Basic Linear Algebra
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Lecture - 17
Row Space, Column Space, Rank-Nullity Theorem - II

So let us look at some examples for that.

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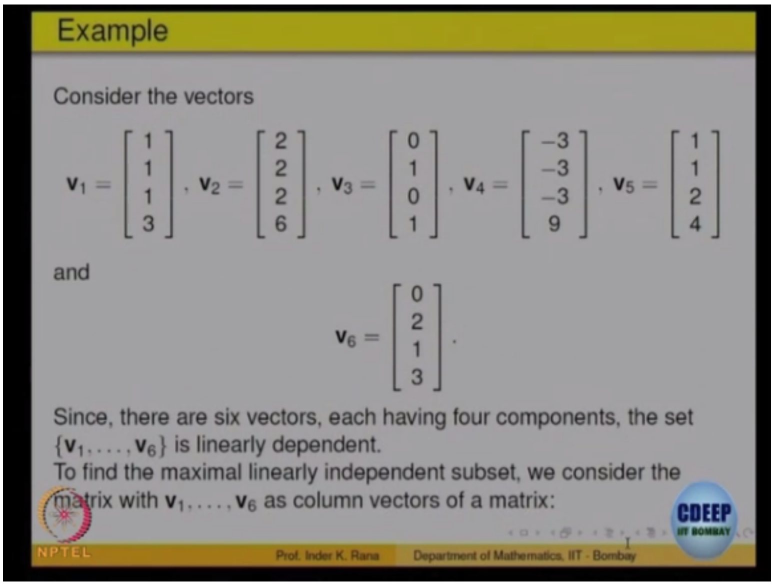

Let us look at I got 6 vectors v1, v2, v3, v4, v5, and v6. How many components are there? Each as vector in R4, so there are 6 vectors in R4, number of vectors is more than right R4 than dimension, so these will be linearly dependence set, straightaway it is a linearly dependent set right. I want to find out; out of all this which ones are linearly independent? Out of the 6 right that is a dependent set.

So probably one of them I can throw, maybe two of them I can throw out or remove three of them, I do not know how many, I want to find out which are the vectors which put together will give me a linearly independent set and that should generate the same vector space as all of them, all the 6 generators have space vector space the linear span right. I want a basis for that vector space right.

So what will be the basis? It should be a linearly independent subset of the generators one and what is the second thing? It should be linearly independent and generator. Two things, we should generate everything and it should be linearly independent. So I want a subset of the 6 vectors which is a linearly independent set and all 6 of them can be represented, each one of them can be represented as a linear combination of those whichever I choose.

So what is the process of doing that? I am going to describe that okay. So let us look at, so these are 6 vectors each having 4 components so it is a linearly dependent set obvious right. So what you want to do? We want to find the maximal linearly independent subset right. We find a part of it which is linearly independent and maximal. So 3 of them I find and 4 of them will not be linearly independent and that will form a basis right.

Definition of basis, either it should be a maximal linearly independent set or a linearly independent set which generates, either of them is okay or it should be minimal set of generators, these 3 either of the conditions I can use criteria, so let us look at. So what is the process? I take these vectors v1, v2, v3, v4, v5, v6 and make them as a column vectors. So the algorithm is the following.

I will put these vectors as a column vectors and form a matrix A right. Now I want to find out which one are linearly independent? So I will reduce that matrix to row echelon form and I will look where the pivots are coming right. Wherever pivot is coming, the corresponding vector I should pick up from the original one the column vector corresponding one I should pick up that collection should give me the basis for the column space right. So from that theorem, this should be the algorithm. So let us do that.

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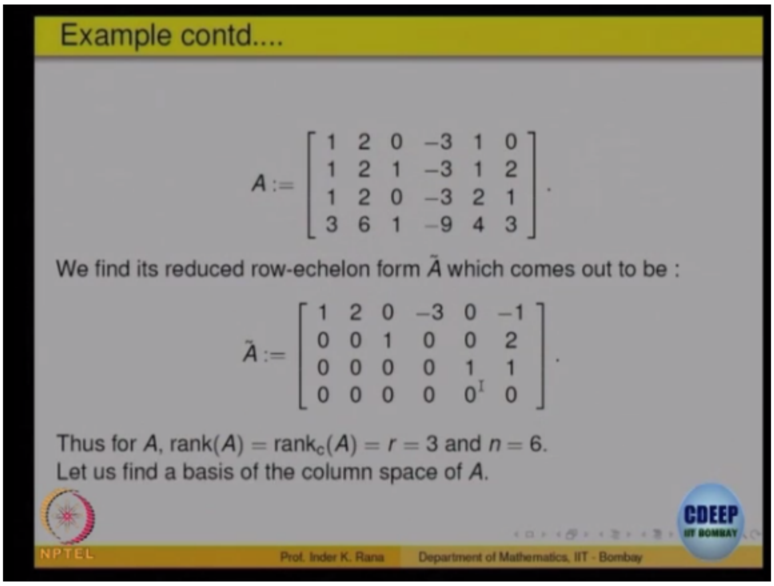

So let us form that matrix, so this is the matrix say by putting right each column is that vector okay, so the first one is 1 1 1 3 and so on, so that is what we have done 1 1 1 3 and so on. So all these 6 are put as columns, so I got a matrix, what is the order of the matrix? There are 4 rows and 6 columns, 4 x 6. I want to reduce it to row echelon form so what I will do? I know the process of doing it right.

I will look at the first column, see where the nonzero entry is coming, make everything below 0 and so on go on doing it. By elementary row operations, that is interchange is allowed, multiplication by a nonzero scalar is allowed and addition of two rows is allowed. So once I do that, so we will not spend time on doing that, so this is what I get. This is what, so this is. Is it in the row echelon form?

The first nonzero entry in the first row is coming at 1, everything below is 0 right. The next in the second row, the first nonzero entry is coming at the third place, everything else below that is equal to 0 right. In fact, above that also is 0 so this is what is the reduced row echelon form right in that column everything is 0. The next pivot in the third one is coming up this place right 1, this is 1 2 3 4 and 5 the fifth place and everything else is 0.

So these are the only pivots right because in the next row everything is 0. So this matrix, we do not have to make bring it in the reduced row echelon form, row echelon from is good enough. What we want to know is where are the pivots coming right. So here the pivot comes in the first place p1 is 1, p2 is 3 and p3 is 5. So we will pick up the corresponding vector in the original collection, v1 v3 and v5 right and anyway here it is rank is=3 total number is 6. So let us find the basis of the column space.

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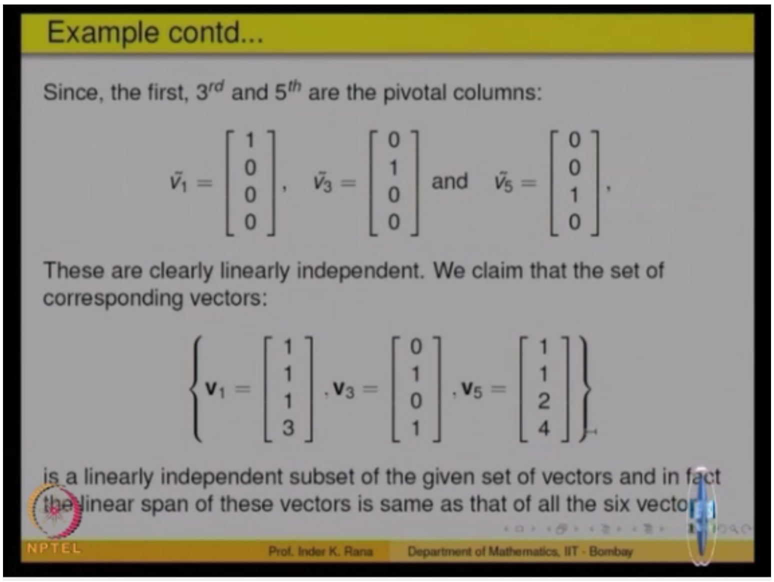

So I pick up v1, v3 and v5, those are the pivotal vectors in the original one. So here is v1 tilde, this is for the reduced form, we have pivotal columns in actual reduced form but original one will be from v1 tilde corresponding v1 v3 and v5, these are the original ones we pick up clear. Reduced row echelon form or row echelon form look at the pivotal columns corresponding columns from A you pick up, the original ones.

Now the claim is these 3 vectors should be linearly independent right and should span everything. I want to check that also right. Theorem says it should be so, so let us try to verify that with this example okay. So I want to verify, it is the linearly independent subset and in fact spans all the remaining ones, so let us check.

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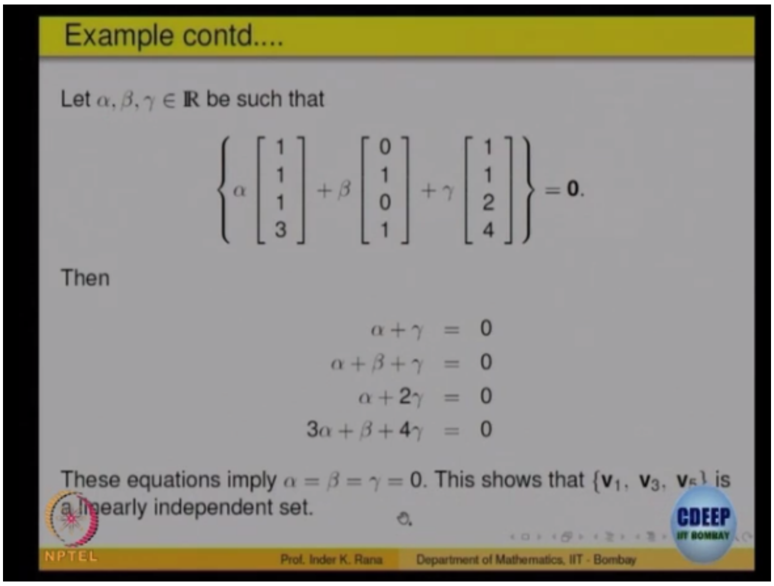

First of all, how do I check? Linear independence of the 3 right. So let us take alpha times the first vector, beta times the second one and gamma times third one right. I should not be putting okay, this bracket is redundant actually, it is the linear combination=0 right. In a mathematical notation, this bracket should not be there okay. If I want I should put on the right hand side also a bracket through 0, left hand side is a set, right hand side is a vector right.

So mathematically typo here, either I should remove these brackets. So basically what we are saying look at the linear combination of the 3 vectors alpha times the first one+beta times the second one and gamma times the third one=0. What will it then be? Alpha*1+beta*0+gamma*1, so first equation is alpha+gamma is=0. So write down these equations, the system of equations now right.

So these are 4 system of equations alpha, beta and gamma. Find out solutions of this right, you can check see alpha+gamma=0 will be this is equal to minus, put the value here and solve it. If you solve these things, it will give you that alpha is=0, it could be gamma=0, all 3 are 0. That means we will check in by the very definition if a linear combination of vectors is 0, then each one of them is=0 by looking at the corresponding system of equations.

So these are linearly independent, that is verified. Next, what should I do? Check that all 6 are linear combination of these 3, these 3 are themselves anyway linear combination of themselves, so remaining ones right are linear combinations, so let us look at that.

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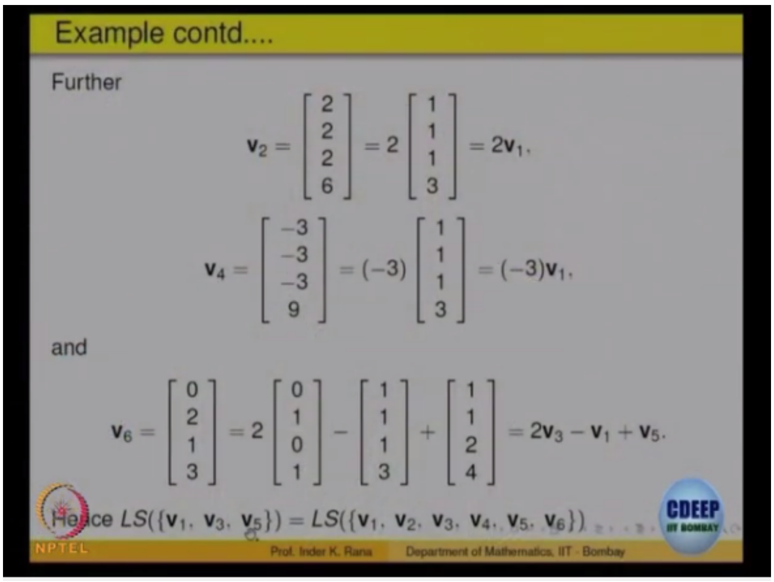

So this is v2, v2 has 2 2 2 6, this is 2 times v1 okay. Next one is v4, v4 is that is -3 times 1 1 1 3 right, -3 times v1 and how did I find out this that there is a linear combination, how will you do it? What we are doing is I want to write v6 as a linear combination of the 3 vectors. So I will take alpha times this vector+beta times this+gamma times this=to this vector now. So that system of equation I will have to solve.

That system alpha beta gamma equal to constant, I will have to solve that, I will get the vectors alpha, beta and so I solve that okay and I got 2-1 and now I can check whether this sum comes out actually equal to this, this linear combination is equal to this or not right. So that means the vectors v2, v4 and v6 which were not selected as part of the pivotal columns are expressed as linear combinations of the pivotal columns of A itself.

So all the vectors you want me to say v6 are linear combinations of those 3 right and those 3 were linearly independent that means those 3 form a basis for the linear span of the columns right. So that is all you find the columns. So linear span of v1, v2, v3, v5 is same as linear span of all the 6 vectors right and each one of them we wrote as a column. So this linear span so what is this v1, v2, v5?

They give me a basis for the column space of the matrix A right because you wrote everything as columns and in the row echelon form whichever pivotal columns are corresponding ones should give me the basis and we have verified that okay right.

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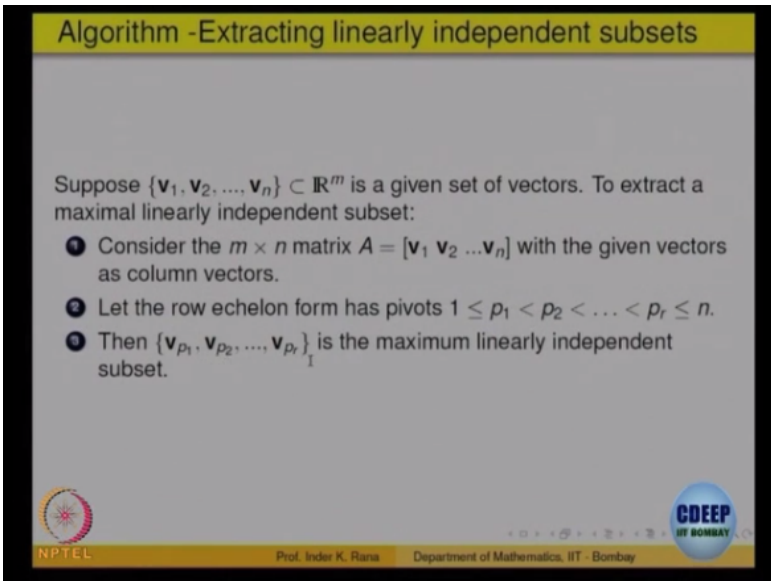

So this is I put the algorithm. Suppose you are given vectors v1, v2, v3, vn in given set of vectors to extract a maximal linearly independent subset out of it right, that is our aim, given a set of vectors, I want to extract a maximal linearly independent subset out of the given set, I want to find a subset which is linearly independent at the larger subset. So the algorithm is consider m x n matrix by writing these as the column vectors.

So given vectors write them as a column vectors right. Once you have written the column vectors, reduce them to the that matrix to the reduce the row echelon form or row echelon form and look at the pivots p1, p2, and pr, r is the rank, so there are r pivotal columns, pick up those columns vp1, vp2 and vpr, those columns is the that collection is the maximum linearly independent subset right.

And this we have verified in one example and this can be put on a computer. This can be put on a computer right. Well matrices can be put on a computer or computation of changing (()) (12:22) to row echelon form is premultiplication by matrices. So a machine can do that job okay but in the exam you will be the machine doing okay. We will have to do it ourselves right. So this is one algorithm.

Now another possibility is that you are given a set of vectors, you are not interested in knowing which one of them are linearly independent but you just want to find a basis for the linear span of those vectors, we want to find some basis, here what we are doing? Given the set of vectors, we are finding among the set of vectors as a basis but sometimes you are not interested in finding basis from the given set, you just want to find a basis.

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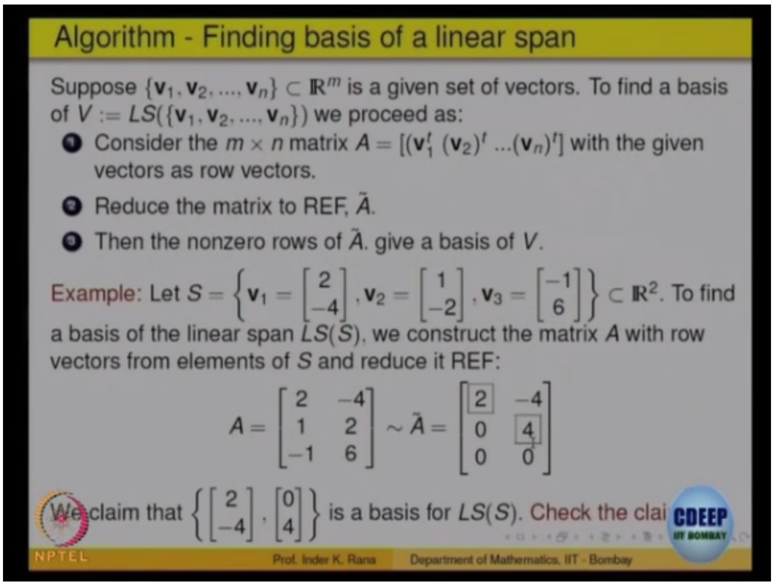

So let us look at the algorithm for that. So that algorithm is you are given vectors v1 to vn right, you look at the linear span, I want to find some basis of it right. So what I do? Write the matrix say now instead of writing these as a column vectors, write them as a row vectors. So form a matrix so why is this transpose because v1 normally is taken as a column, I want to write it as a row vector.

So form a new matrix A whose first row is the vector v1 written as row vector, second v2 written as row vector. So write these as row vectors, will get a matrix A right. Reduce it to row echelon form and look at the nonzero rows. What is the property of the nonzero rows? They are linearly independent and give the row space right. Row space is same as the space span by all the vectors.

So if you want to find just basis of the linear span of vectors finite number of them, write them as the row vectors, reduce it to row echelon form and pick up the top right. Pick up the rows in the row echelon form not the original one where you pick up the ones in the row echelon form. They will be linearly independent and give you a basis okay. So let us look at this example.

I have got 3 vectors, v1, v2 and v3 in R2 right. There are 3 vectors in R2, so obviously they are linearly dependent set. So if we are taking the span of them right, that should have less than number of elements than 2, <or=2. Any basis of the span that is S, span of S so linear span of the set S can have at the most 2 vectors right, not all 3. I want to find some two vectors which will give me the span of S.

So what I do? As I said, let us write that matrix. So what is the matrix? A. What is the first vector? 2 -4. So instead of writing as a row, I am writing it as a column, I am writing it as a row. So 2 4 right our general convention is writing as a column vector but for this algorithm we are writing as 2 -4 as row and next one is 1 -2, I missed -2 here right typo and next one is -1 and 6, so write these vectors as the row vectors, reduce it.

When you reduce it, you get reduced form is this, so there are only two rows which are nonzero. So what should be the basis? The vector 2 -4 and 0, 4, so these two should be give you the linear span. So the two methods both require reducing the matrix to row echelon form. When we write as columns, the pivotal columns picked up from the original one are linearly independent.

When we write them as the row vectors, then the nonzero rows in the row echelon form are linearly independent. In the first one, the advantage is you are getting a subset which is linearly independent right. In this form, you are only getting some basis; it is not the original ones right but the span of the original set of vectors you are finding a basis for that, that is all. So depending on what you want to do, whether you want to find among the given ones a basis or you just want to find a basis, you write it as a row or the vector as row or column.

But the process will again involve reducing it to row echelon form, that is crucial, without that you cannot do nothing. So all along till now what we have done is we have tried to reduce everything to computation to row echelon form or reduced row echelon form right. The key lies in that; you should be able to compute in that right. So that part I am leaving for you to check.

That means what? I should be using these 2, I should be able to write all these 3 vectors as linear combinations. So solving 3 sets of equations right, you have to solve that. So do that for the practice.