Basic Linear Algebra
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology- Bombay

Lecture – 11
Solvability of a Linear System, Linear Span, Basis II

Can you think of some other subspace in R2, only every line through the origin is the sub space, this was one of them, I fix X0, X0 could be anything, right, so there is another possibility that means R lines passing through the origin in R2 are vector spaces in R2, right, some other example; so I am just looking at R2, simple one, you can visualise.

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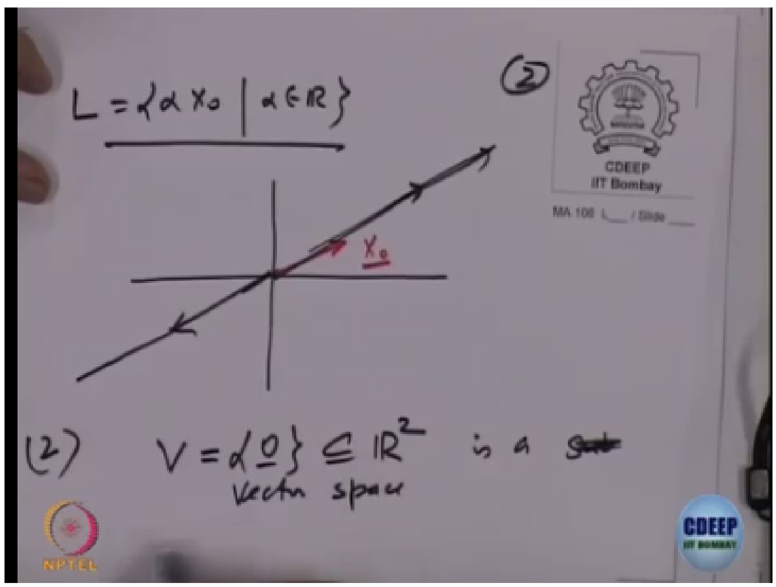

So, let us look at another example, let us look at the zero vector alone, so that is V, is that a vector space? Only one element is there, what all you multiply, it will always be 0 and it will be staying inside, so is a subspace, so oh sorry is a vector space.

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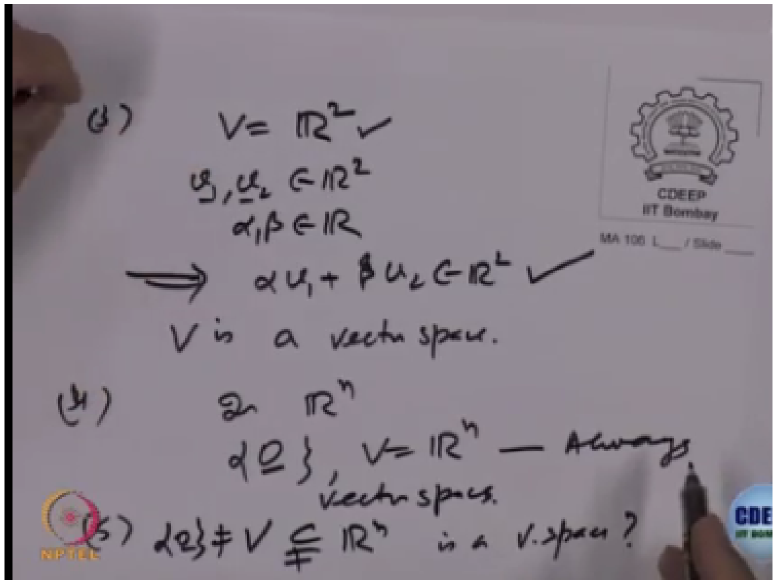

Let us look at the whole space plane or so let us look at third example, V = R2, V is the substrate of R2, right, V is the substrate of Rn, so I am looking at specialising n = 2, what about V = R2, is that a vector space, yes it is because if we take any 2 vectors, right, multiply alpha, so right, if V1, V2 belong to R2 and alpha, beta belong to R, does this imply that alpha V1 + beta V2, is again in R2; yes, it is the vector again, right.

So V is; V= R2 is a vector space, so in R2 or in Rn, so you can write generally in Rn, look at the zero vector and look at V = Rn, these are always vector spaces, right, so these are called trivial vector spaces, you do not have to verify anything they are obviously given right, there obvious examples of vector spaces, interesting thing; it becomes when you want a proper subset of Rn and you want say it is a vector space.

So, interesting thing is V proper in Rn and it should not be equal to 0 because 0 is always; right, right, A is a vector space, when is it a vector space? So, the examples I have been trying to give you are in R2, every line passing through origin is a vector space, zero; origin itself is a vector space; zero vector, whole plane is a vector space, you ask a question is there any other subspace, is there any other vector space in R2, is there any vector space in R2, other than 0, other than the line passing through origin and the whole space.

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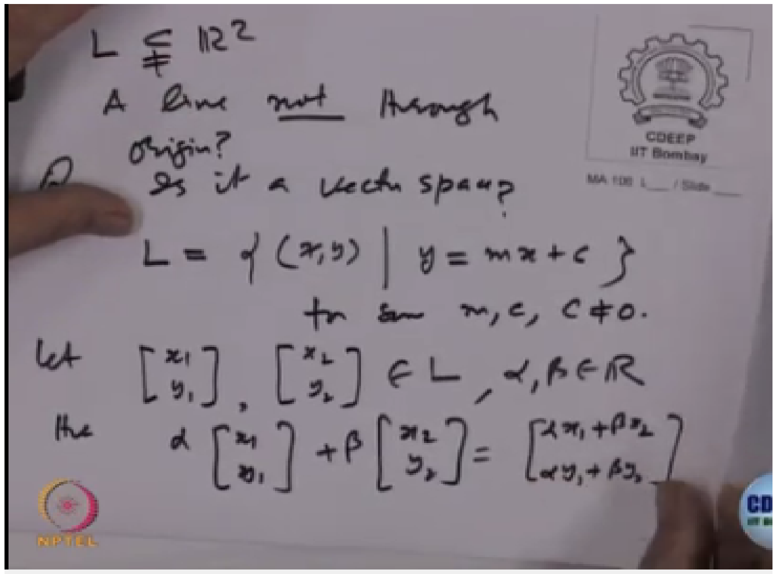

For example, let us look at this; this is an interesting example to consider, so look at a line not passing through origin, so L is the proper subset of R2, a line not through origin, so question is; is it a vector space, so that is the question, so is it a vector space, so how do you describe a line as a set of vectors, so I can write this line L as; I want to write set of vectors in R2, so how were been writing the vectors?

Normally, points in R2 are written as ordered pair; x, y you can write also has matrices, right, column vector x, y, whichever way, so let us write it as points for the time being, this x, y such that what is the relation between y and x, if they have to lie on a line, right, so y = some mx + c, right for some m, c and if you want not so origin, then c should not be equal to =0, right, so that is all points in the plane which lie on the line are given by these 2 constants m and c.

So, I want to check, right, so whether if I take 2 points here, whether the combination, right, so as vectors what you will write; x1y1, other vector would be x2 y2 on the line, so let belong to L and let us take alpha and beta belonging to R, then alpha of x1y1 + beta of x2y2, I want to check whether it belongs to L or not, what is that equal to; by our vector addition that is = alpha x1 + beta x2 and here it will be alpha y1 + beta y2, right, vector addition.

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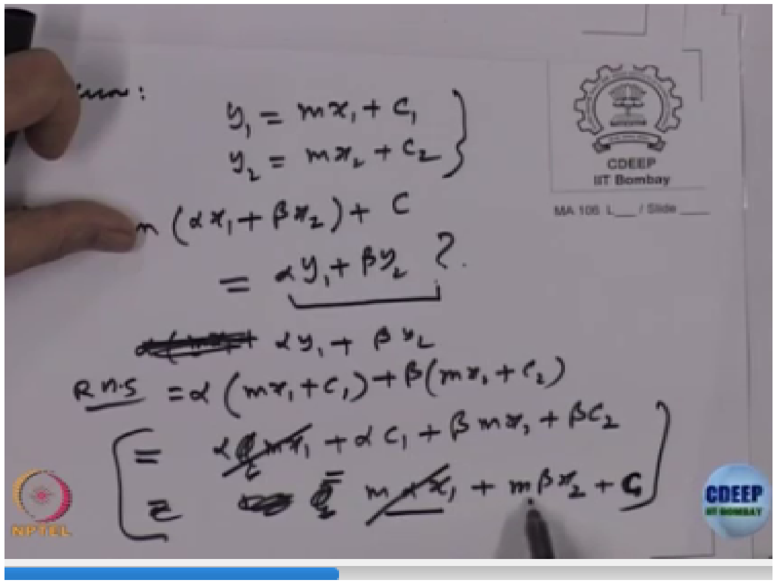

So, now let us go a step further and try to simplify this, what is the relation I know, so we know, what x1y1, x2y2 they belong to L, so what is y1 =; mx1 + c1, what is y2; mx2 + c2, right, so given this is the following true that means, alpha x1 + beta x2, right that is x squared, so whether can I say m times this + c = alpha y1 + beta y2, is that okay. If this point; s2 belong to L, then what should be the condition?

The y component should be = mx component +c, right, so can you say this is true then, right, so what is this, so this is alpha of mx1+; I do not; do you think okay, so let me just ask you instead of saying, do you think this will be true because what is y1; so, what is the right hand side? Let us look at the right hand side of this, what is that; alpha times y1, what is y1? X1 + c1 + beta times y2 that is mx2 + c2, so what is that =?

So that is alpha times mx1, okay alpha times mx1 + alpha times c1 + beta times mx2 + beta times c2, so I can write this as, right, so I want whether this is = this one; m alpha x1, so whether this is you want to know whether this is = this y1 y2 = m times alpha times, see this is y1, this alpha y1 + beta y2 that is = this that is a right hand side is = left hand side, what is the left hand side?

Alpha mx1 + m beta x2, you know there is no; m alpha + m beta x2 + c2, so you want to know whether these 2 are equal or not, right, so do you think they are equal, where is c, yeah, so that is c, right, see left hand side is alpha mx1+ beta mx2 + c, right, so what happens; alpha mx1, I can cancel out this, beta m beta x2, I can cancel out this.

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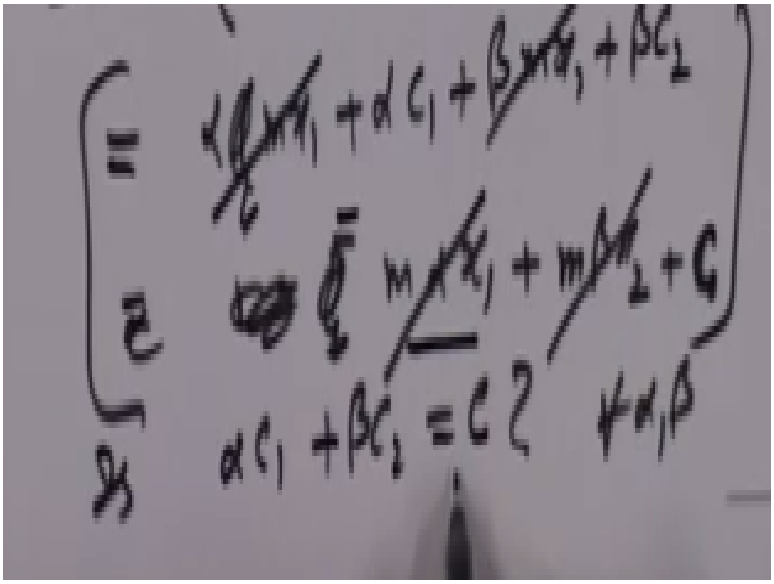

So that means this is asking for, is it true that alpha c1 + beta c2 = c, for every alpha, beta, right then only left hand side will be = right hand side but that is not true for every alpha beta, for example, alpha you can choose 0, beta you can choose 0, that means c should be 0 but c is fixed that is given to me, I cannot say it is 0, right, so this may not happen always. So that means what, so that is a contradiction, not possible always.

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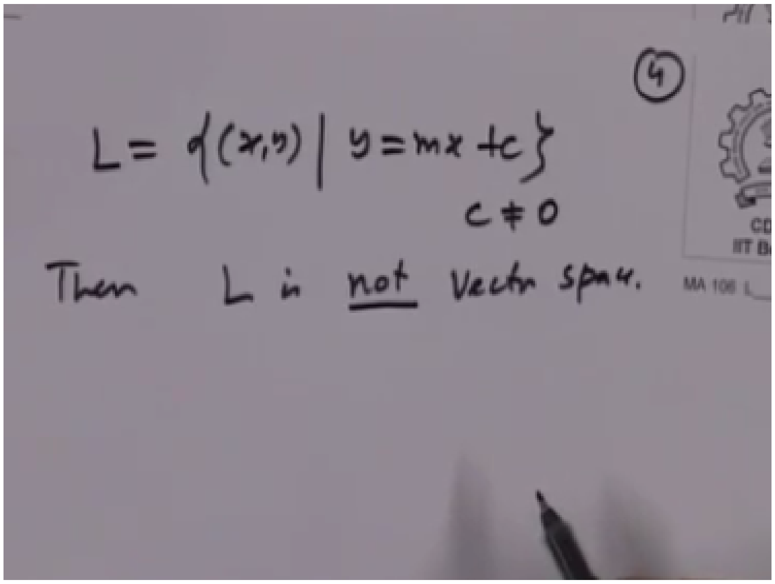

So, what does that mean? That means what is the conclusion; so the conclusion is that if I take L = x, y, where y = mx + c, c0 =0, okay then L is not vector space, then it is not a vector space because if I take 2 elements and take scalar multiples of them and add that is not again in the same right, so that means in the plane every line through a origin is a vector space, every line which is not through a origin is not a vector space, right.

So, the question still remains, can you say these are the only subspaces or these are the only vector spaces in R2, I cannot answer that question at present but that it is a true statement, so we will show later, right these are the only vector spaces in R2 either it should be a trivial that is 0 or it should be the whole space R2 or it should be a line passing through the origin, at present we have only shown these are vector spaces and no line not through a origin is a vector space.

But we will say these are the only ones evict later, okay, right, okay, so the line passing through the origin of slope and;

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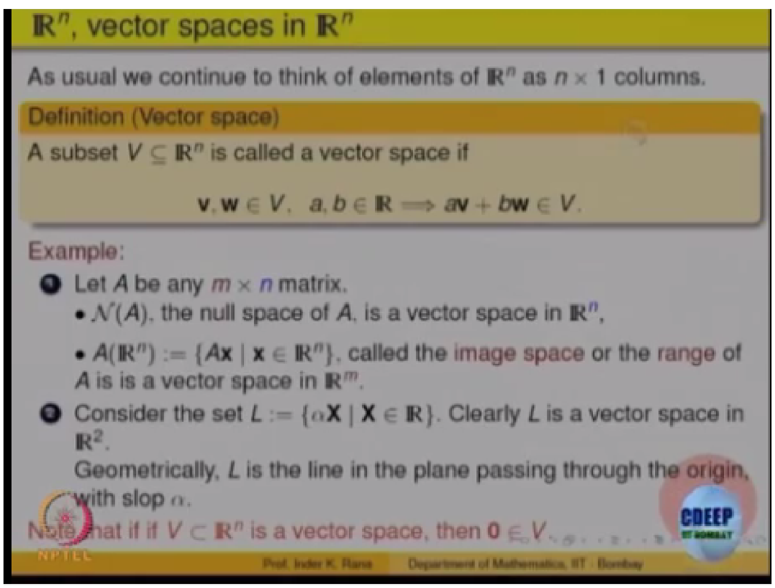

Now, this is an observation we said if a line does not pass through origin that is not a vector space in fact that is a property of every vector space that means if V is a vector space in Rn, then the 0 vector should be inside it, why, why should 0 vector be inside every vector space, so what is the definition of a vector space? It says alpha V1 + beta V2 should belong to V for every V1, V2 in V and alpha beta scalar.

But who stops we are taking alpha = 0 and beta = 0, that means the 0 vector should be in V, right, alpha V1 + beta V2 should belong to V for every alpha, beta and for every V1, V2 in V, in particular, if alpha =; beta = 0, then what is alpha V1 + beta V2 that is 0, so 0 should belong to V, right, otherwise it is not true, right that property say so that means, it is a necessary condition for a subset V in Rn to be a vector space that 0 must belong to it.

Otherwise, it cannot be a vector space, so that is what happened in the plane, no line which is passing through the origin, that means if a line is not passing through a origin, then zero vector is not part of it so, it cannot be a vector space and that is what we proved, checking actually, right, okay.

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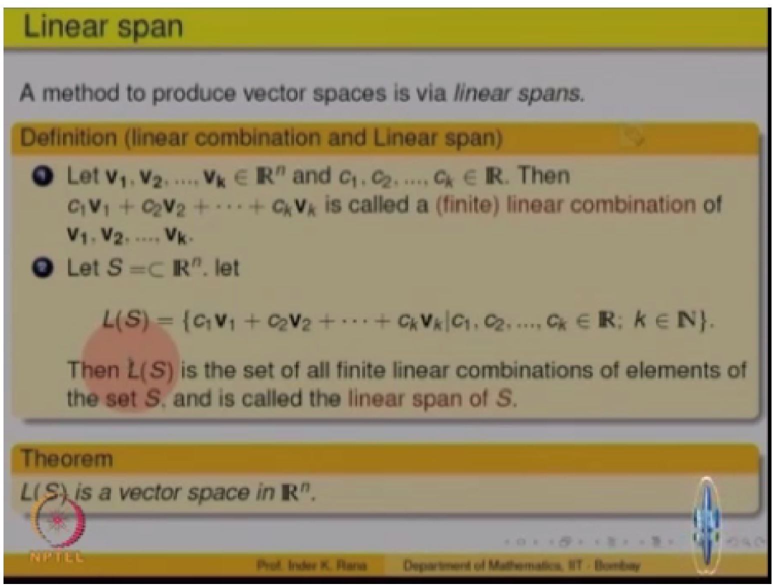

So, let us start with the some finite number of vectors in Rn, okay, L scalars R and look at this new vector alpha v1 +; not alpha c1; c1 v1 + c2 v2 + ck vk, so what I have done; I have scaled every vector v1/c1, v2/c2 and so on and taken a some of them, that is again a new vector right, so this thing; c1v1 + c2v2 + ckvk is a new vector in Rn, so such a vector will call as a finite linear combination of v1, v2, vk, just a name.

Why finite? Because the number of vectors is finite, you can just call it as a linear combination, okay, but to stress that we are taking only finite number of them; we will get a finite linear combination of these vectors. So, now comes a general definition, okay, so let us look at a set; the symbol has not come out S as a subset of Rn, look at a subset S of Rn, now what I can do?

I can pick up any finite number of vectors from S and form a linear combination of those vectors, right, so I choose any vectors v1, v2, vk in, right in S and look at scalars c1, c2, ck in R and form a; so, where are these v1, v2, vk, they are coming from right, from the set S, so that should have been written here, pickup v1, v2, vk in S, scalars c1, c2, ck in R and look at this linear combination and make a set consisting of all these linear combinations called that as LS.

So, what is LS? LS is a set of all finite linear combinations of vectors taken from S, right, is it clear, this k is not fix, k is not fixed, you can pick up only one vector, you can pick any 2, any 3, any 4, any finite number of them, form a linear combination by picking vectors finite number of them picking finite number of scalars making a linear combination. For all such possible linear combinations, right and call it as L of S.

For example, in that example in the plane, we took a fixed vector x0 and multiply alpha by it, so we took all linear combinations of the only one vector as far as only single vector that was a line, so this is called a linear span; LS is called a linear span of the set S; set S is the set of vectors, it may not have any property, right, you pickup elements from S, any finite number form a linear combination you get a new set, right of vectors that is called the linear span of set S, so that is ruled by L of S.

The theorem is LS is always a vector space as may be any set but LS is always a vector space, so let us check that why it is always a vector space, it is a simple proof but let us check it, okay, right.

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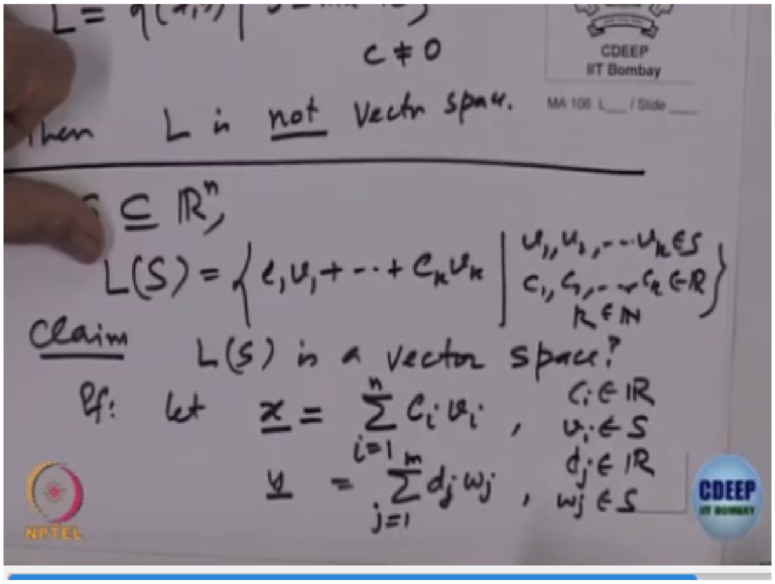

So, S is a subset of Rn, what is LS; LS is c1 v1 + ck vk, right, where v1, v2, vk's, they belong to S and c1, c2, ck's they belong to R and k is not fixed, so k is any natural number that also varies, is it clear what I am saying, we are not choosing only k vectors, we are choosing k is anything, I can choose 1 vector, I can choose 2 vector, 3 vectors, right, any finite number of vectors form a linear combination and put them in a box and that box is called L of S, right.

So, the claim is; L of S is a vector space, this is a vector space, so let us verify, so what we have to verify, right, so what is the proof, how do I verify this, to show it is a vector space, what I have to do? Take 2 elements of LS, take scalar multiples of them and add that should again be in LS, right. So, let us write, so let one element, let us call it as a vector x be = sigma, instead of writing dot, dot, dot, we will start using this notation, Sigma, it is ci vi; I = 1 to some n, right, where ci's are real numbers, vi's are vectors in S, right.

Is it okay, let us take another vector y which is something, some other scalars diw; djwj; j may be from 1 to m, we do not know from where to where, right, there may not be same vector chosen again, for y we would be choosing some other vectors, some other m number of them, so where dj's belong to R and wj's belong to S, so that is what is given to us, so this what we have to check?

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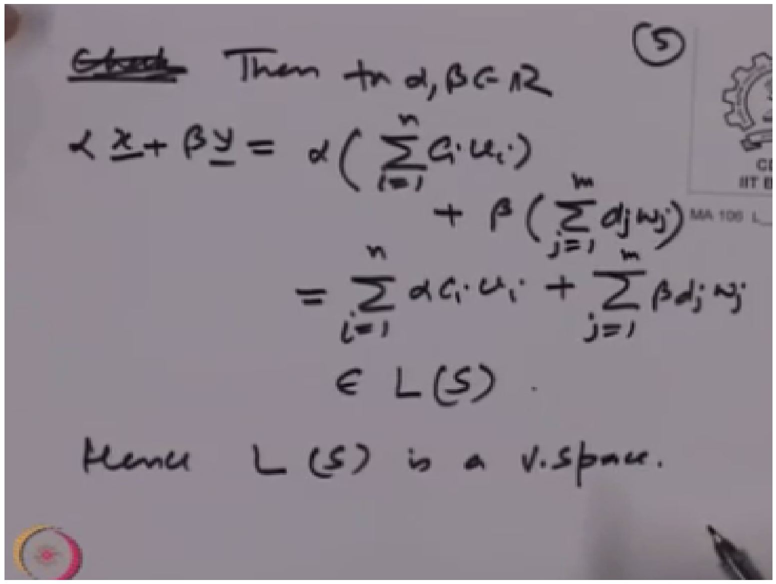

So, check or let us write then, what is alpha x + beta y, then for alpha and beta belonging to R, I should add them and check it is again a linear combination right but what is this =; this alpha times i = 1 to n ci vi + beta times j = 1 to n dj wj, right but what is this; this is same as I can take alpha inside, so this I = 1 to n, alpha ci vi + sigma j = 1 to m beat dj wj, is that okay but what is this?

That means I have chosen vectors; v1, v2, vn up to vn, I have chosen alpha vectors, right, so my new collection of vectors is probably n + n, each one is multiplied by some scalar and add it that is all, right, so is it clear that it belongs to LS, yes, it is again a linear combination; finite linear combination of elements from S that is all nothing more than that right, it is the matter of only writing, so these okay, so hence L of S is a vector space; L of S is a vector space.