# Basic Linear Algebra
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## Lecture - 13
## Linear Span, Linear Independence and Basis - I

So let us begin our today's lecture. We will just recall what we have done earlier. We have looked at what is called the linear span of set of vectors. So given a subset S of Rn, you define what is called the linear square. So this is the linear combinations of the vectors in S that is c1, v1 and so on okay.

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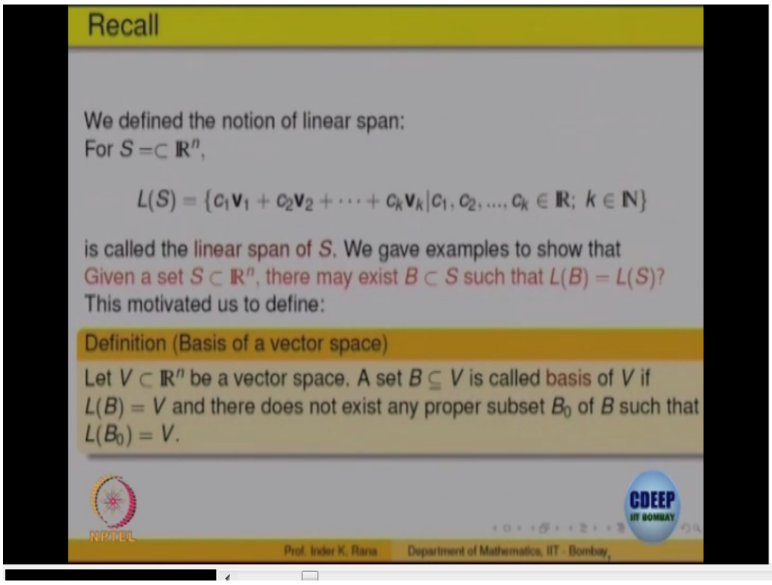


So this we called as the linear span of the vector, so we looked at some examples of the linear span and then we defined what is called the basis for, before defining the basis we showed that there can exist more than one sets such that the linear span of set B is=linear span of S. That means if a set spans a vector space, it is possible that a subset also spans that vector space right. So this motivated us to define what is called the basis of a vector space.

So basis was defined as a set of vectors whose linear combinations give the whole vector space and there is no proper subset of it which gives the linear combinations as the vector space. So basis is a minimal set of generators right. Given a set, it should generate everything, so every vector in V should be a linear combination of it and no proper subset of it should generate it, so no proper subset of it should give linear combinations all of that.

So such a thing we called as basis of vector space and we gave examples of basis of vector spaces.

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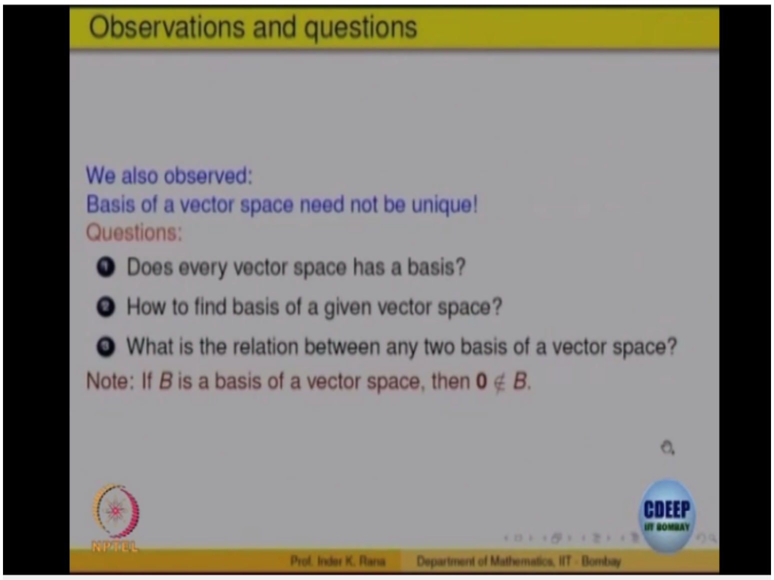


So basis of a vector space need not be unique. That is what we said, so given a vector space there could be more than one set which form a basis. So the questions arise does every vector space has a basis? How to find a basis of a given vector space and what is the relation between any two basis of the vector space right? So given a vector space first question will like to answer is does it have always have a basis right.

Second, does it how to find that basis given vector space and what is the relation between any two basis of the vector space. So let us observe one thing to start with that basis if a vector space if set B is a basis, then 0 cannot belong to it right and this 0 cannot belong to the basis right. If it is there anywhere because it is not going to do, so we will remove it. So basis will always have because it is a minimal set.

So if B is the set of vector which is a basis and contain 0, then we can delete 0 right because any vectors we can take a linear combination 0 times v1+0 times that will give you 0. So linear span will always have 0, so if 0 is a part of a basis then it is not minimal, so we can always remove it to make it a minimal right and definition of basis is the minimal set. So 0 will never belong to the basis of a vector space okay.

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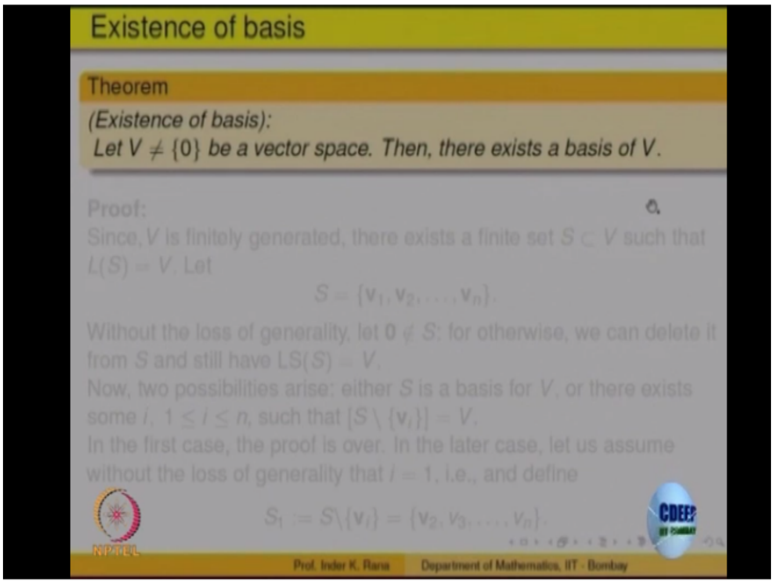


So we want to show that every vector space has a basis. So the idea is following. So let us look at the proof of this.

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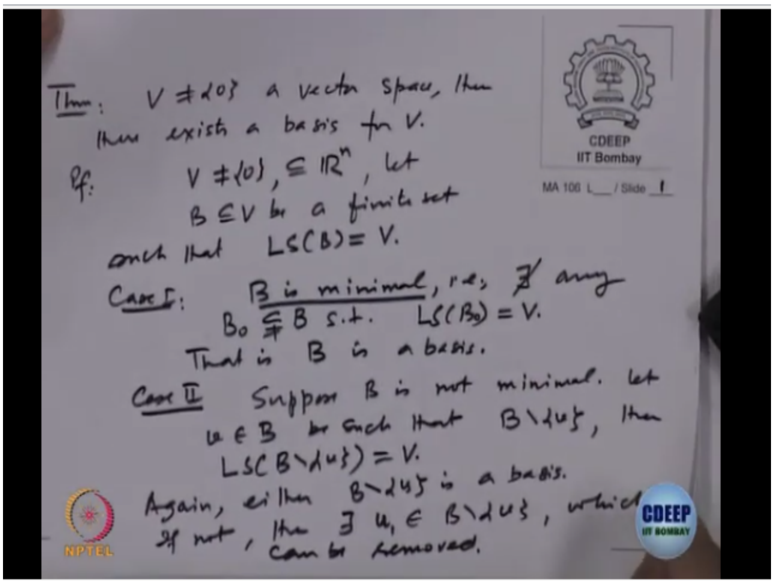


So theorem is if V is obviously a nonzero vector space, then there exists a basis for V right. So let us look at a proof. See so V is!=0 and V is contained in Rn right. So we start with other set of generators right. So let B contained in V be a finite set such that a linear span of B is=V right. We have got a set of generators and we want to check whether it is basis or not. So what is the possibility?

So one possibility is case I that B is minimal that is what is B minimal means, there does not exist any proper subset such that linear span of B0 is=V. Then, what will happen? That means we are saying that this set itself is a basis. So that is B is a basis, so what is case II? So case I was it is minimal. So what is the second possibility? It is not minimal. So suppose B is not minimal, so let that means what?

If it is not minimal that means at least there is one vector right which you can remove. So let v belong to B be such that if I take the set B and remove from it v, then the linear span of B-this v will be=V right. If it is not minimal, then at least one vector is redundant, we can remove it, it is not required. So let us say that is why vector is v which is in B which is not required right.

So look at so it is not required that means what? B-that vector should give me everything right. So again two possibilities either this B-v is a basis, once we have removed that one vector which was not required so one possibility is after removing one vector it becomes a basis. If not, again in so if not then there is there exists some say u1 belonging to B-v which can be removed right like we removed earlier one vector.

So there is one possibility again that there is a vector which we can remove. So if you go on doing this process what will happen right, the set B is a finite set right, the basis is a finite set. We removed one vector, we removed another vector right so eventually you cannot remove everything, eventually there will be one vector left at least in the last stage if possibly then that single vector will give everything right.

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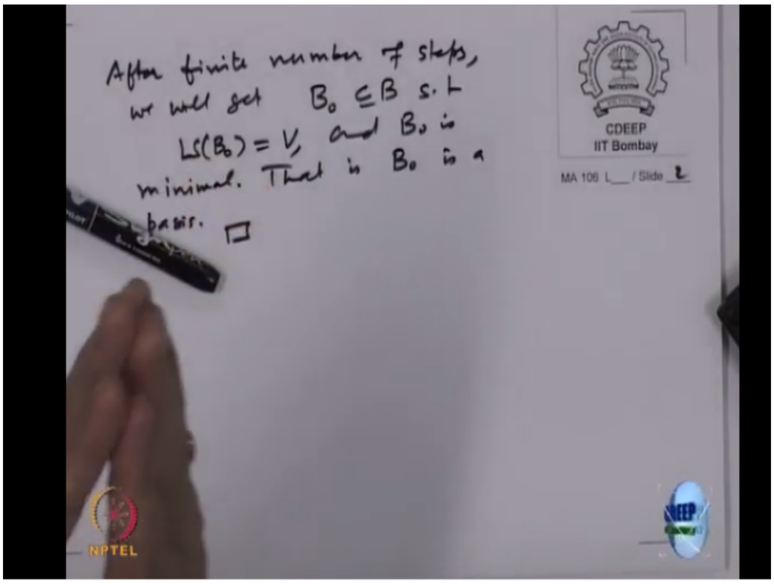


So after finite number of steps, so let us write after finite number of steps, we will get set B0 which is contained in B such that linear span of B0 is=V and B0 is minimal. That is B0 is a basis. So what are we saying? We are saying that we have got a vector space which is generated by some set of vectors B that may not be minimal; I can make it minimal by removing one at a time which are vectors which are not required.

So that will say that eventually we will get a set which is minimal and will generate. So that is essentially the idea of saying right. So let us I wrote B and here in the slides it is S.

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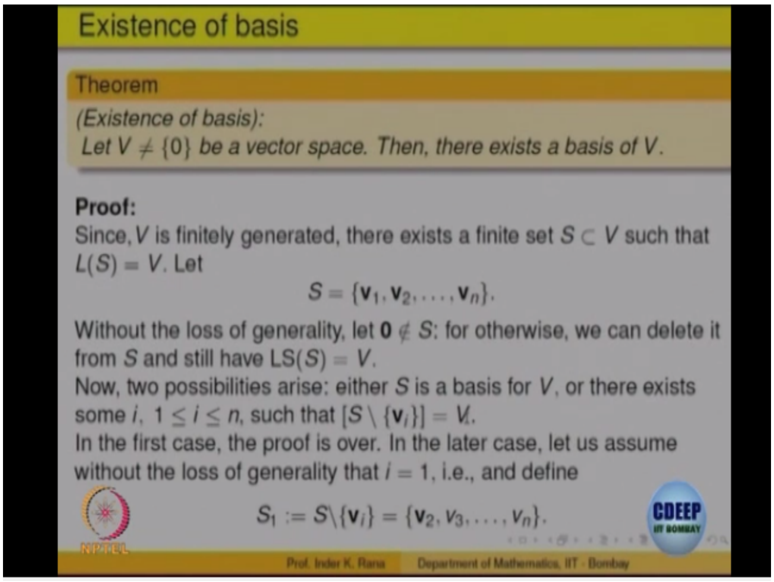


S is a set v1, v2, vn such that the linear of span of S is=V right. Obviously, we can assume 0 does not belong to it because it belongs we can remove it okay. Now two possibilities arise S is a basis that set itself is a basis if not then what will happen at least one of these vectors v1, v2, vn is not required, so let us say that vector is vj. So remove from S that vj and the span of that, so the square bracket also indicates the span of linear span of S-the vector vj is V right.

So once we have removed one, again check whether that is the basis or not. If not, remove another one if required and go on doing it.

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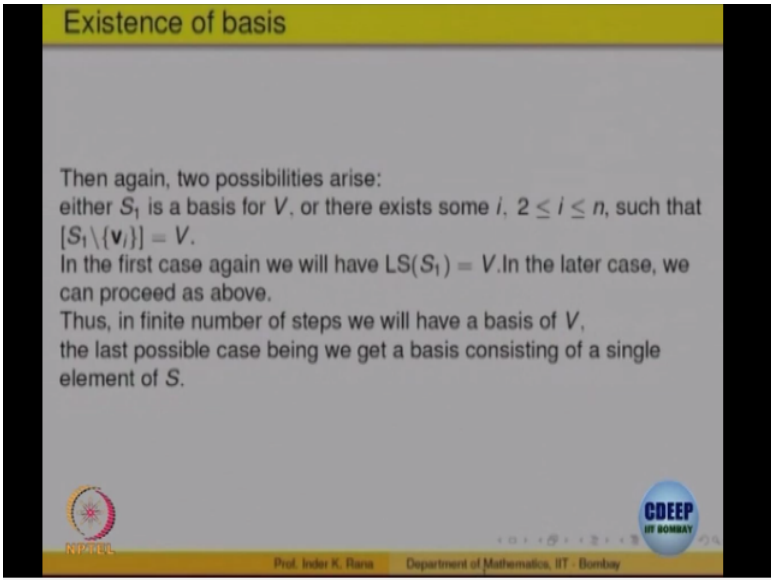


So after finite number of steps right, in finite number of steps we will get a subset of S which cannot be reduced further and that is the basis because S is only a finite set, so you cannot remove everything, one by one we could remove at the most n-1 and the process will stop. So that shows the existence of a basis. So basically what we are saying is if V is a vector space in Rn, it has got a set of generators which is finite.

Then, you can go on removing which are not required to get a minimal set of generators and that is a basis right.

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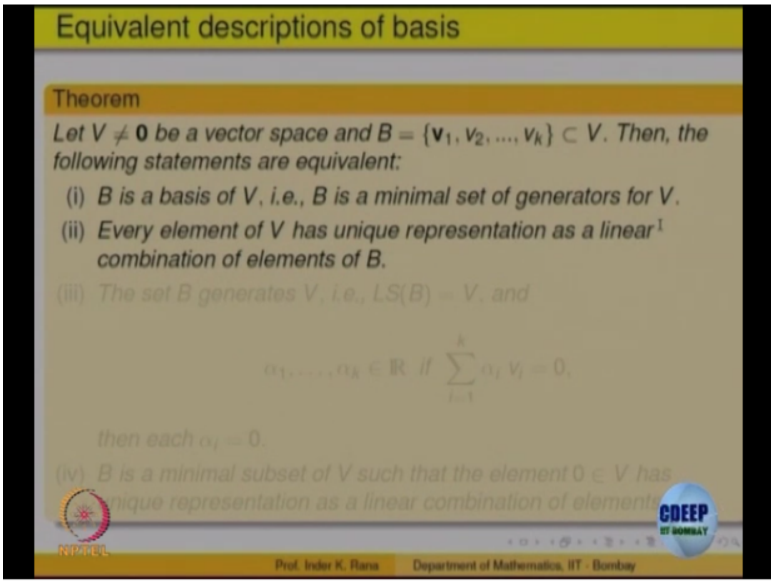


So let us give some equivalent ways of describing what is the basis. So V is a vector space which is of course nonzero and B is a set which is v1, v2, vk contained in V. So we want to give different ways of looking at when the set B can be a basis. We have already defined one definition as minimal set of generators. So the first one is B is the basis that is it is the minimal set of generators.

And we are going to show it is equivalent to saying that every element v of the vector space V has a unique representation as a linear combination of elements of B. So let us prove that. Let us prove that the I and II are equivalent.

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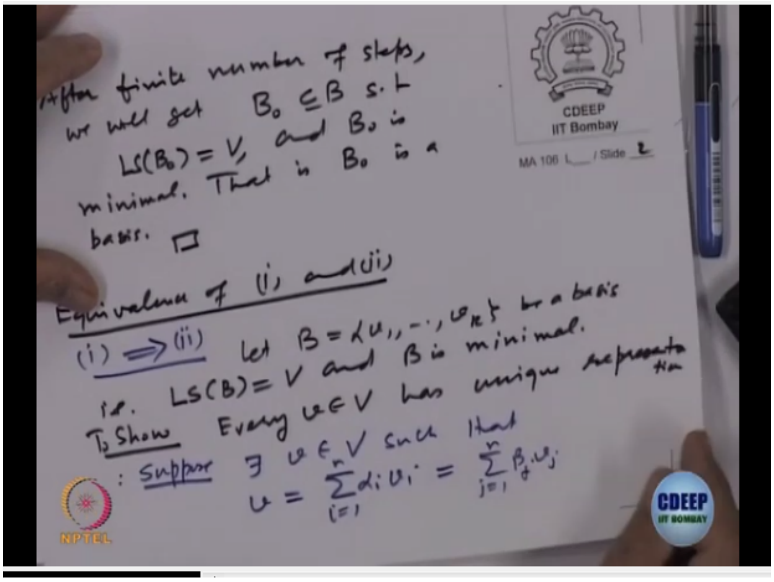


So equivalence of I and II, so what does the equivalence mean? That means if I assume I, I should be able to prove the statement II. If I assume statement II, I should be able to get the statement I from it. So let us look at one way implication. The first one I implies II. Let us look at that. So what does the statement I gives? So let us say let B is=v1, vk be a basis, so that is so what is this is what is given to us.

That is the linear span of B is=V and B is minimal right. To show, what is to be shown? Statement II and the statement II is every vector v belonging to V has unique representation right, so it does suppose not, so suppose not. So suppose so what is the negation of the statement II? Anyway, there is at least one vector which has two different representations.

So suppose there exists v belonging to V such that this v is=sigma alpha i vi and also equal to some other scale as beta j vj j=1 to n okay. So we are assuming there are two positive I representations okay. So what is our aim to show? It is not possible that means each alpha i must be=the corresponding beta i right. So let us assume.

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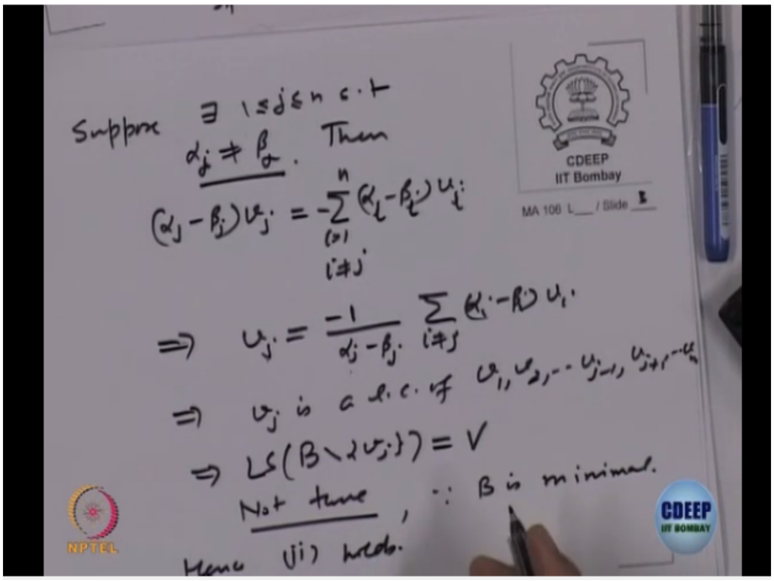


So suppose there is some j between 1 and n such that alpha j!=one of them alpha j!=beta j. Then, from this equality we will have alpha j-beta j vj, take out that where these two are not equal. I can write it as sigma of –sigma of alpha j-beta j of vj or let me write i, i=1 to n but i!=j. one term I have taken in the left side, everything else on the right hand side. Is it okay? From this, this is equal to this, so if I bring it on this side, so alpha i-beta i vi summation is=0 right.

So is it right? So I take the other things on other side i!=j right. Is it okay? I am just taking one term on one side, everything else on the other side right. So what does this imply? This implies vj can be written as -1/alpha j-beta j summation i!=j alpha i-beta i vi, not that alpha j is!=beta j, so I can divide by that right. That is what we assume so but that means what? That means vj is a linear combination of v1, v2, vj-1, vj+1, vn right.

This vj is a linear combination of the remaining vi's, so that means what? That means from the set B if I remove vj, I do not lose anything right, I can get everything. So linear span of this is=the whole of V right because vj itself is a linear combination of remaining, so all the remaining ones will do the job but that is the contradiction, not true. Why, what is the contradiction?

Because what was given to us was I, it is a minimal set. I cannot remove anything from it. So not true because B is minimal. So hence II holds. So what we have shown is I implies II. Let us show the other way around will be II implies I. So let us prove the other way around statement,

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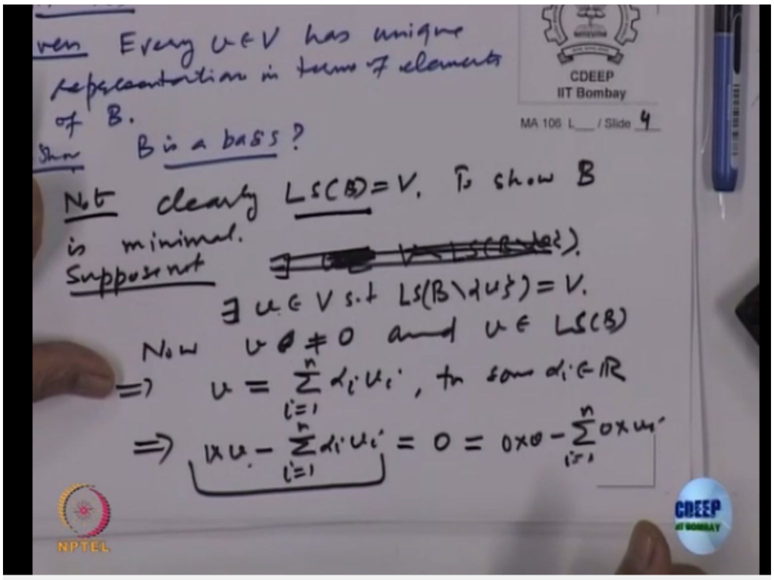


So II implies I, so given what is the statement II? Statement II says every element has unique representation right n terms of elements of B. What you have to show? B is a basis that is to be shown right, that is statement I that B is a basis. So note, first of all we note that every element has got a unique representation, so note clearly linear span of B is=V right because every element is representable in fact uniquely.

So it spans anyway, so what is left to be shown? To show it spans and to show it is a basis that means we have to show it is minimal right. To show B is minimal, suppose not that means what? There exists a v belonging to V such that if from B if I remove v and take the linear span that is=V right okay. So now look at this vector v, this v is!=0 first of all, why it is not 0? Because from B we have removed right v cannot be 0 because if 0, B did not contain 0 so nothing is removed then.

So it is not 0 and v belongs to LS of B right v belongs to the linear span of B right implying what is v=it should be a linear combination of something. So let us say it is sigma alpha i vi i=1 to n, for some alpha i is scalars right but that means what if I take everything on one side v-sigma alpha i vi i=1 to n is=what it is equal to 0 vector right, everything else I transpose on one side, shift everything on one side is 0 but what is 0 also=it is 0 times v right-0 times vi right.

I can write 0 as a linear combination of anything by putting all square as 0. So but here it is 1 right, V is 1 x v, so I have got on this side a linear combination where all the coefficients are not all the constants involved in the linear combination are not equal to 0 is equal to another linear combination where everything is 0. So what does it contradict? That means 0 has got two different representations. Its representation is not unique but that contradicts II right.

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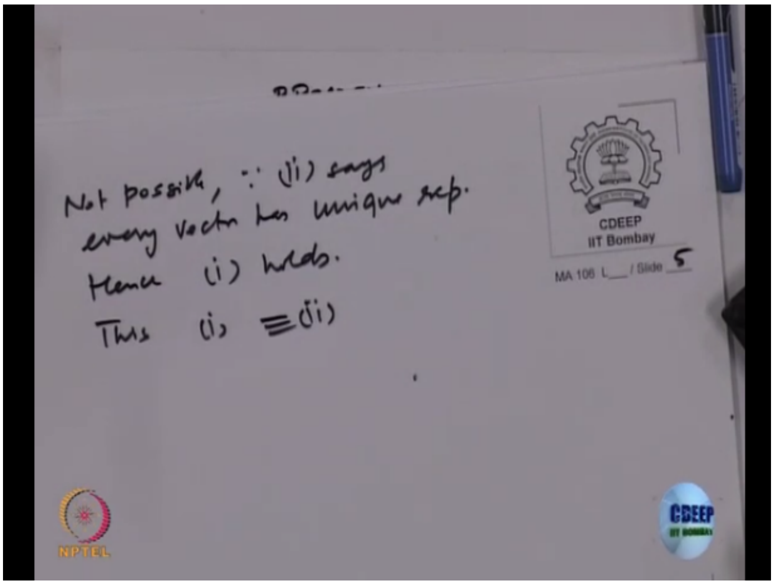


So not possible because II says every element has unique representation and we are given for 0 two different representations. So hence right so it says hence I holds. So this proves, what does it prove? This proves I is equivalent to II, I implies II and II implies I. So that was the statement we wanted to show here.

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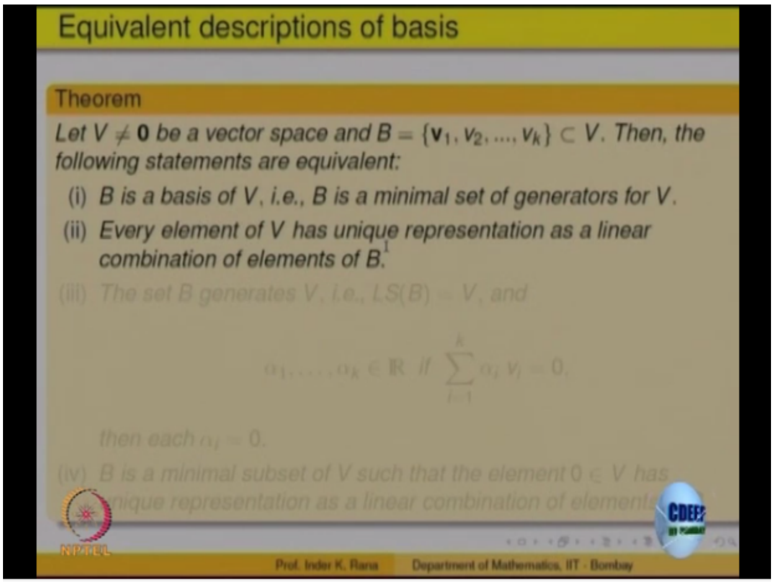


So in the theorem, the first statement is B is a basis, it is a minimal set of generators and were shown it is equivalent to say that every element is representation as a linear combination of elements of B and that representation is unique, there is only one way of doing it.