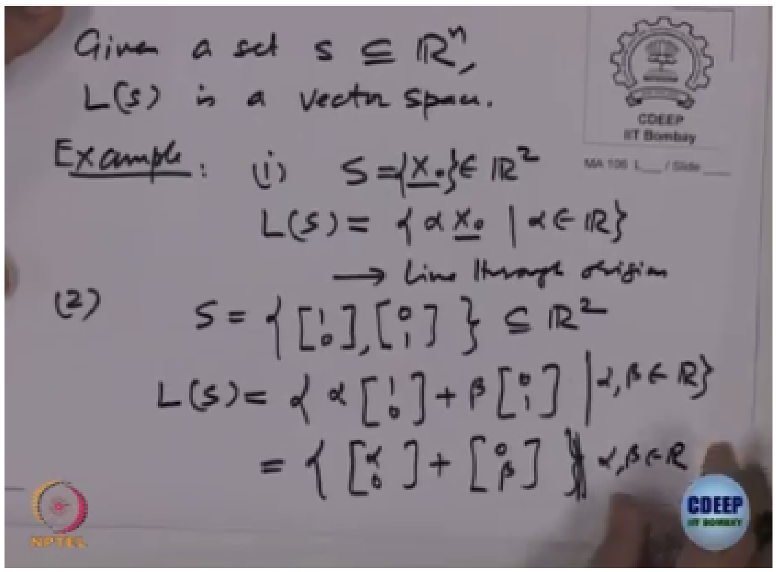
Basic Linear Algebra
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology- Bombay

Lecture – 12
Solvability of a Linear System, Linear Span, Basis III

(Refer Slide Time: 00:37)



So, given a set S contained in Rn, L of S is a vector space, right, so let us go back and write that example with is just now said, so examples; so let us take S = x0 belonging to R2, right, then what was LS? I should as it was a single vector x0, it is a set, so it is x0, I should write it properly, it is the set consisting of only one element right, x0, so how will you form linear combinations of it?

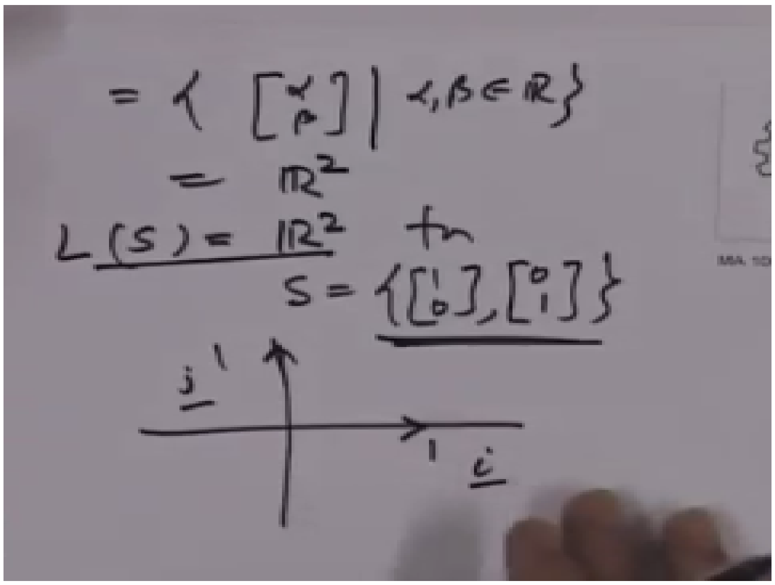
Only one vector to choose, multiply by a scalar, so what is LS; alpha times; alpha belonging to R, is that okay that is the only combination possible, so that was line through origin, right, is it okay, okay. Let us look at a second example, so let us look at S to be a set, let me take some particular vectors, okay, this is; this you all have done it, let us look at 1 0, and 0 1 in R2, I am taking 2 vectors 1 0 and 0 1, right.

So, what is LS? It should be alpha times 1 0 + beta times 0 1, where alpha and beta both belong to R, right, inner combinations, if I take alpha = 0, I am only taking one of them, if I take beta =

0, then I am taking only the first one, if both are non-zero, I am taking a linear combination of both, so I can by choosing alpha and beta suitably, I can take one of them or either of them or none of them.

But none of them means what; alpha =; it is not none of them, it should be a 0, so 0 vectors comes inside it, right, is it okay, what is that equal to; let us write, simplify, so that is alpha 0 + 0 beta, where alpha and beta both belong to R and what is that? So, what is that?

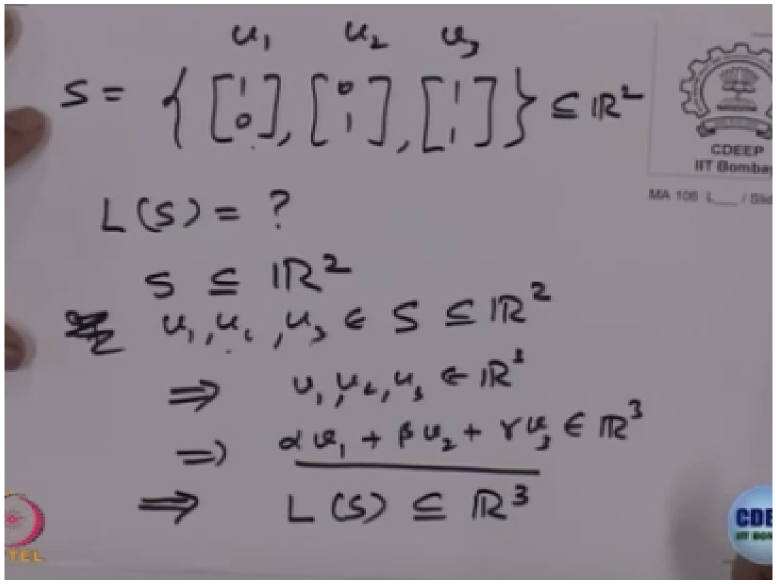
(Refer Slide Time: 03:28)



So, that is equal to alpha beta, alpha and beta belonging to R, right, what is this, if alpha, beta are arbitrary, what is this =; this is precisely R2, the whole plane, right, alpha and beta are arbitrary, so every vector in R2, I can choose it and make it = this, right, so LS = R2 for S = 1 0, 0 1, right, we recognise this already, do you recognise what I am doing, I am not doing anything new, it really says, what is 1 0, so this is this vector, what is 0 1; that is that vector.

So that is what you call normally, you call it as I, you call that as j, what is every vector, some alpha xi + beta of j, right, so that is precisely (()) (04:52) done in a different language that is all, right, so that means a linear span of these 2 vectors = overall of R2, right, let us take one more example in this to understand what we are doing.

(Refer Slide Time: 05:13)



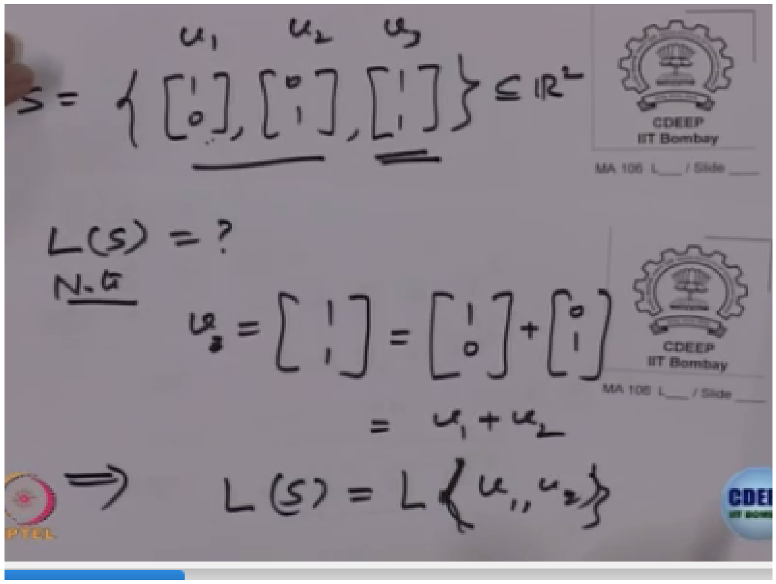
So, let us take the set S to be =; I am taking again 1 0, 0 1 and let me take it 1 1; 3 vectors in R2, right, I have taken 3 vectors in R2, what is LS? S, okay, S is a subset of R2, this set is a subset of R2, right, so if you call this is as v1, called this as v2 and call this as v3, just for the sake of it, instead of writing every time matrix, so v1, v2, v3 they belong to S which is a subset of R2, so what does that mean?

That means, v1, v2, v3 also belonging to R3, right, it belongs the subset or belong to whole space anyway, right a set, if something is element of a subset is also the element of the whole thing, so, v1, v2, v3 are elements in S, there also elements of R2 but that implies, where is alpha v1 + beta v2 + gamma v3, where is that? That is again in r3, right, if you take 3 vectors or any number of vectors and take linear combination that is again going to be an element in; this again a vector right, go on adding.

So, you can add first 2 and add in the next one, again you will stay in R2, right, so that means what, what does this imply? This implies but this where is this element, it is a linear combination of elements of S, so that implies L of S, is a subset of R3, L of S is a subset of R3, you cannot go outside R3, you will be inside R3 only, everything is R2/1 right, R2, everything is in R2, linear combination of adding 2 vectors will again be in R2, so will stay in R2.

So, L of S is a subset of R2, right, we want to know, can we say something more about it, okay, so let us write.

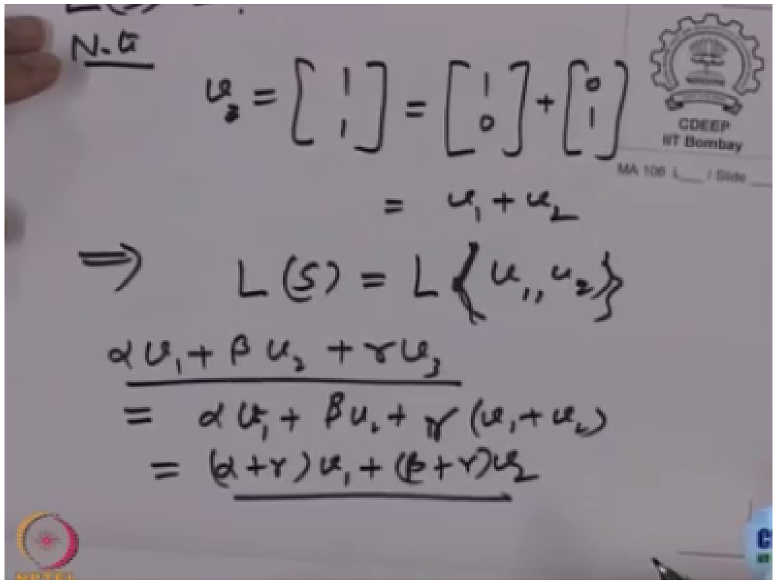
(Refer Slide Time: 07:47)



So, LS, we want to find out what it is, okay, so let us note one thing let us look at v3, what is v3 that is 1 1 right, v1 is 1 0, v2 is 0 1, I can write this as 1 0 + 0 1 right and that is same as v1 + v2, right that means this v3 is already a linear combination of v1 and v2, so this implies what does this mean, that means linear span of S is same as the linear span of v1 and v2, I do not need v3 because v3 itself is a combination of v1 and v2, right.

So, I can get v3 from v1 and v2 by a linear combination, so again by taking a linear combination, I do not have to repeat the process of adding v3 again, there is no need for that is it clear to everybody, yes, (()) (09:07) not clear, let me expand a bit more.

(Refer Slide Time: 09:10)



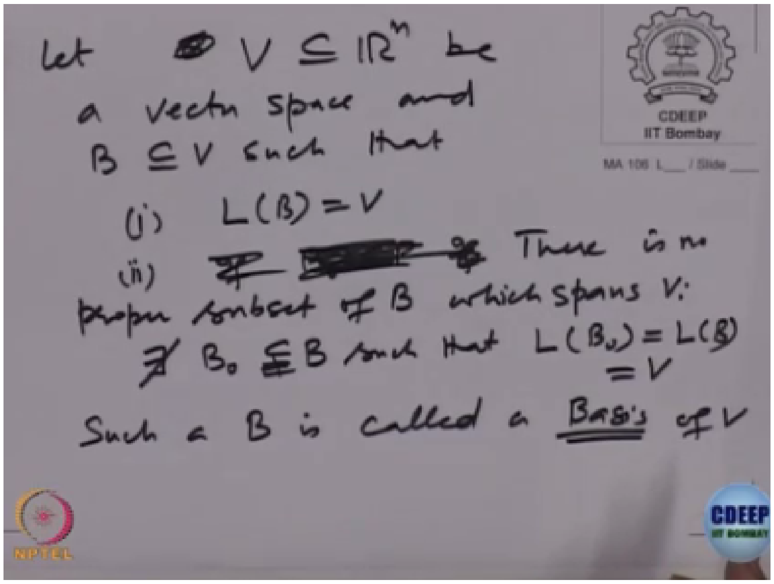
So, let us look at alpha v1 + beta v2 + gamma v3 that is in the element of L of S, right, so what is that equal to; it is alpha v1+ beta v2 +; what is v3; is v1 + v2, so it is gamma times v1 + v2, so what is that =; is alpha + gamma of v1 + beta + gamma of v2, so any linear combination of v1, v2, v3 is actually a linear combination of v1, v2 only, right, I can expressed it as linear combination.

So, what does that imply that implies a linear span of S, which was v1, v2, v3; 3 vectors is same as a linear span of first 2 only, right so that means v3 was redundant in finding what is a linear span of set S, right because one of them was already linear combination, right that means what; given a set of vectors, it will take a linear span of them right, then probably, some of the vectors you can cut down, remove them, right.

You can get a smaller subset of the original set, which will span the same thing and probably, you can go on cutting, can you remove everything, no you cannot remove everything, there has to be something, right so that means what that means given a set S and a linear span of it, possibly you can have a subset of it which gives the same span but there should be something called minimal probably, the smaller subset which will give me that right.

And idea should be like when this example we have done it, we should be trying to do it every time, right, so that motivates a definition what is called the bases of a vector space.

(Refer Slide Time: 11:15)



So, let us define, okay, so let; so let us take let v contained in Rn be a vector space and B, a subset of V such that 1; a span of B = V; V is the vector space, which is the span of V, right, is it okay, second property want to say is; I want to minimise, I want to say this is the smallest that means how should I write it, so if B0 is a subset of; B0 is a subset of, I want to say it is a smallest.

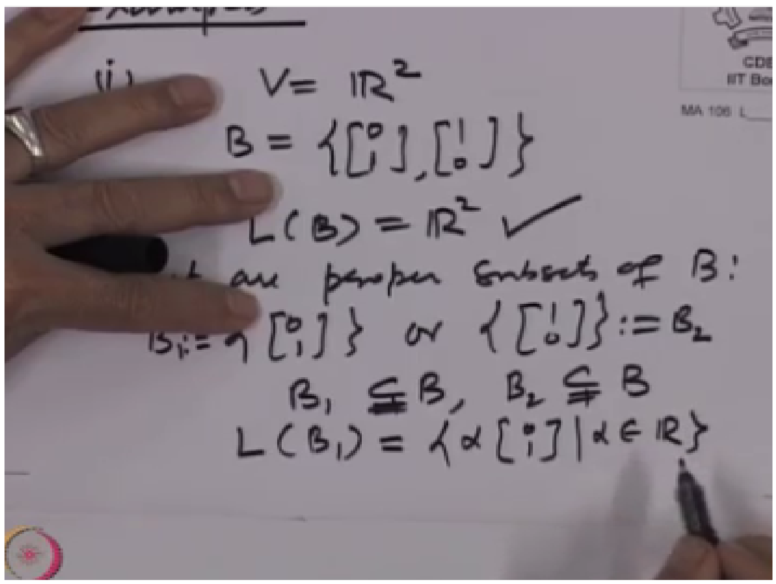
So, I shall write, so if is a vector space such that and okay, I think it is the better way, there is; let me write English first, there is no subset probably, I want to should say proper subset of b which spans that means what that means, there does not exist B0, a proper subset of B such that linear span of B0 = linear span of B which is = 1; 1 B spans B, right, I said B spans and from B, I should not be able to cut down something, so that the smaller things also spans.

So, here is the property of B, I am saying, such a B is called a basis of V, such a B is called a basis of the vector space B that means what; so, once again, let us try to understand this concept, V is a set; V is vector space and you are saying a set B is a basis means what, it should have 2 properties, one; the span of B should be V, every vector should be a finite linear combination of elements from that set B, right that is one.

And from B, I should not be able to cut down something and make it smaller and still getting the all of B that means no proper subset of B will span the vector space B, is it clear or explain, there is; B is some subset of V and when I take a linear combinations, I get V but this B, I cannot shrink it smaller, so when I take linear combination, I can still the whole thing again that should not be possible.

That means B is in some sense are smaller set we generates, right, smallest in the sense no proper subset will give me span which is everything, further let us look at some examples to understand this concept, okay something to understand.

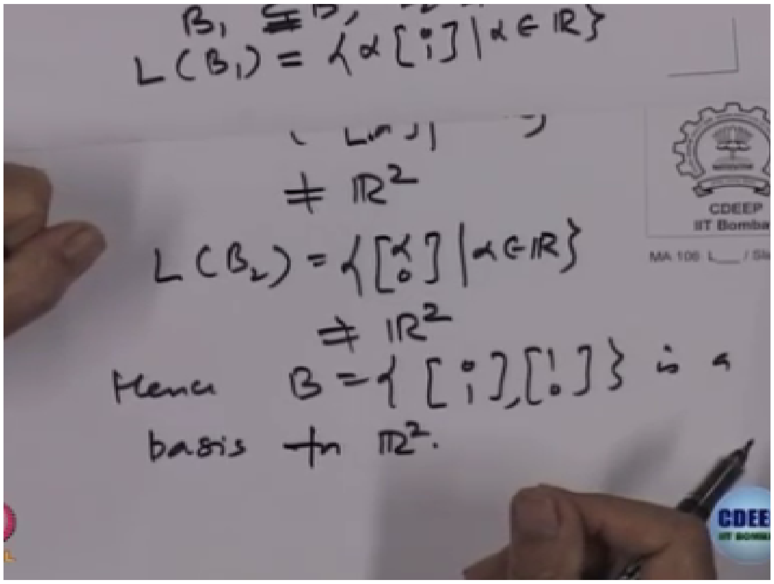
(Refer Slide Time: 15:16)



So, examples, I think best is plane itself, so let us look at V = R2, right and let us look at B which was just now we said, let us look at 0; 1 0, okay 0 1 and 1 0, right, so L of B = r2, is that okay, we just now said every vector in R2 is a linear combination like this, right, so what are proper subsets of B? It has 2 elements right, if you want a proper subset, their possibilities you take 0 1 or 1 0 right.

And if we want to say proper subsets will has a non-empty also, if you; otherwise empty, nothing will be there, right, so proper non- empty we are assuming that okay, now so call this is as B1 and call this is as B2, B1 is a proper subset of B, B2 also is a proper subset of B, right and these are the only once, now what is linear span of B1, you want to check is it = whole of it or not, what is the linear span of B1, what is this; alpha times 0 1, right, where alpha belongs to R.

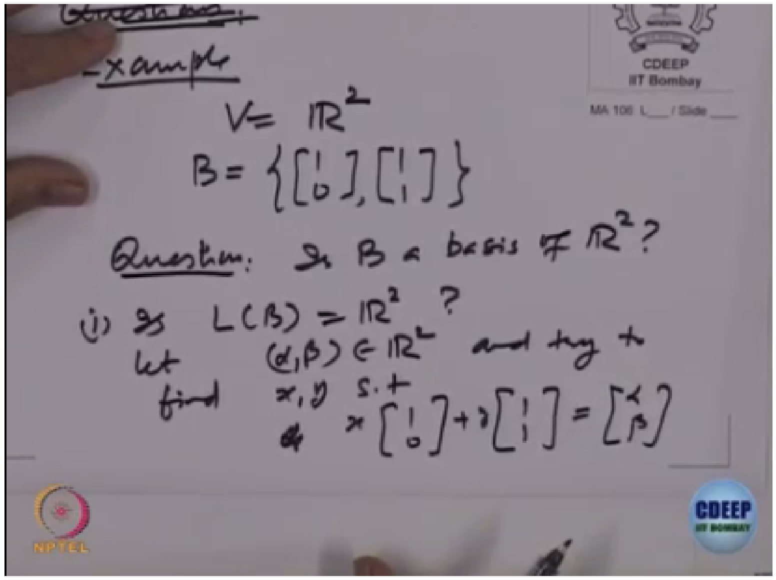
(Refer Slide Time: 17:29)



Then, what is that equal to; so that is equal to 0 alpha; alpha belonging to R, what is this; this is the x axis, this is y axis, this is not all of R2, so != R2, what is similarly L of B1; B2 as 1 0 so that will be equal to alpha 0; alpha belonging to R, right which is again != R2, so that is a sense we are saying it is the smallest, so that means what; that if I look at this B, it has a property L of B is span of B is whole of R2.

And even if I cut one of the elements that does not span whole of R2, so we will say, so hence B which is = 0 1, 1 0 is the basis for R2, this is the basis for R2, right, so now you understand, what is the definition of basis, it is the set of vectors in that vector space right, so that their linear span gives me everything and no subset give me the span as the whole thing and such a thing will be call as a basis of the vector space.

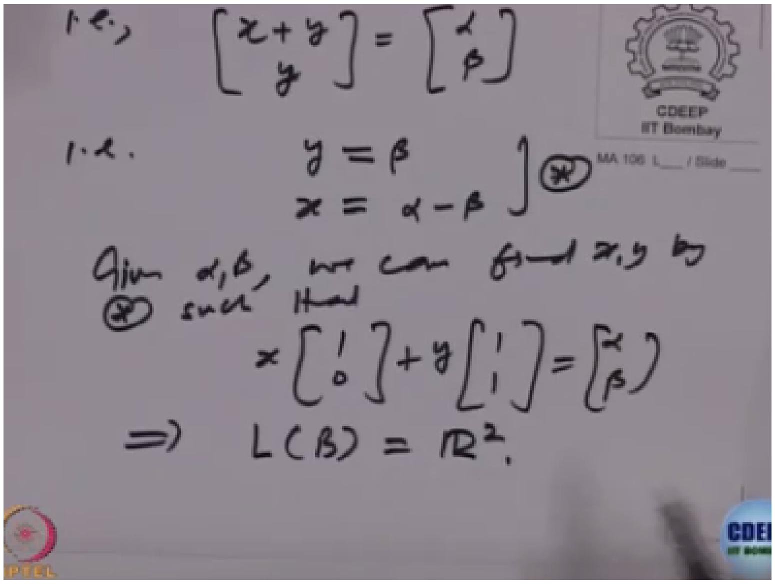
(Refer Slide Time: 19:22)



So, what are the questions one would like to answer, so here are the questions we will like to answer, so questions, okay before even questions, let us look at one more example, okay, you understand what is happening again, let us take V = R2, right, I let me take now the said B2 be = say 1 0 and 1 1 right, so the question is; is B a basis, is this is a basis of R2, so if we want to check it is a basis or not according to my definition, one, which should span R2 right.

That means, every vector in R2 should be a linear combination of these 2 vectors and second; if I drop one of them, then I should not be able to get the whole thing, right, okay, so let us check; so first is L of B = R2 that is the question that means what? So, let us take alpha and beta belonging to R2 and try to find the x and y such that alpha of; x of 1 0 + y of 1 1 = alpha beta, right that is what it will be in saying that R2 can be obtained as a linear span of the vectors in B right.

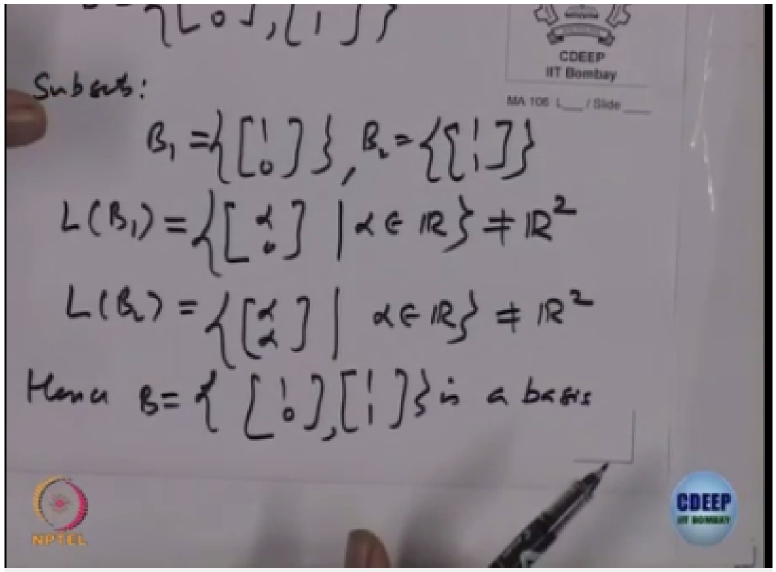
(Refer Slide Time: 21:41)



That is what this means, so let us check what is this means, is it possible or not let us check that so, that means, what shall I will be doing, I should be; what is the left hand side that is so, x and that will give me x + y and y = alpha beta, right, is that okay, I have simplified the left hand side that is; what should be y = beta and what should be x = alpha - beta, so a given alpha beta, we can find x and y by this equations by star such that x times 1 0 + y times 1 1 = alpha beta, right.

So, given any vector with components alpha beta, I can find scalars x and y, say that this is the linear combination give me, so implies that L of B = R2, first one condition I verified, right, to check that whether this B is the basis or not I verified the condition, what is the second condition I should verify? I cannot make it smaller, right, so second.

(Refer Slide Time: 23:15)



What is B? That was =; that was B that was 1 0 and 1 1 right, subsets; proper subsets, what are proper subsets? B1 that is 1 0, other one; 1 and 1 right, so let us see what is L of B1; what is L of B1; that is precisely alpha 0, alpha belonging to R, right, everybody following, yes, what is L of B2, so alpha, alpha; alpha belonging to R, these are my spans of B1 and B2, what is L of B1? If you look at geometric point of view that is x-axis, so that is != R2.

It is a proper subset of R2, it does not give everything, what is alpha, alpha? There is a line y = x, so that is again != R2, so I take B and cut any one of the vector, I do not get span everything, so hence 1 0 and 1 1, so this B is a basis, so what we have done, for R2, we are produced 2 different basis, one was 1 0 0 1 right, just now we proved that and now I have gotten 1 0 1 1 that also is a basis.

So, what does that mean; that gives an indication that the basis for a vector space need not be unique, you can have more than one basis for a vector space right but the property should stay, it should span and it should be the smallest spanning rate right, you can have more than one possible in fact in the; in R2, if you take any 2 lines which are intersecting not parallel not coincident I mean there should be 2 lines.

Any 2 lines if you take that will form a basis that is something saying similar to it is not mentioned anywhere in books or schools, when you have coordinate system, you always take x-axis and y-axis right and you say everything else can be obtained by moving parallel to x-axis and then going parallel to y-axis to reach any point, so any vector is a linear combination of x coordinate and y coordinate.

But why should I take to be perpendicular, I can take any 2 lines, the only thing would be I should be moving parallel to these lines now right, so a coordinate system that is what coordinate system is that it should not be a perpendicular always right, so that is what we are saying that 1 0 and 1 1, they are not perpendicular that still form a basis right, advantage of those perpendicular once is because of Pythagoras theorem and computation became simpler that is a other story.

But as such basis, any 2 lines which are non-parallel, right otherwise is the same thing basically we are saying that they should intersect right, any 2 intersecting lines will give you a basis, so take vector here and take a vector here, right, they will give you a; okay. So, basis we have defined, so what we will have done today is; we looked at a system of linear equations, Ax = b, we looked at the solutions set of the homogeneous system.

And said that solution of the general system can be obtained from homogeneous system right, by taking a particular one and adding it to the solution of the homogeneous one, so solution of homogeneous becomes important and solutions of the homogeneous system we can call it as the null space and that has a property of being a; that motivated us to define what is the vector space, right, so null space has the property, if we take 2 vectors and take linear combination that is again in the null space, right.

So, we define what is called a vector space, we will look at various examples and what we are looking at is; how to generate a vector space as a linear combination of a subset of it and we have come to the notion of basis, the basis is a set of vectors by which whose linear combination of elements of which gives you the vector space and it is the smallest one, nothing smaller will 1 right, so next lecture, we will look at how to find basis of a given vector space, okay.