Basic Linear Algebra
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Lecture - 16
Row Space, Column Space, Rank-Nullity Theorem - I

Okay so welcome to this today's lecture. We will start recalling what we have done earlier.

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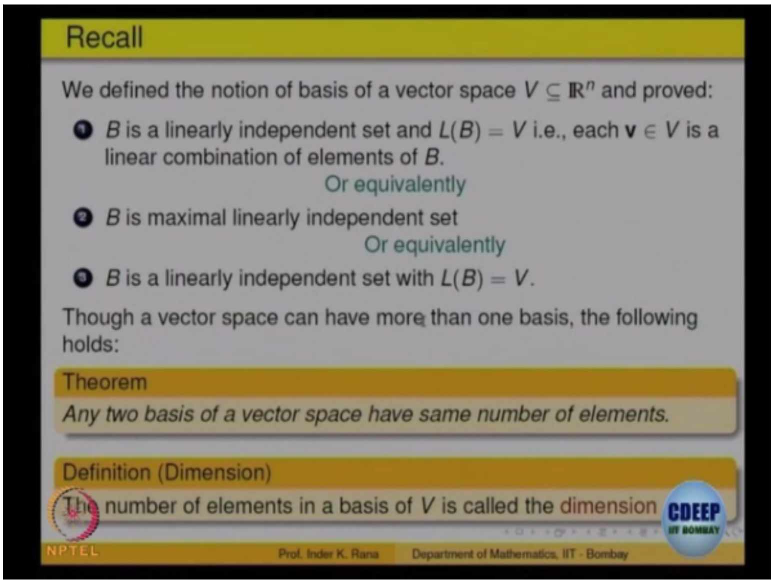

So we defined the notion of a basis of a vector space V in Rn and proved the following facts about the basis. B is a linearly independent set and generates V. That means every vector in V is a linear combination of elements of B and it is a linearly independent set or equivalently B is a maximal linearly independent set. So you can replace this condition that it spans by the maximal property.

And it is equivalent to saying it is a linearly independent set such that LB of that, that is same as the earlier one or it is a minimal set of generators that is another property that we proved last time. We also said will assume this theorem that any two basis of a vector space have the same number of elements. So that gave rise to the definition of dimension of a vector space. So dimension is a number of elements in any basis of a vector space.

There can be different basis for the number of elements and each base will be same for a given vector space, so that we called as the dimension.

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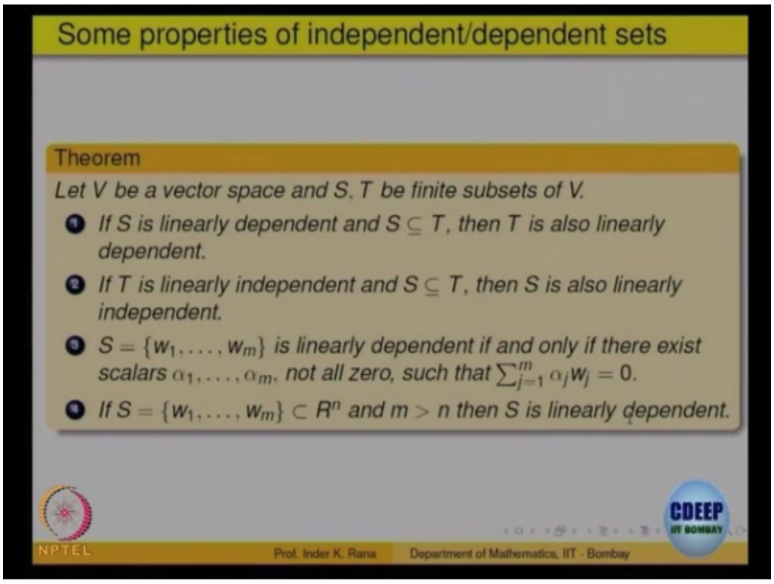

Some useful properties of independent and dependent sets. If a set is linearly independent, if a set S is linearly dependent and S is a subset of T, then the bigger set also is linearly dependent. What does independent mean? Here that is where the linear combination of elements of S which is 0 but not all coefficients are 0 but elements of S are also elements of T, so that means there is a linear combination of elements of T which is 0 but one of the coefficient is not 0.

So T is also dependent, so if inside a given set you find a subset which is linearly dependent, then the bigger set itself is linearly dependent and equivalent way of saying the same thing would be if T is linearly independent the bigger set is linearly independent, then every subset also is linearly independent because the property of independence is that if a linear combination is 0, all the scalars must be 0 right.

If it is true for elements of T, then obviously it is true for elements of the subset also. Another way of saying the linear dependence is a set is linearly dependent if and only that is a definition actually of linear dependence but you can also write one element at least one element of S is a linear combination of elements of the remaining one and this is another property that we proved namely you have got m vectors in Rn right.

You have got m vectors in Rn and the number of vectors is bigger than the number of components then that is always linearly dependent. So we observed that okay. So in R2, we have got 3 elements, 3 vectors, they have to be linearly dependent right, they cannot be independent, so that property we saw.

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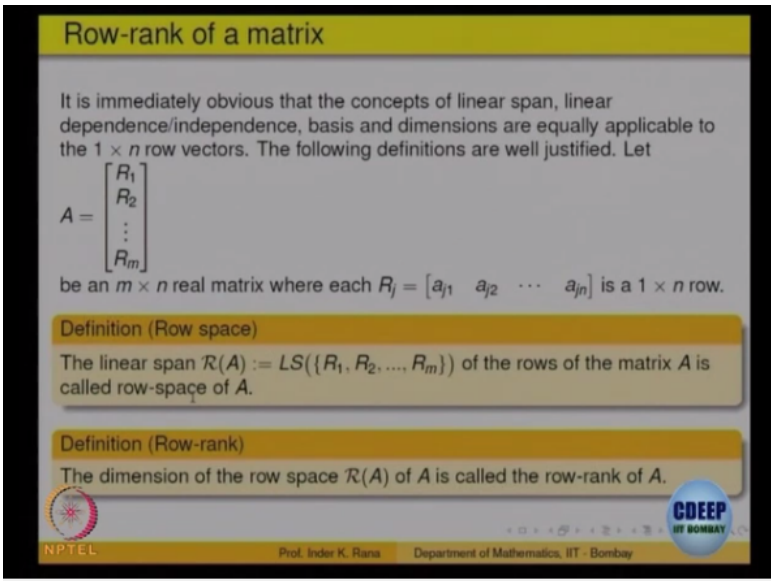

Let us look at some vector spaces associated with the matrix. So let us take a matrix A and these are the row vectors R1 R2 and Rm, there are m rows and each row is a right vector of length n okay, so this is the m rows and n columns. Each row vector has got n components right, so this is m x n matrix. So these vectors R1 R2 Rm if you put them together in a set and generate a vector space, so that means a linear span of the row vectors R1 R2 and Rm is called the row space of the matrix A.

So given a matrix, look at the row vectors right, so look at the row vectors there are m of them, so the subspace of Rn each vector is of length components n. So if you generate a subspace that will be subspace of Rn. So row vectors generate a subspace that is called the row space of, so that vector space has got a dimension right. So that dimension is called the row-rank of the matrix A that is called the row-rank of the matrix A.

So what is a row-rank? Now we are defining a new term, earlier we had the notion of rank. So what was the rank? In the row echelon form, the number of nonzero rows right that was, now we are defining something new but will show it is actually equivalent to what the earlier definition is that in terms of vector spaces look at the row space that is the space span by the row vectors, look at its dimension and that dimension is called the row-rank of the matrix.

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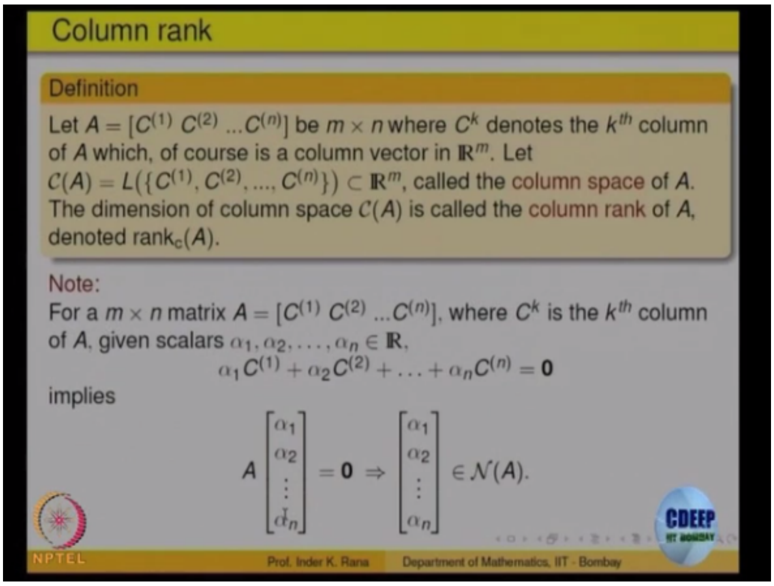

And similarly we will have for the columns, we shall take the matrix and write the column vectors C1, C2, Cn, so these are the column vectors right. There are n of them, m x n matrix, so there are how many columns are there, n of them, each column has got m entries right. So this is you can write a matrix in the column form, so this we call the column form of a matrix, matrix with only columns are written down.

So this is a column vectors, so look at the space span by the column vectors, each column vector is a vector in Rm right, m components are there, so you will get a subspace of or you get a vector space which is inside Rm, so that is called the column space. So column space is a vector space in Rm and the row space is the vector space in Rn okay. So these are the two subspaces and they play some important role. We will see what are the roles they play.

But just one observation here, once you have written in the column form a matrix, if you take a linear combination of that=0, these are column vectors right of a matrix A. If I take some linear combination and say it is 0, then that automatically implies that the A applied to the vector alpha 1, alpha 2 and alpha n is 0 right. This equation linear combination can be written as this right and that means what?

That means that the vector alpha 1 alpha 2 alpha n is a solution of a homogenous system right. A applied to alpha 1 is 0, so that means this vector will form to alpha n belongs to the null space of A. Remember what we define a null space, all vectors say that Ax is=0, so a linear combination of columns=0 gives you an element in the null space, the observation which we will use it later on.

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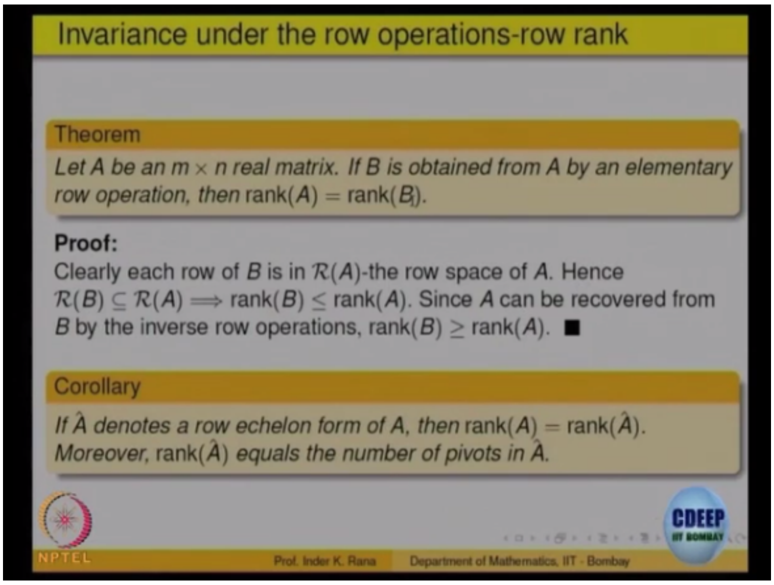

So here is the first theorem about the row-rank. Supposing A is a given matrix and you apply row of ratios to it, it will change to something, rows will change. Now what the elementary row operations were? One was interchange of two rows, other one was adding one row to another or multiplying a row by a nonzero scalar right. Now if you take a linear combination of the rows and transform them by elementary row operations what you will get?

We will get some new rows but they are linear combination of the earlier ones anyway. Probably, if R1 and R2 are interchanged you will not change any linear combinations right. If you have taken R1+R2, we will get a new row right but in the new row it is R1 applies R2 which is obtained from R1 and R2 itself by linear operation right. So what we are saying is the row space of a matrix does not change if you apply elementary row operations to the matrix.

You have got a matrix A with rows R1 R2 right, rows last wise Rm or if you apply row operations to this matrix, the rows will change right, R1 tilde R2 tilde and so on but the row space of the transform matrix is same as the row space of the earlier one, row space does not change, rows are changed because each row in the transform matrix is a linear combination of the earlier rows.

So it is not going to change in the vector space property right, in a vector space if you take two elements and take linear combinations that stays again in the vector space. So the basic fact is the row operations changes the rows, transform the rows but it does not change the row space of the matrix. So that is what once the row space does not change dimension will not change, so row rank of A is same as row rank of B if B is obtained from A by elementary row operations right.

These are just written there in the proof okay. So as a corollary of this you are given a matrix A and you have brought it to row echelon form. So what is a rank of the row echelon form? Rank of the echelon form that is the nonzero rows we defined earlier and what will be a row-rank of the row echelon form that is same as the row-rank of the matrix A. Let us try to observe that if you take the nonzero rows of the row echelon form, then that is a linearly independent set.

The nonzero rows in the row echelon form is a linearly independent set right, is very easy to write the proof but will just skip the proof, will look at examples and see how it works okay. So this is a fact that if you take a matrix A take it row echelon form then you get nonzero rows right. The row space of the two is same, row space of A is same as the row space of the row echelon form.

And in the row echelon form the nonzero rows, the vectors they are linearly independent. They are all linearly independent and bottom rows are 0, bottom rows are 0 in the row echelon form, only nonzero rows stop and we are saying they are linearly independent. So what will be the dimension of, so these nonzero rows will form a basis for the row space of row echelon form because they are linearly independent.

These are the only rows which are available which are independent, so what is the row-rank of the reduced row-rank of the row echelon form? That is the number of nonzero rows right and that is what we have defined as the rank. So what we are saying that rank that we defined as the number of nonzero rows is same as the row-rank of the matrix A is same as the row-rank of the row echelon form of the matrix. All three are same.

So let me repeat the reason again. Under row operations, the row space does not change first fact. That means the row-rank of A is same as the row-rank of the row echelon form. Second fact, the nonzero rows in the row echelon form are linearly independent and how many are there? (()) (12:53) the rank right. So they form a basis, they are linearly independent and form a basis, so that is a dimension.

So R and rank of the matrix is also the dimension of the row space. So rank is=row-rank is=row-rank of A tilde that is the row echelon form, all three are same quantities right. There are different ways of looking at it but you just see look at the number of nonzero rows or look them as a basis for the row space of the matrix A okay. So this is one observation.

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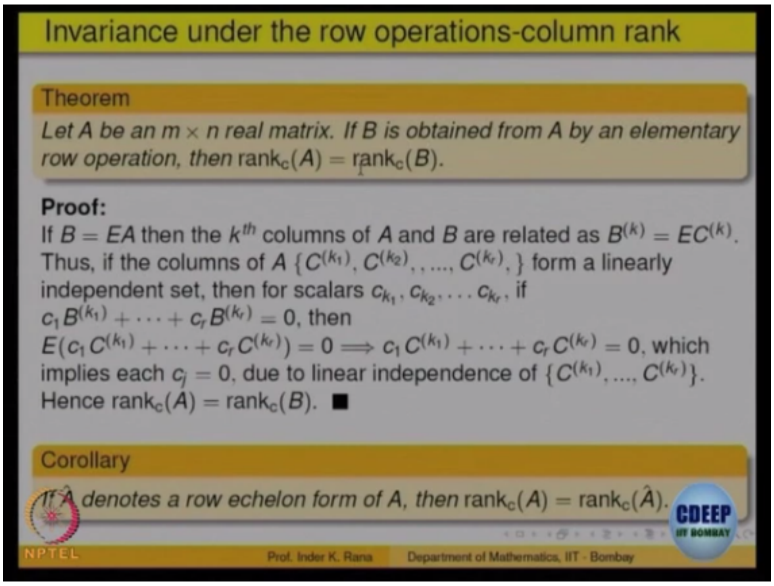

Similarly, if you look at the column space, we want to claim that the column space of a matrix A is same as the column space of the row echelon form of the matrix. If you perform that means how do you get the row echelon form? By applying row operations right and in terms of matrix multiplication what is a row operation? Premultiplying it by an elementary matrix and those are all invertible matrices right, we observed that.

So we will use that fact now. So if B is obtained from A by a row operation then B must be equal to E times A right. Now look at the kth column of the both sides, the kth column of B is precisely the kth column of A premultiplied by E, that is the block matrix multiplication right. So if that is the case okay, so what will happen? If the kth column of B is premultiplied by this okay then what we wanted to say is that if there is a linear combination, column rank does not change means what?

If there is a linear combination which is independent in A and apply row operation right, columns will change. We want to say that whichever columns were independent after row operations, they still remain independent. That will show that the column space of A is same as the column space of the row echelon form. Is that observation okay for everybody? If some elements, some columns were independent in A, I apply the row operations, columns will change right.

But suppose I start with independent columns, I know that some columns are independent right and I transform them by the row operations, I will get some transform columns. If I want to show that they also still remain independent. If we start with independent, they still remain independent, the transform (()) (15:45) independent, I want to show that.

So let us show that, so let us take a linear combination okay, c1 c2 ck, c1 of Bk1 and cr of so I am not taking all the columns, I am taking the k1th column, k2th column and I have picked up some r columns and I am taking a linear combination that is=0 in B okay, that is 0 in B. So what are Bk's? They are obtained from the corresponding column of A by premultiplication by the elementary matrix.

So what is Bk that we saw it here, that kth column is E times Ck, so I put that value so that means E times everything. So E is taken out right, distributive property of matrices, so this but if this is=0 what is E? E is an invertible matrix, so fact that every elementary matrix is invertible. So that means if E applied to a matrix is=0, then that itself should be 0 because E is invertible, I can operate on both sides by E inverse if you like okay.

So that means see the linear combination of c1, linear combination of these column vectors of A is=0 right. So what is if a linear combination of column vectors in B is 0 then the corresponding linear combination in A is also 0 and if they were linearly independent if these were independent in A then what will that imply? That is all the scalars c1, c2, cr must be 0, so what you are shown is if c1Bk1+crBkr is 0 then all are 0 provided the original ones were linearly independent right.

So these are proved by saying that if you start with some collection of linearly independent columns of A and transform them by elementary row operations, the transformed corresponding columns are still linearly independent. They still remain linearly independent right. So what does that mean? That means the column space of A is same as the column space of the transform matrix right, independence is not going to change right.

So that precisely says there is a column space of A is same as column space of B if it is obtained from A by elementary row operation. So if the space is same, dimension remains the same as for the row space we saw, so dimensions are same. So as a corollary right if B is the row echelon form, so we take a matrix A, bring it to row echelon form, then the column space of A is same as column space of the row echelon form and as the dimensions are same right, they do not change.

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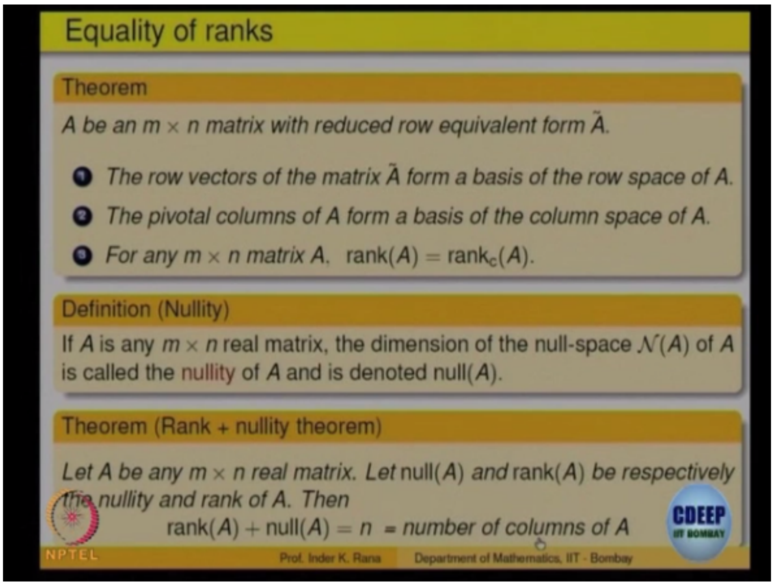

Here is a theorem which is based on these things only, what we have just now discussed. So it says if A is m x n matrix right, the row vectors of the matrix A tilde that is the row echelon form, they form a basis that is what we said earlier right. If we take a matrix A, bring it to row echelon form, then the nonzero rows form a basis right that we said with the rank is=the row rank okay.

So they form a basis and similarly if I look at the pivotal columns of A, so what are the pivotal columns of A, you take the matrix A, bring it to row echelon form, you see where the pivots are coming right. So knock down p1 the first pivotal column, p2 the second pivotal column, so look at the original columns right of the matrix, p1th column, p2th column and prth column, then they are also because independence is not going to change right.

So they are independent and span the column space, so that is second observation okay. They form a basis of the column space of A. So I am trying to tell you how to get a basis of the row space, how to get a basis of a column space. How do you get the basis of the row space? Take the matrix; bring it to row echelon form, the first the nonzero rows, top nonzero rows will give you the basis for the row space right.

The second one the column space, take the matrix, bring it to again row echelon form, look at the pivotal columns now, look at the pivotal columns of A original matrix. Then, they are going to give you the basis for the column space right and the third the rank is same as the rank of the column that you already observed and the rank of the matrix is same as the rank of the column, space is same as the rank of the row space.

One introduces that null space remember, Ax=0 right all vectors x so that is called the null space. So dimension of the null space is called the nullity of the matrix A. So that is the dimension of the null space okay and there is a theorem which says that given m x n matrix, if you add up the rank of the matrix with the nullity either sum of these two that is=n the number of columns of A, these two always add up and give you number of columns.

Again proof is easy but some technical writing is involved, so we will not do that. We will use this result and verify in examples right. So what you have said? Till now let me summarize before I start looking at examples. Given a matrix A right there are 3 important subspaces associated with A, one is a row space that is the space span by the row vectors of the matrix.

There is a column space span by the column vectors of the matrix right and what we are saying is the dimension of the row space is same as the dimension of the column space is same as the rank that we defined earlier, the number of nonzero rows in the row echelon form of a matrix right. So these are 3 important subspaces. How do you get their basis and dimension?

So second step, the basis of the row space is vector space span by the rows. The basis is the nonzero rows in the row echelon form; they give you the basis okay and then once you have found the basis you have found the dimension also the number of that. For the column space, again look at the row echelon form look at the pivotal columns, note down at what places the pivots are occurring.

The corresponding columns of A you pick up, they give you the basis of the column space. So what will be the dimension of the column space? Number of pivots and that is same as equal to the rank anyway, so column rank is same as the row rank okay but we also get a basis by picking up the corresponding column vectors right, so these are the ways of getting the basis for them okay.