Design Making With Spreadsheet
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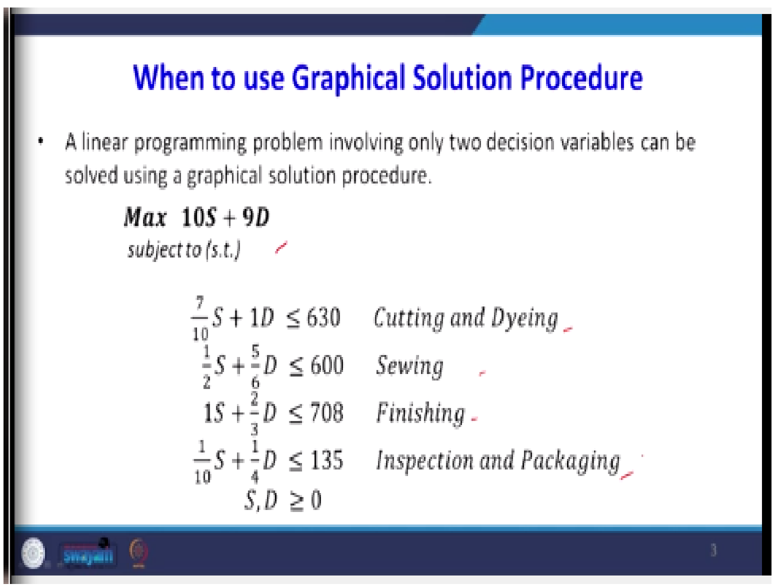
Lecture-04
Linear Programming Problem–Graphical Solution Method

Welcome students to another lecture on the course decision-making with the spreadsheet. In the previous lecture, I talked about how to formulate a linear programming problem. In this lecture, I will explain how to solve a linear programming problem. One method for solving a linear programming problem is using a graphical method. So, the agenda for this lecture is the graphical solution for a maximization problem.



This is the problem that I have brought here, which I have taught in the previous lecture. A linear programming problem involving only 2 decision variables can be solved using a graphical solution procedure. So, this problem, maximization of profit = 10S + 9D subject to there are 4 constraint is there. I will explain how to solve this problem using a graphical method.

What are the graphical solution procedures? The graph will have a value of S on the horizontal axis and the D on the vertical axis. So, in the horizontal axis, I have taken S, and in the vertical axis, I have taken D. Any point on the graph can be identified by the S and D values, which indicate the position of the point along the horizontal and vertical axis, respectively.

Because every point S and D corresponds to a possible solution, every point on the graph is called a solution point. The solution point where S = 0 and D = 0, this point is referred to as the origin. Because S and D must be nonnegative, the graph in Figure 1 only displays positive solutions. Because the S is greater than or equal to 0 and D is greater than or equal to 0, so we will not consider any negative solutions.

For example, if you consider this point, say S = 200 D = 800, and if it is this point S = 400 and D = 300, this is the 1 solution point.

The next stage is plotting all the constraints into the graph. So, let us see how to plot constraint number 1, which is cutting and dyeing. So, I have taken the constraint number 1: (7/10) S + 1D <= 630. To show all solution points that satisfy this relationship, we start by graphing the solution point satisfying the constraint as an equality. Even though it is less than or equal to a sign there, we will consider it equal to a sign.

So, those are the points where (7/10) S + 1D = 630 because the graph of this equation is a line, it can be obtained by identifying 2 points that satisfy this equation and then drawing a line through the points. So, how can we get the 2 points? In this equation, if you substitute S = 0, you will get your D point. So, if you substitute S = 0, say D = 630, we get 1 point (0, 630). The second one you substitute, D = 0, will get the value for S.

So, when you substitute D = 0, the value for S is 900, so we get another point (900, 0), thus the second point satisfying this equation S = 900, D = 0. Given these 2 points, we can now graph the line corresponding to the equation. So, we got the 2 points; now we can draw a line.

Because all the points on the line satisfy (7/10) S + 1D = 630 , we know any point on this line must satisfy the constraint. So, all the point above this line will satisfy the constraint. But our constraint is less than or equal to 630. But where are the solution points satisfying where the constraint is less than or equal to 630? So, how do we identify which side of this line satisfies our constraint?

You have to take any point below this line and another point above the line. For example, I have taken arbitrarily 200 and 200; you have to substitute these values into this equation. If this point is satisfied, you have to shade on the left-hand side that is towards the origin. If it is not satisfying, you have to shade on the outside the side; on this side, you have to shade it.

Consider 2 points S = 200, D = 200 S = 600, and D = 500, which I have explained in the previous slides. You can see from Figure 2 that the first solution point is below the constraint line, and the second point is above the constraint line. Which of these solutions will satisfy the cutting and dying constraint? When you substitute S = 200 and D = 200, we get the values of 340. So, this 340 is less than 630, so the point which is on the left-hand side of this line satisfies our constraint.

Because the 340 hours are less than 630 hours available, so S = 200 and D = 200 production combination or solution point satisfy our constraint. Let us take another point, S = 600, D = 500. When you substitute this, we are getting the value of 920, but the 920 >= 630, that is violating our constraint. So, the region that satisfies our constraint is on the left-hand side, this region.

If a solution point is not feasible for a particular constraint, then all other solution points on the same side of the constrained line are also not feasible. If the solution point is feasible for a particular constraint, then all other solution points on the same side of the constraint line are possible for that constraint. So, what is the point here suppose you take any one point if it is satisfying over constraint, so this is less than or equal to, this equal to represents only the straight line. When you write less than or equal to, this shaded line is shaded in blue color. So, what this point says is that if any one point satisfies this region, all other points below this line will satisfy our constraint.

So, one has to evaluate the constraint function for only one solution point to determine which side of a constraint line is feasible. Obviously, we have to see which side of this constraint line is feasible, whether it can be either on the left-hand side or the right-hand side. In Figure 3, we indicate all the points that satisfy the cutting and dyeing constraints of the shaded region. So, this region is the region that satisfies our constraint, so this will be less than or equal to 630.

In the same way, I have taken another constraint, which is the sewing constraint. So, this is <= 600, <= 708, and <= 135. So, these points satisfy our constraint.

If I superimpose all the constraints, I will get here this kind of figure. So, the area that is common for all the constraints is called the feasible region. So, this is the region that is common and satisfies all the constraints, so that region is a feasible region. Why are we calling it a feasible region? If you take any point, that point will satisfy all our constraints.

Feasible solution and feasible region: the shaded region in this figure includes every solution point that satisfies all the constraints simultaneously. Solutions that satisfy all the constraints are termed feasible solutions, and the shaded region is called the feasible solution region or simply the feasible region. Any solution point on the boundary of the feasible solution or within the feasible region is a feasible solution point. So, in this boundary, this is a feasible solution point.

Finding the optimal solution by trial and error. One approach to finding the optimal solution would be to evaluate the objective function for each feasible solution; what is the meaning? Every solution is a feasible solution; we can substitute these values in our objective function. The optimal solution would be the one yielding the largest value. Suppose you take this as point number 1, point number 2, so you will get Z1, Z2. If the Z2 is larger than the Z1 because it is a maximization problem, the Z2 is the point corresponding to Z2, maybe say somewhere here S2; D2 may be your optimal solution.

The difficulty with this approach is that an infinite number of feasible solutions are possible. So, checking all the points and verifying which is the optimal value is a very difficult one. Because one cannot possibly evaluate an infinite number of feasible solutions, this trial-and-error procedure cannot be used to identify the optimal solution, so the trial-and-error solution will not be used.

So, we are going to find out another method; what are we going to do? We are going to select some arbitrary value for our profit contribution. So, select an arbitrary value for the profit contribution and identify all feasible solutions S and D that yield the selected value. For example, which feasible solution provides a profit of 1800 dollars? So, I have fixed; suppose I want to have my profit contribution at 1800 dollars. What should be the value of S and D? These solutions are given by the values of S and D in the feasible region that will make the objective function.

Solution for an arbitrary value of 1800 dollars of your profit contribution. Suppose I want to make a profit contribution of 1800 dollars, so this is the line of 10 years + 9D = 1800 dollars. If I want to make this much profit contribution, I have to find out what the value of S and D is because it intersects on the x-axis and y-axis. I can find out where it is intersecting; otherwise, in this equation, if I substitute S = 0, I will get the D value, and if I substitute D = 0, I will get the S value, so that (180, 0) one value is, then (0, 20 is another value.

This expression is simply the equation of a line, in which one 10 years + 9D = 1800, all feasible solution points S, D yielding a profit contribution of 1800 dollars must be on the line. So, for all the points, there is a point mean value of S and D that will satisfy our objective function; it is 1800. The procedure for graphing the profit or objective function line is the same as what I have explained previously. So, when you substitute S = 0, the D will be 200, so we get one point, this point (0, 200).

Similarly, when you substitute D = 0, we will get another point of (180, 0. So, drawing the line through these 2 points identifies all solutions that have a profit contribution of 1800 dollars. So, the graph of this profit line is presented in this figure, so this line is our profit line.

Because we have taken only one value, I can take another profit contribution value arbitrarily, which is 3600, 5400. Then I can plot this line: 10 years + 9D = 3600, 10 years + 9D = 5400. The objective function is to find a feasible solution that yields the largest profit contribution. Let us proceed by selecting a higher profit contribution and finding the solution yielding the selected value. For instance, let us find all solutions yielding profit contributions of 3600 dollars and 5400 dollars. To do so, we must find the S and D values that are on the following lines. So, these are the 2 lines, we have to find all the points in these 2 lines.

Using the previous procedure for graphing the profit and constraint line, we draw the 3600 and 5400 dollar profit lines as shown in the graph. Although not all solutions point to the 5400 dollar profit line or feasible region, at least some points on the line are, and it is, therefore, possible to obtain a feasible solution that provides a 5400 profit contribution. When you look at this line where it is said this line, it is not covering; see, this is covering only from this point to this point. This region is not covered by this line, but all the points in this region will satisfy your objective function.

Can we find a feasible solution yielding an even higher profit contribution? Please look at Figure 8 and see what general observations you can make about the profit lines that are already drawn. There are 2 points; point number 1 is the profit lines are parallel to each other; look at this one: all the points are parallel to each other. Second, higher profit lines are obtained as we move farther from the origin.

So, when we move farther from the origin, we get a higher profit line. So, when you go this way, it may be 5600; this may be something else. So, what are the 2 points? All the objective functions are parallel lines, and when we move away from the origin the value of the objective function increases.

These observations can also be expressed algebraically. Let P represent the total profit contribution that is a P = 10S + 9D. So, I am going to write this one y = mx + c because I want to know what the slope is here because D, this is D, this is S. So, if I write in this form, I can find out this is the slope of that objective function, what is that? The slope is -10 by 9.

So, when P = 1800, when you substitute in the previous values, you will get (-10/9S)*200. When you increase P = 3600 dollars, again, this is the equation. So, what is the similarity you are saying? Even though the value of the objective function on the right-hand side is increasing, the slope remains constant at (-10/9S), (-10/9S), and (-10/9S). So, this implies that the objective function lines are parallel to each other.

So, the slope (-10/9S) is the same for each profit line because profit lines are parallel. Further, we see that the D intercept increases with larger profit contributions. Thus, higher profit lines are farther from the origin we have understood algebraically.

Next, we will look at how to find out the optimal solutions. Because the profit lines are parallel and higher profit lines are farther from the origin. We can obtain a solution that yields increasingly larger values for the objective function by continuing to move the profit line farther from the origin in such a fashion that it remains parallel to the other profit lines. So, what does it mean? Suppose we have the objective function like this: when we draw parallel lines and when we move away from these origins, there is a possibility the value of your objective function will increase. However, at some point, we will find that any further outward movement will place the profit line completely outside the feasible region.

So, when we move away from the origins, we should be careful that we are not completely moving away from the feasible region. Because solutions outside the feasible regions are unacceptable, the point in the feasible region that lies on the highest profit line is the optimal solution. So, what do we have to do? You have to draw different objective function lines; we have to keep on moving away from the origin. At the particular point, the value will be highest, so that corresponding point is your optimal solution.

So, what can you do? You can take a ruler or the edge of a piece of paper and move the profit line as far from the origin as you can. So, what can you do? You can move this way, what is the last point in the feasible region that you reach? So, this point, which is the optimal solution, is shown graphically. So, this is the extreme point is our optimal solution. So, what is this point, and how are we getting this point? There are 2 constraints that intersect. One is finishing, and the other is cutting and dyeing. The 2 lines that intersect that corresponding point are called your optimal solution; when you substitute that optimal solution with your objective function, you get your objective function as 7668.

How do we get those intersection points? So, there are 2 constraints, so wherever the right-hand side is 630, which one? That is a cut dyeing constraint. So, from this (7/10)S + 1D = 630, even though it is less than or equal to because I want to know the value of S and D, I am taking equal to sign only, not less than or equal to, because otherwise I cannot solve it. So, (7/10)S= 630 - 1D.

So, from this, if you find the S value, S = 900 – (10/7)D, this is our value of S. Another constraint that is intersecting is your finishing constraint, so I have taken 1S + (2/3)D =708. So, instead of S I am going to substitute this value, so that I have taken here. So, 900 – (10/7)D + 2 (2/3)D = 708. So, what can you do? You can bring this 900 on the right-hand side. It will become -192, so the remaining is – (10/7)D + (2/3)D.

So, D is common, you take on the left-hand side, the remaining is – (10/7)D + (2/3)D; when you simplify this, you will be getting –(16/21)D = -192. So, with the D value at the end, you are getting 252. So, we got the D value; when you substitute this D value into this S, you will get 540. So, the optimal point here is 540 - 252, which is where the 2 constraints intersect.

So, this point, this point is (540,252); how did I get this point? After drawing the objective function line, when I kept on moving away from the origin, I found the extreme right side edge is this one, so this point is formed by the intersection of 2 constraints. So, I have solved the 2 constraints to get this point, so (540, 252), when you substitute this value, you will be getting 7668 dollars.

A note on the graphic line: suppose a company manufactures 2 models of small tablet computers. Assume that there are 2 models; one is an assistant model, another one is a professional model. Management needs 50 units of the professional model because it is its own salesforce and expects sales of the professional tablet to be at most one-half of the sales of the assistant tablet computer. So, the constraint enforcing this requirement is (P – 50) because this 50 is consumed by their own salesforce, and it cannot exceed one-half of the assistant tablet. So, less than or equal to (1/2) A, when you simplify this you will be getting (2P – 100) <= A.

So, this is a constraint I need to plot, so substitute P = 0 will get A = -100, so this is the point. So, when you substitute A = 0 you will get P = 50, so this is another point. So, what is the point you have to notice? Already, we have defined that the solution will be only on the positive side, but there may be a situation for drawing here this line. You may get your point that point may lie on the negative side. But at the time considering the feasible region, you have to consider only the positive side, the positive value of A and P; that is a point you should note.

There may be another situation, R > = T, so R - T greater than or equal to T, so that is R = T, so that is this point where it is a 45-degree line. So, if R = 100 T and also 100, that is a way of drawing the constraint.

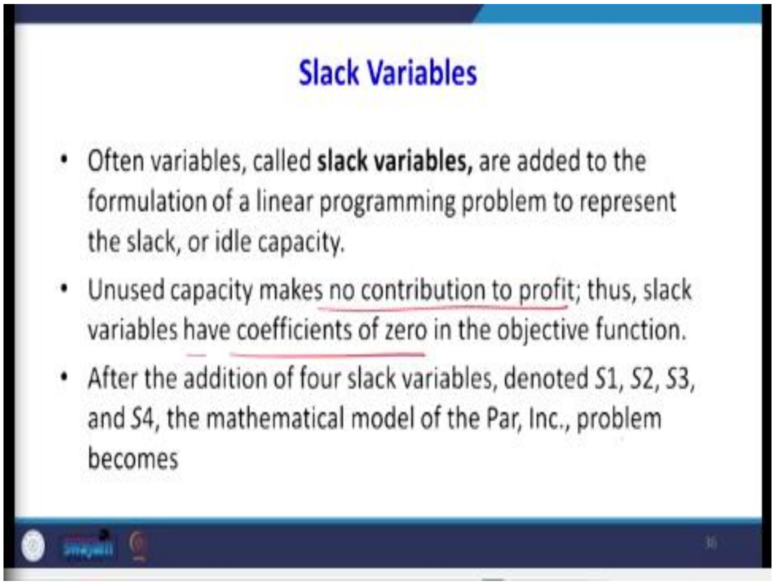
Next, we will be explaining an important term in a linear programming model, that is, the slack variable; otherwise, it is another name for unused resources. So, this was the given problem; we got the value. What is that optimal value? (S = 540 D = 252). So, in constraint number 1, when you substitute it, the resource used is 630, but the hours available is 630. So, all the resources are fully utilized, that is, unused resources.

So, the slack variable for this first constraint is 0. Look at the second constraint, the sewing constraint. When you substitute (540, 252), the resources consumed are 480, but the available resources are 600. So, 120-unit hours are not utilized, so here, the slack variable for the second constraint is 120. For the third constraint, the hours required is 708, and the hours available are also 708, so unused resources are 0; the resources are fully utilized, which means there are 0 slack values.

The last constraint hours required are 117, but hours available is 135; there is a positive slack variable. So, what does slack variable mean in the linear programming context? The unused resources is called our slack variables.

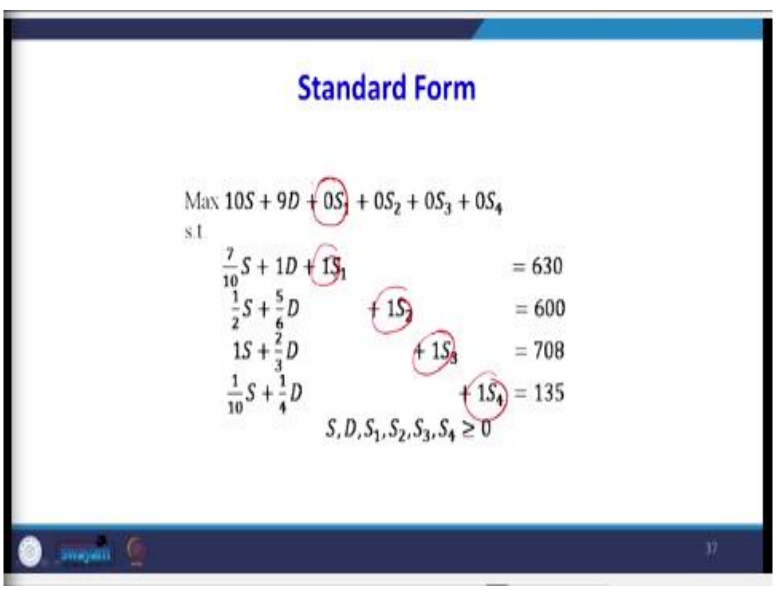
Thus, the complete solution tells management that producing 540 standard bags and 252 deluxe bags will require all available cutting and dyeing, and finishing time. That means there will not be any unutilized resources. So, in this department, all the resources will be fully utilized because the value of the slack variable is 0. While (600 – 480) = 120 hours of sewing time and (135 – 117) = 18 hours of inspection packaging time will remain unused.

So, what do we understand of these 4 constraints? We have 0 slack variables for 2 constraints; for another 2 constraints, we have some positive slack variables. If it is a positive slack variable in that department, there are unutilized resources available. So, 120 hours of unused sewing time and 18 hours of unused inspection packaging time are referred to as slack for the 2 departments. In linear programming terminology, any unused capacity for a constraint is referred to as slack associated with the constraint. So, in a linear programming model, if the constraint is the type of less than or equal to type, the corresponding unutilized resource is called a slack variable.

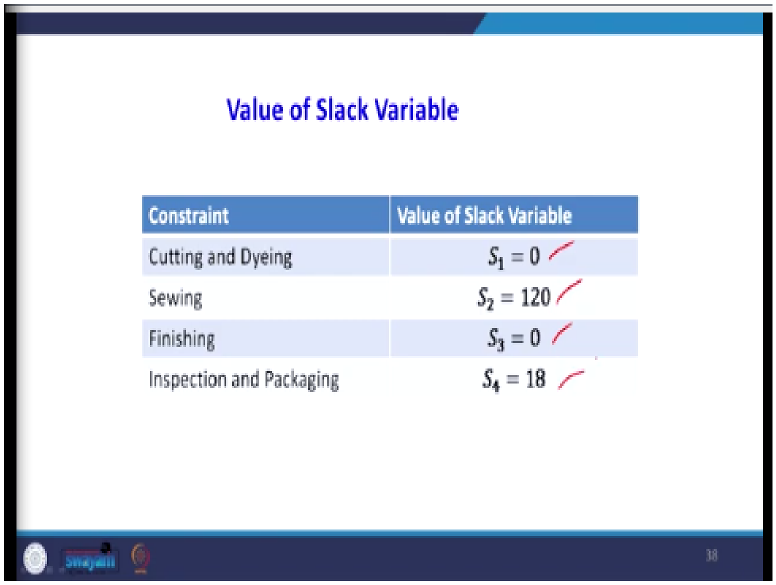


Often variables called slack variables are added to the formulation of a linear programming problem to represent the slack or ideal capacity. So, the slack unutilized resources have to be added to the linear programming problem. So, unused capacity makes no contribution to the profit because it is idle, and that is not going to help in achieving your profit. So, thus the slack variable has a coefficient of 0 in the objective function.

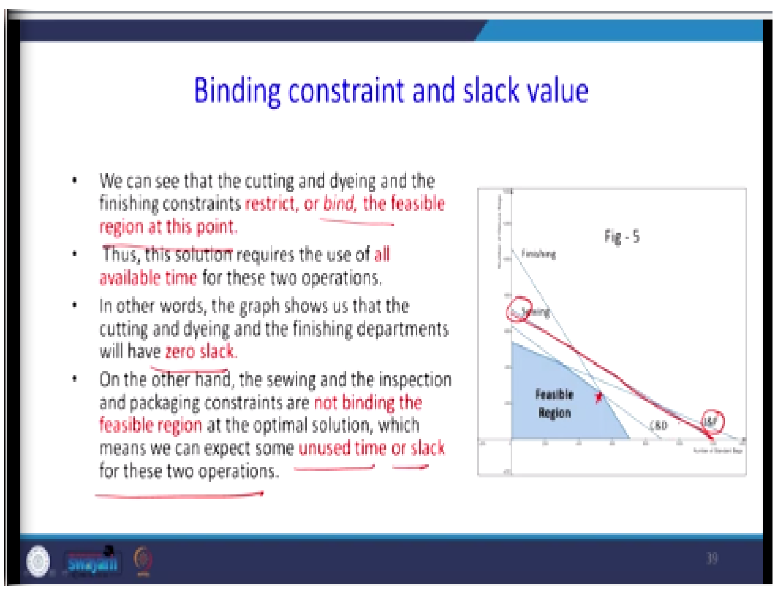
So, when you introduce the slack variable, the contribution of the slack variable is 0, so in your objective function, that will have the 0 coefficient. After the addition of 4 slack variables, denoted as S1, S2, S3, and S4, the mathematical model of the problem becomes like this.



See that we have added +S1, +S2, +S3, +S4 and it is equal to sign because the contribution of this S1 is 0, so in your objective function the coefficient is 0S1, 0S2, 0S3 and 0S4.



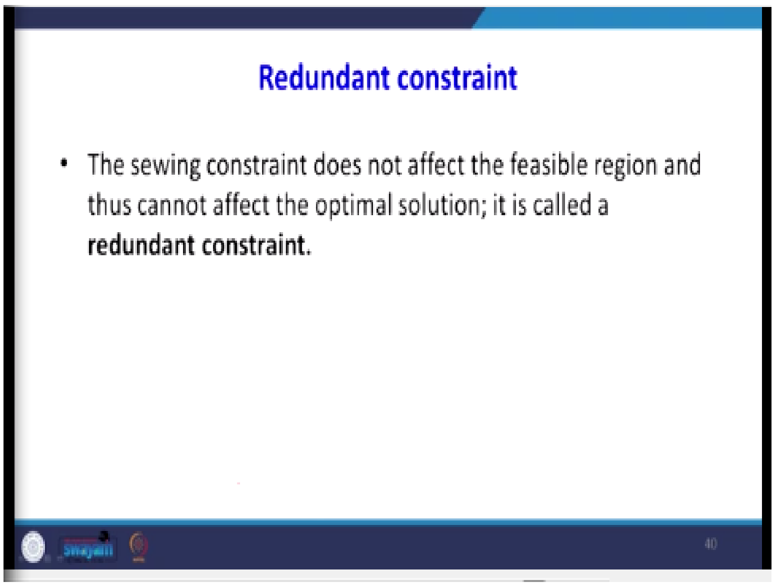
So, this is the value of a slack variable. So, the cutting and dyeing department has 0 slack variable, the sewing department 120, finishing 0, and inspection packaging 18.



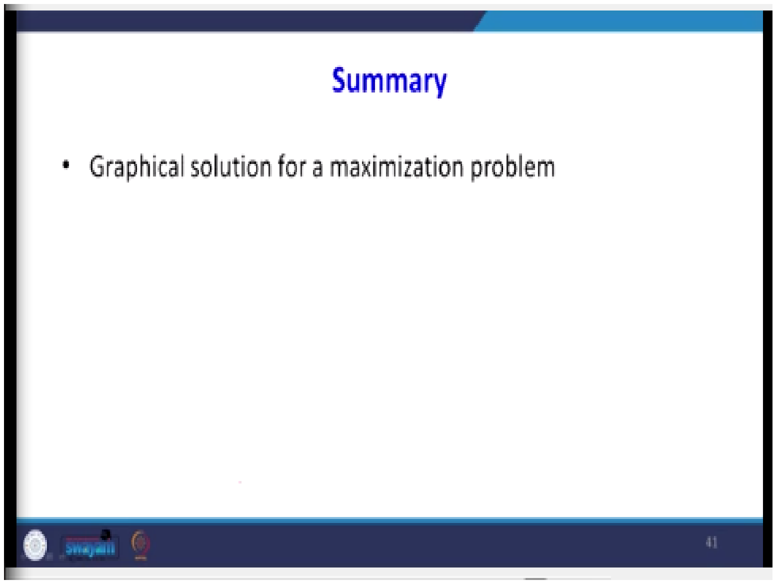
Now I will introduce another term called binding constraint. What is the binding constraint? We can see that the cutting, and dyeing finishing constraints restrict or bind the feasible region of this point, so this point. So, this point is formed by finishing constraint and cutting and dyeing constraint. So, this is restricting our optimal solutions beyond which it will become not optimal. So, this corresponding constraint is called a binding constraint.

Thus, this solution requires the use of all available time for these 2 operations, what are they? Finishing and cutting, and dyeing. In other words, the graph shows us that the cutting and dyeing and the finishing department will have zero slack. For any constraint, if it is zero slack, that constraint is called a binding constraint. On the other hand, sewing and inspection packaging constraints are not binding the feasible region; see the other constraint; what are they?

This inspection and this sewing do not bind the feasible region at the optimal solution, which means we can expect some unused time or slack or slack for these 2 operations; which one? Sewing and inspection.



The next one is the redundant constraint, another term. The sewing constraint does not affect the feasible region and thus cannot affect the optimal solution. It is called a redundant constraint.



When you go back, what is the redundant constraint? You see, this one sewing this constraint is not forming the part of the solution, so this constraint is a redundant constraint. Even though it is redundant, we have to keep it in our LP model because when the resources of any constraint change, the redundant constraint may be a binding constraint at a certain point in time. So, even though it is redundant it is a custom to mark that redundant constraint also.

In this class, I explained to students how to solve a linear programming problem using a graphical method. I have taken a problem; I have solved that problem using a graphical method. In that I have explained what is the feasible region, I have explained what is the optimal point and what is the optimal solution. Then I also explained what the slack variable is, and what a redundant constraint is. In the next class the same problem I will solve with the help of a graphical calculator called Desmos. After that, I will use an Excel solver to solve this problem; thank you very much.