP model that can be used to schedule regular and overtime production for every 3 months. Then what is the production schedule, how many units are carried in inventory each month and what is the total cost? Is there any unused production capacity? If so, where? So, these are the questions we need to answer. First, we represent the problem in the form of a network.



So, what I have done? The production scheduling problem I have represented is in the form of a network. What is that? R1 represents regular time production period 1, how much we can go for regular capacity maximum 275. Then O1 represents overtime for period 1, so how much can we go to a maximum of 300? So, 50 represents the regular time production cost per unit, 80 represents the overtime production cost per unit, and 150 represents demand for period 1.

This 20 represents if you produce in month 1 and you store it for the next month. The inventory carrying cost is 20 dollars per unit. Similarly, I have written for period 2 and period 3 also. So, here, R1, R2, and R3 represent regular time production in months 1, 2, and 3. O1, O2, and O3 represent overtime production in months 1, 2, 3, D1, D2, and D3 represent demand in months 1, 2, 3.

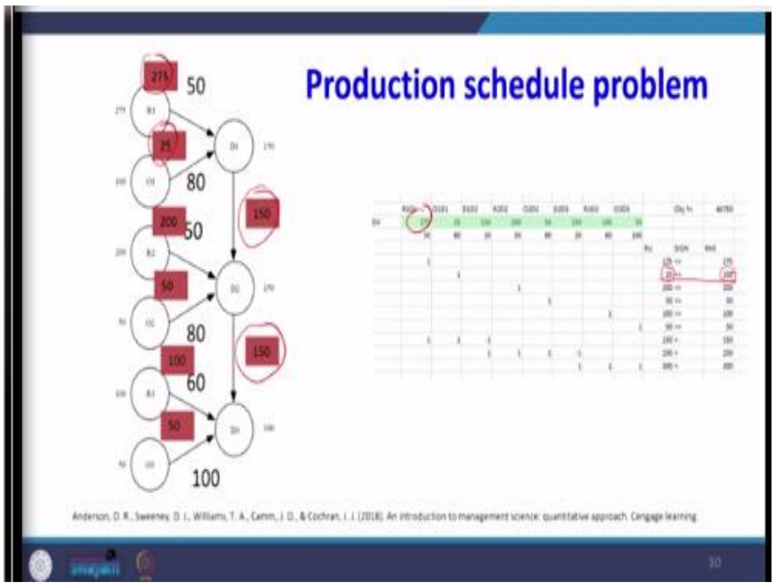


Now, the LP formulation for this model is we have to minimize production cost and inventory carrying cost when I say production cost, that includes regular production cost and overtime production cost. For example, the month 1, R1D1 represents the number of quantities produced by regular production, that is, R1D1, so 50. O1D1, that is 80, is the number of units produced over time. This 20 D1D2 represents inventory carrying cost, so this term is for period 1.

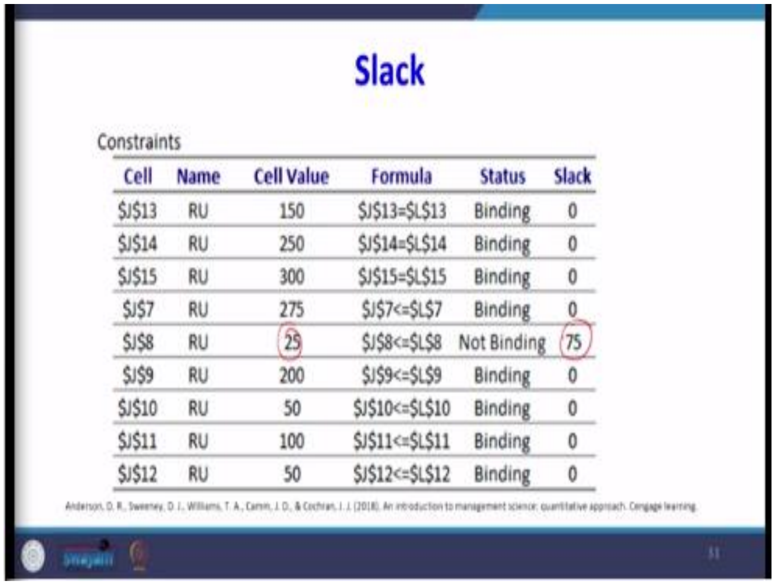
Similarly, for period 2, similarly, these terms are for period 3, 60 R3D3 and00 O3D3. In the same way, there is a production constraint and demand constraint, production constraint for regular R1D1 maximum 275, this 275, O1D1 100. So, this is for months 1, month 2, month 3 production constraints. Then what is the demand constraint? As usual, R1D1 + O1D1 will be used to meet this 150 demand, and any additional units that will be D1D2 will be stored as an inventory.

So, that is brought on the left-hand side, so -D1D2, similarly I have written for period 2 and period 3, so this is our complete production scheduling problem. Now, I am going to solve this problem with the help of a solver. Now, the problem that I have formulated is I am going to solve with the help of a solver. So, I have written, as usual, the coefficient of the objective function, the value of decision variables and the constraints, and the objective function.

So, when I go to data, I go to the solver, so this is a minimization problem because we have to minimize production cost, overtime cost, and inventory cost. The changing cell is B4 to I4, then I have written demand constraint and production constraint; this is a linear programming problem. If I solve, so the value of the objective function is 46750. So, I got to 275 in R1D1, which means the regular time production for period 1 is 275. O1D1 overtime production for period 1 is 25, like what I got for period 2 and period 3. So, I will explain with the help of the network.



Now you see that I have written my solutions here. So, 275 is this one, so I should go for producing 275 units through regular, 25 units through overtime, and 150 should be used as an inventory. Similarly period 2 also 150 should be used as an inventory. So, if you do that the maximum cost is 46750.



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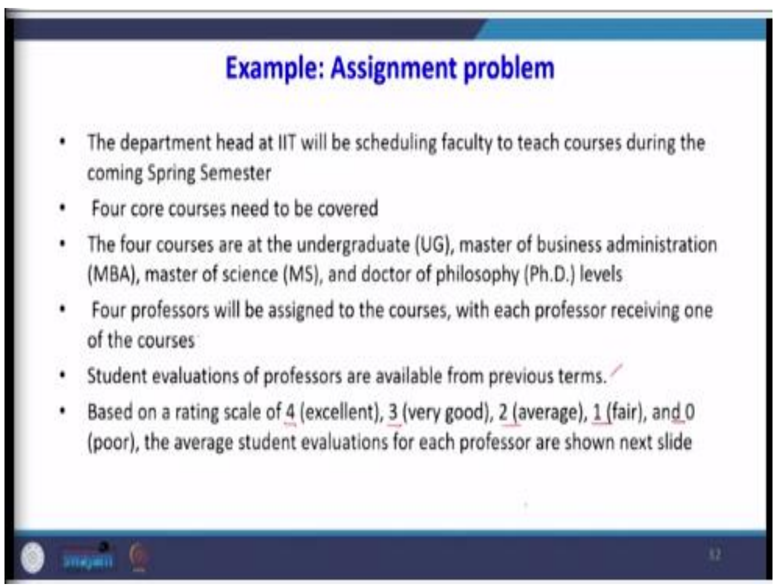
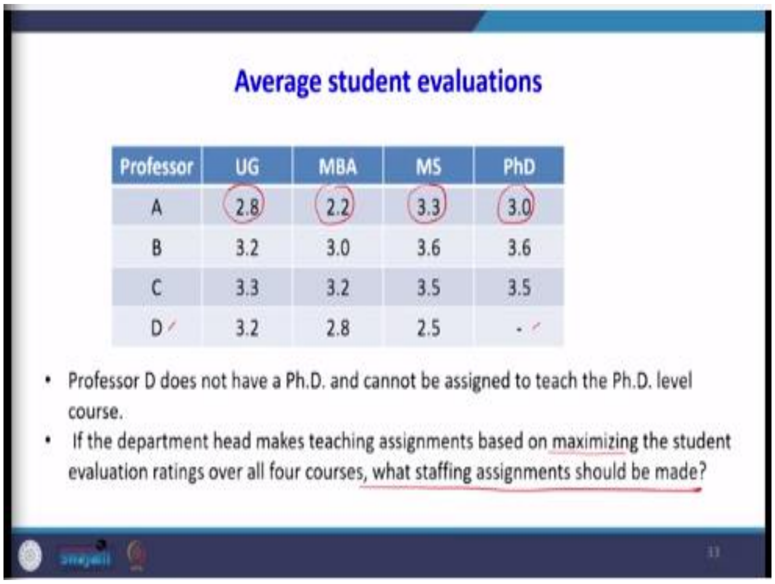
$J$15	RU	300	$J$15=$L$15	Binding	0
$J$7	RU	275	$J$7<=$L$7	Binding	0
$J$8	RU	25	$J$8<=$L$8	Not Binding	75
$J$9	RU	200	$J$9<=$L$9	Binding	0
$J$10	RU	50	$J$10<=$L$10	Binding	0
$J$11	RU	100	$J$11<=$L$11	Binding	0
$J$12	RU	50	$J$12<=$L$12	Binding	0

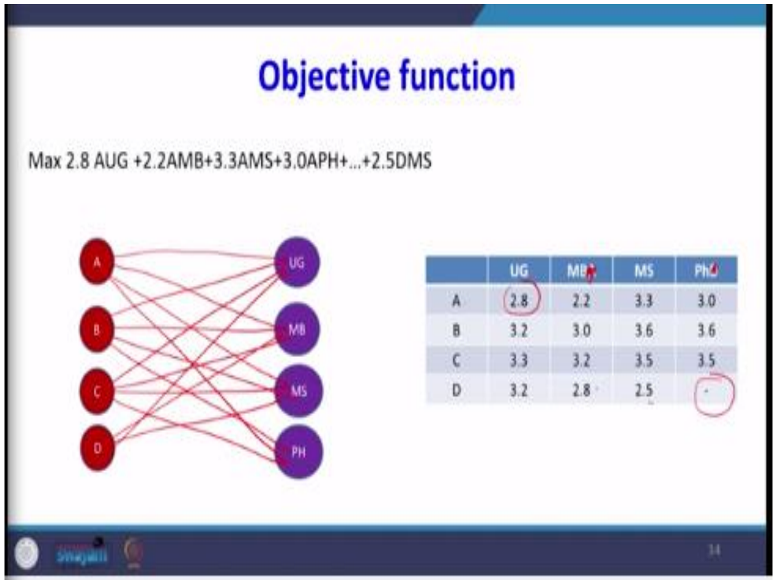
Now you see that there are some unused facilities there. So, here the maximum overtime is 100 but we produce only 25, so there are 75 unused overtime is available. So, we can see from here also, you see this one in period 1, the maximum overtime production is 100 but we are producing only 25, remaining 75 capacity is not utilized, that is the meaning of this 75.



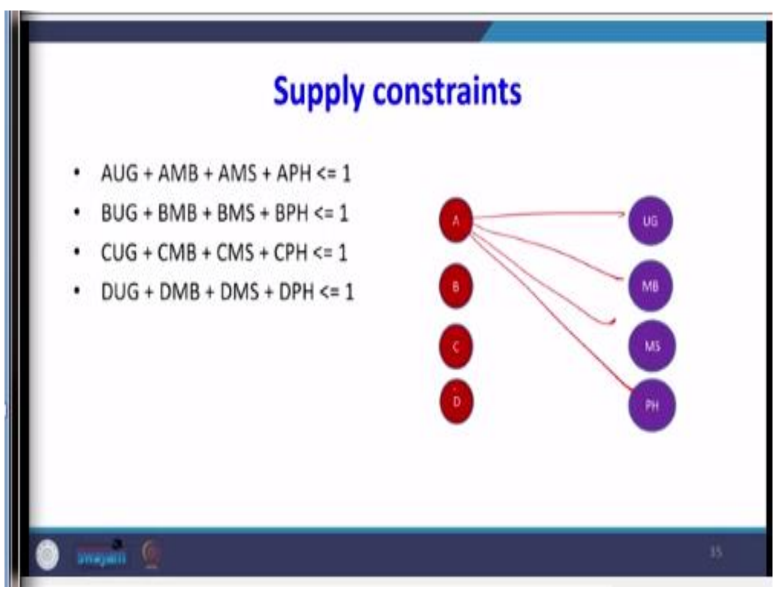
Now I am going to solve an assignment problem with the help of network models. The department head at IIT will be scheduling faculty to teach courses during the coming spring semester. Assume that the 4 core courses need to be covered; the 4 core 4 courses are at the undergraduate level, at the MBA level, at the master of science level, and at the Ph.D. level. So, the 4 professors will be assigned to the courses, with each professor receiving one of the courses. So, student evaluations of the professors are available from the previous term. So, the student feedback is available on a scale of 1 to 4. Based on the rating scale, 4 means excellent, 3 means very good, 2 means average, 1 means fair, and 0 means poor. The average student evaluation for each professor is also shown in the next slide, like this.



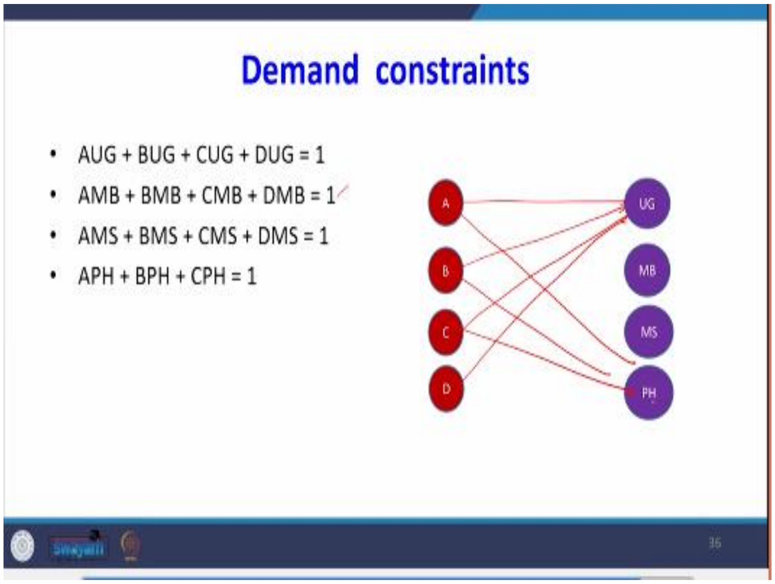
So, what is this meaning? Professor A's average teaching score is 2.8 for undergraduate courses. So, for an MBA, it is 2.2; for an MS, it is 3.3; for a PhD, it is 3, like this for all 4 professors is given. But you see, Professor D is not teaching any subject for PhD level courses, so Professor D does not have a PhD and cannot assigned to teach the PhD level courses. Now, the problem is, if the department head makes teaching assignments based on maximizing the student's evaluation ratings over all 4 courses, what staffing assignment should be made? If the HOD thinks that student satisfaction has to be maximized for each professor, which course has to be allotted? Whether Professor A should be allotted for the UG course or he should be allotted for the MBA course, or the MS and PhD. So, now this is an assignment problem.



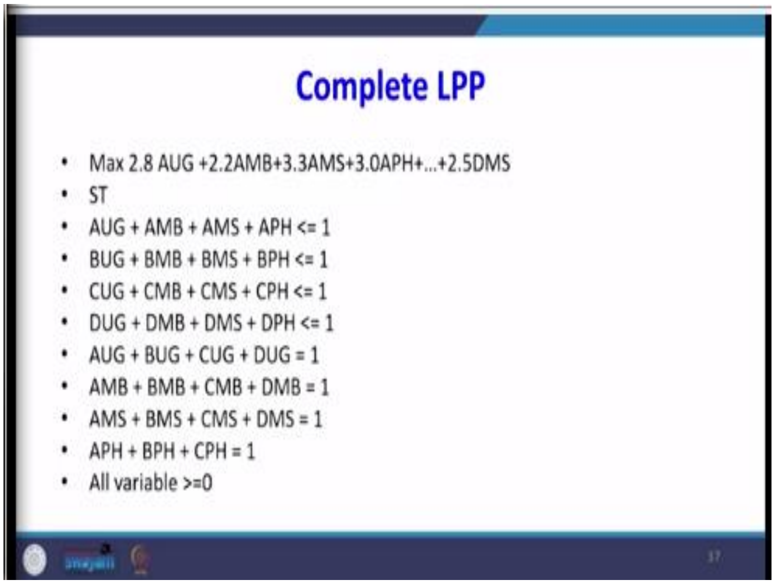
So, I am going to explain this with the help of a network. Say, for example, Professor A can take UG, MBA, MS, and PhD, and B can also go for UG, MBA, and MS, and Professor C can also teach UG courses, MBA courses, MS courses, and Ph.D. courses. But Professor D can teach UG, MBA, and MS, but not PhD. So, this is the representation of network models for an assignment problem. So, how I wrote this 2.8? If professor A is teaching for UG, this will be AUG is 2.8, so AMB, do not write this AMB is 2.2 MB. So, 3.3 AMS is only a PhD, so for a PhD like this, I have written up to 2.5. But you see that here we did not include that variable, so this is my objective function.



And what is the supply constraint? One professor can teach only one course, whether he can teach for UG or PG or MS or PhD. So, that is why this AUG, AMB, AMS, and APH should be less than or equal to 1. Similarly, Professor B can teach any 1 course, similar to Professor C and D.



And another thing is the demand constraint; what is the demand constraint? So, here, the UG course required 1 professor, so that means AUG, BUG, CUG, and DUG should be equal to 1 similarly MBA also records 1 professor, so AMB, BMB, CMB, DMB = 1 for MS also. But the last one, you see that APH, BPH, and CPH should be equal to 1 because Professor D cannot teach Ph.D. level courses; this is equivalent to our demand constraint.



So, now this is my complete LP problem for the assignment case. This I am going to solve with the help of a solver, and then I am going to interpret the result. Now, I am going to solve the formulated problem with the help of Excel. So, I have written decision values, coefficient objective function, and the value of objective functions, as usual. Here, when I solve with the help of a solver, so data and solver, I have to maximize student satisfaction.

So, it is a maximization problem; the changing cell is D4 to R4, so even though I have written less than or equal to, I can write equal to also. So, when I solve this one, you see that the value of the objective function is 13.3. So, the average student feedback will become 13.3, We see that for Professor A, the MS course has to be allotted, for Professor B, the PhD course should be allotted, and for Professor C, the MBA course should be allotted.

You see that this is an assignment problem; we did not consider the integer constraint. Because the value of constraint is integer in nature, it automatically takes the solution, which also comes as an integer. But if you want you can make a restriction but there will not be any problem with this. Suppose we are getting a, for example, 0.7 over 0.6, so we can make a restriction that makes it an integer. Because already the values are in the integer form the solution also will get in the integer, yes

This is an optimal solution, so what are we seeing? As usual, for Professor A, we are allotting an MS course; for Professor B, we are allotting PhD course; Professor C is allotted for the MBA course, and Professor D is allotted for the UG course. So, in this lecture, I have had a maximum flow problem, then I have taken a production and inventory problem, then I have taken one more problem on schedule, and one more problem on assignment also. I have formulated all these problems by using the concept of network models, solved them with the help of a solver, and then interpreted the result. Thank you very much.