Basic Linear Algebra
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Lecture – 02
Introduction II

Now let us look at slightly more complicated things namely if we are given more than one equation in 3 variables, what could be possible solution of that.

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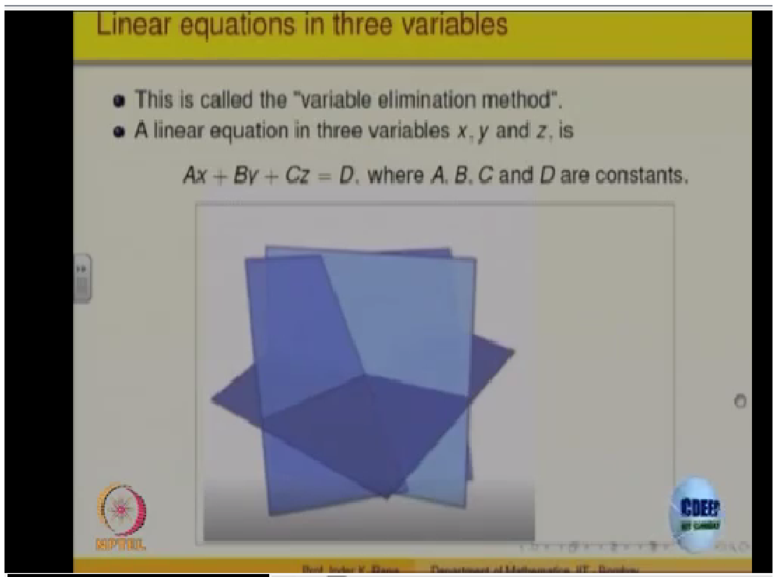
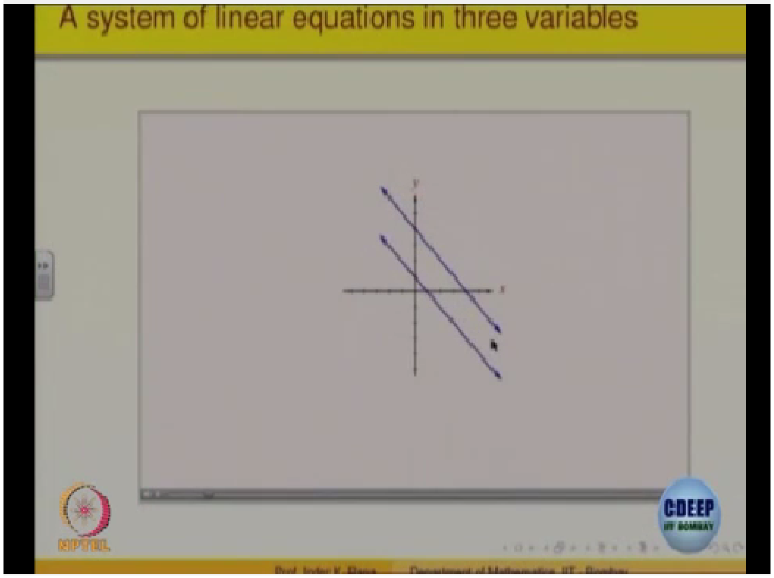


So let us look at, which again geometrically it is very obvious. **(Video Start Time: 00:40 - Video End Time: 04:06)**



There are 2 linear equations in 2 variables. There are 3 types of solutions there. Depending upon the vertices in parallel, so there are no points of intersection and the system has no solution. And if the lines are identical, the 2 perhaps coincide intersecting at an infinite number of points. The system therefore has an infinite number of solutions. Likewise, systems of 3 linear equations and 3 variables can have a single unique solution, no solutions, or an infinite number of solutions depending on ways in which the 3 planes are oriented.

Let us consider all possible ways in which 3 planes can intersect or not intersect. We will number these planes 1, 2 and 3. One possibility is that no 2 planes in the system are parallel. Then all 3 planes intersect at only 1 point. In this case, the system will have 1 unique solution. The second possibility is that 2 of the planes are not parallel and so their points of intersection (()) (01:52). If the third plane intersects the other 2 along the same line, then the common points of intersection for all 3 planes are represented by that one.

The system will therefore have an infinite number of solutions corresponding to every point on the lock. The third plane is not necessarily have to be the same for both of the other points. If the third point is that integral for one of the other 2 planes, then the points come into all 3 planes are still represented by the same line of intersection and the system will have an infinite number of solutions.

An infinite number of solutions will also exist if the system consists of 3 identical planes. In this case, any point which lies on one plane is come into all 3 planes. The system will therefore have an infinite number of solutions corresponding to every point on the plane. In addition, there are several orientations of the plane before result in a system with no solution. Any time the system has 2 distinct parallel planes, there can be no solution.

Since distinct parallel planes have no points in common regardless of the orientation of the third point, there can be no point which lie on all 3 planes. The third plane can be distinct parallel to the other 2 identical to one of the other 2 or not parallel to either of the other planes. Thus intersecting third point.

Regardless of the orientation of the third plane since there are no points in common to all 3 planes, the system has no solution. One additional configuration of the planes which will result in the system with no solutions is that the planes are oriented so then there intersection points lie along 3 distinct parallel lines. Once again since there are no points in common to all 3 planes, the system has no solution.

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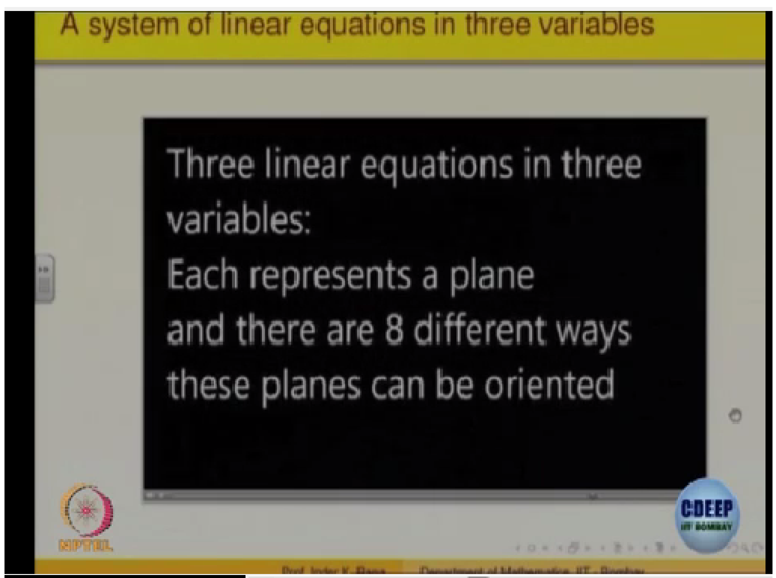

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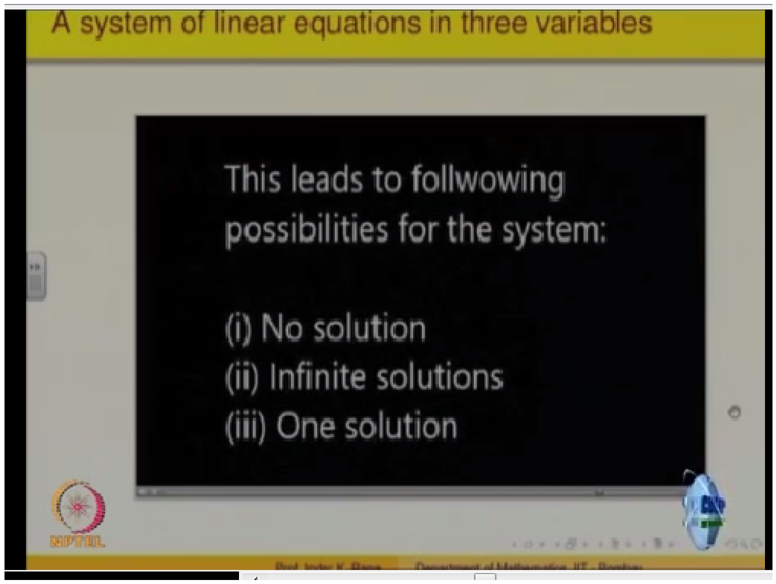

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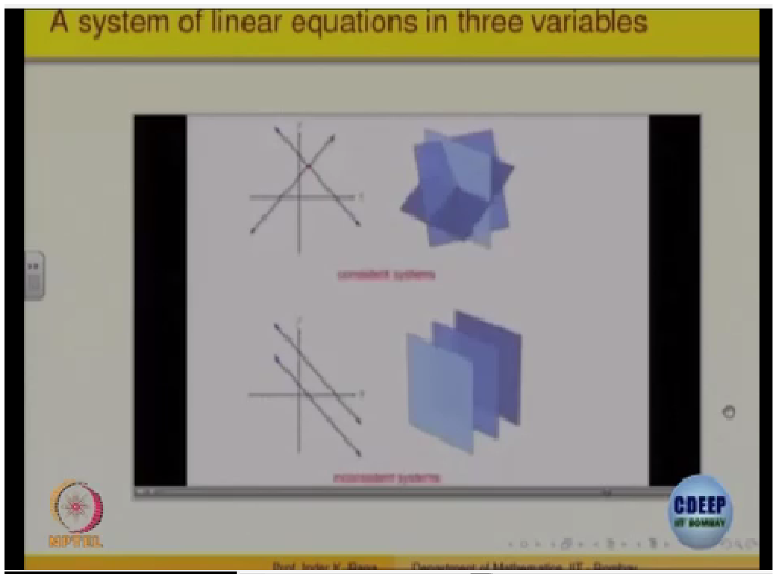

Right, so let us just summarize what we have done till now. We have looked at system of a linear equation in 2 variables. And then we looked at 2 equations in 2 variables and then we looked at the system of 3 equations in 3 variables and possible solutions. In either of this dimensions, either the system has no solution or it has infinite number of solutions or a unique solution. How does one go about analyzing when the number of variables increase?

There is no geometry available. What do we do about it? So what we will do is, try to convert these geometric pictures into algebra and then try to see whether we can extend that algebra for more number of variables.

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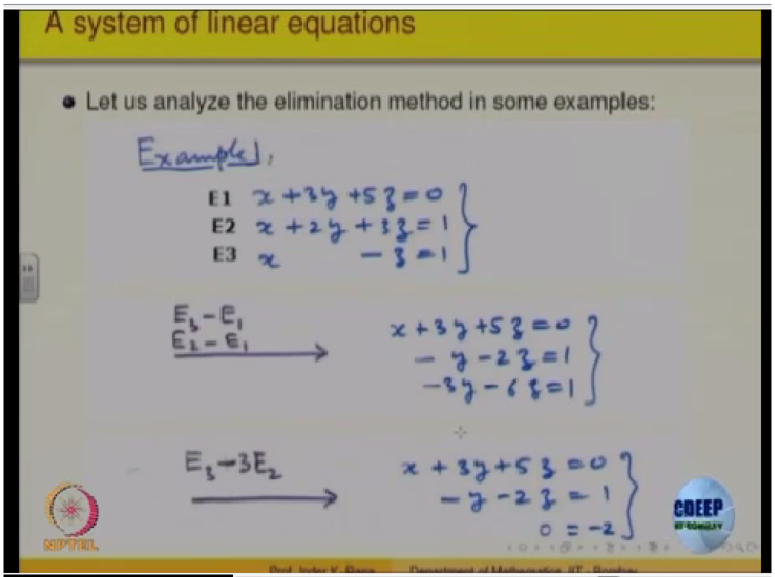

So let us analyze this method of variable elimination method. So let us look at this example, simple example and then we will try to formalize this. So example is x+3y+5z=0, x+2y+3z=1 and the third equation is x-3=1. So I have labelled this equations as E1, E2 and E3. So the idea is try to eliminate. Already in the third equation, right, y is missing. So let us try to eliminate the variable y from the other 2 equations also, okay.

So that is the idea. So what we do is, we do these operations, okay. So here for example what we have done is we have eliminated x from the equations. The first equation remains as it is. The second equation what we have done is E3-E1 and then operation is E2-E1. So we have just taken a linear combinations of the equations. So the solutions will not change. The idea is whenever you take a linear combination of any 2 or more of the equations, the solution does not change.

But it results in a system which may have lesser number of variables. So what we have done is, we have eliminated the variable x and we have gotten x+3y, right, so this system. And now from these 2 equations, I can try to eliminate again one of the variables. So let us do that. So E3-3E2, so that operation is done.

So I get the system of equations which is equivalent to the earlier one because I have just taken linear combinations, done nothing else, right. So solution of the last system should be same as solution of the original system. So that is the idea. But in the last system, I get an equation 0=-2, that means that is not possible, that is absurd, right. So that means this system has got no solution, right. So that is algebraically solving.

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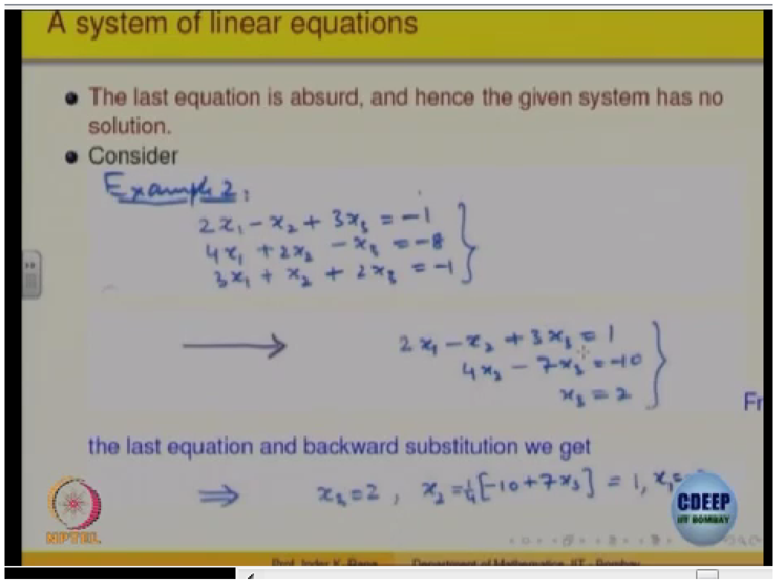

So this system has no solution. Let us look at this. 2x1-, so this is a system you can read that. I get 3 equations and we tried to do the same, right, try to eliminate the variables and I get this kind of, so 2x1-x2+3x3. I have not written down the operations. That is simple because 3 variables and 3 equations one can do that. So that leads to these kind of a new system. This has got all 3 variables present in the first equation, okay. In the second one, only 2 are there.

And in the third one, only 1 is there. So this system is equivalent to a given system. So solution of this should be same as the solution of the original one. And here, I get x3=2, last equation. I put the value of x3 in the previous equation, I get the value of x2 and I put these 2 values in the first equation, I get the value of x1.

So I eliminate and then substitute. So this method is called elimination and substitution method or backwards substitution and that gives me the solution of the system. So this system has got a unique solution, right. Earlier one, had no solution. This one has unique solution. And let us look at another one, this one.

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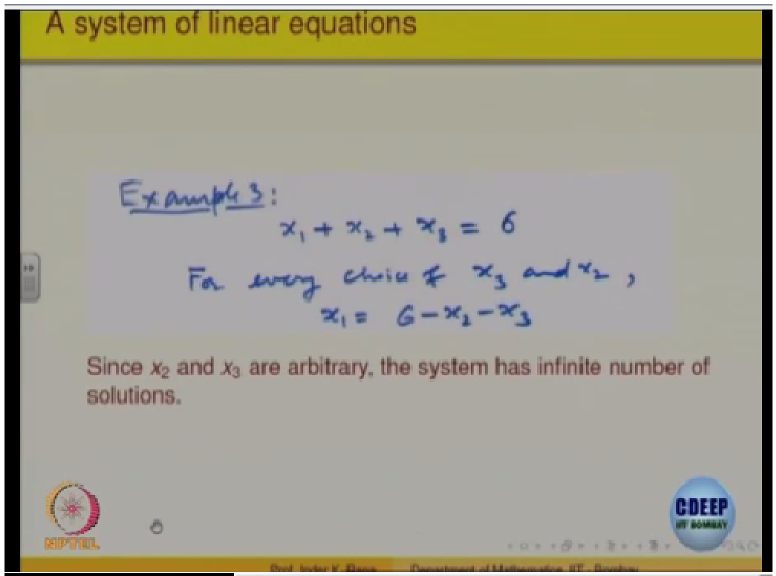

So there is only 1 equation x1+x2+x3=6 that because geometrically is a plane, right. How many points are there in the plane? Infinite number of them. So this system, one equation in 3 variables will have infinite number of solutions possible. How I can write them? One possibility is you can write x1=6-x2-x3, right. So that means what? x1 can be determined in terms of values of x2 and x3 and x2 and x3 are free to choose any values, right.

So I put different values for x2 and different value for x3, I get different value for x1. So for x2 and x3 can take infinite number of values, so that means x1 also has infinite number of values. That means the system has got infinite number of solutions which we are verifying algebraically. Geometrically it is plane, we know it is an infinite number of solutions. So 3 possibilities. I have given you 3 examples where the system had no solution, system had exactly 1 solution and the system had, okay, infinite number of solutions.

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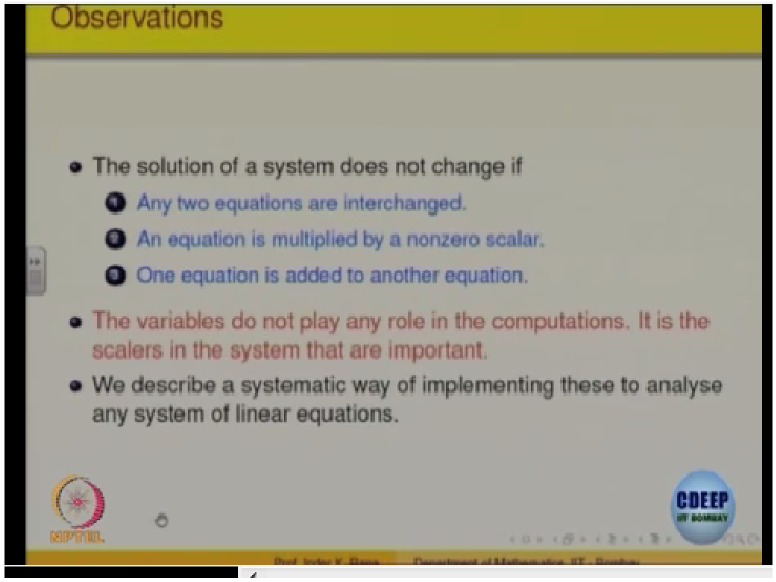

What is we observe in solving all these things? The solution of a system does not change whether it is 2 variables 2 equations, 1 equation. Any 2 equations are interchanged, what will happen? If I interchange any 2 equations, right, instead of saying this is line 1 and this is line 2 in the plane, I am saying this is line 1 and this is line 2. I am just renaming them, right. The solution does not change if I change the order of the equations, right.

An equation is multiplied by non-0 scalar. If I take a line, right and multiply it by a non-0 scalar everywhere on the left hand side as well as right hand side, does the equation change? Equation remains the same. Line remains the same, right. So solution does not change, right. So why non-0? Because when I multiply by 0, then all the information is lost, 0=0, that is nothing, equation is gone, right.

So multiplication by a non-0 scalar leaves the equation. So solution are unchanged. And the third one is, one equation is added to another, that we had already seen, right. If one solution is there, right and you add one equation to another scalar to multiple of that and now of course not 0, right, then the solution does not change, right, that we saw. In the 2 variable thing, we saw that if you add scalar multiple of 1 linear equation to another, the point of intersection does not change.

Only the inclination of the line changes, right. So these 3 are basic operations that do not change the solution of a system of equations, right. So that is one observation. And then the other observation is that the variables really do not play any role in all the computations. x1 remains the x1, x2 remains the x2, x3 remains x3. It is only that constants which are in front of them or on the right side of the equality, right.

They change when you do something with them, scalar multiple and do something, right. So we can forget when we write. We can forget about these variables. Only we should keep track that base is the coefficient of the first variable. This is the coefficient of the second variable. This is coefficient of the third and so on, right. So variable do not play any role. So let us try to describe this in an abstract study without observations.

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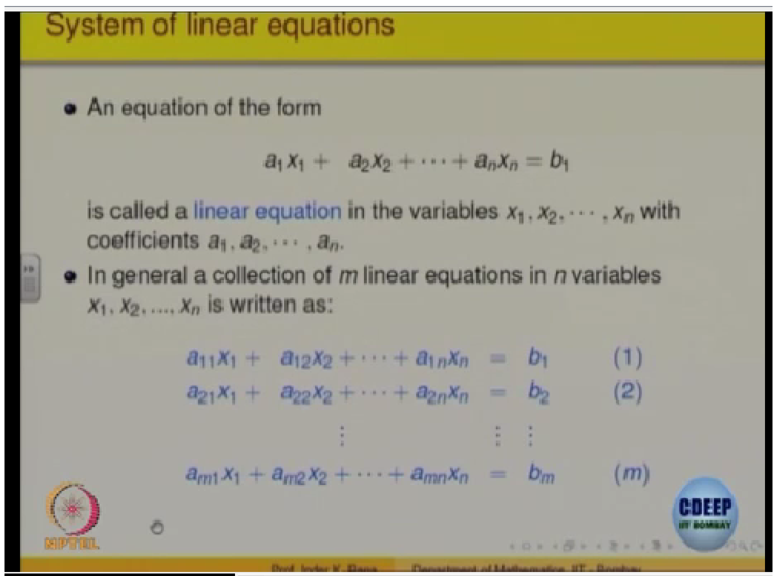

So we will define formally an equation of the form ax1+ax2+...anxn=b1 is called a linear equation in n variables x1, x2. Number of variables is n, only 1 equation. Why it is called linear? Because the powers of all the variables is 1. That is why it is called the linear equation in n variables. So more than 1, so you can have a system of m equations in n variables. So a system of m equations in n variables, we can write them as, right, to have a strategy or systematic way of writing but the first equation, the coefficient a11, right, x1, a12.

So the first letter indicates the equation. Second indicates the variable, right. And similarly, for the last one, mnth equation am1, coefficient of the first variable; am2, coefficient of the second variable on the right hand side, bm. So this is abstract notation of writing m equations in n variables, right. And for computational purposes, we do not want x1, x2, xn coming into the picture, right. We do not want them.

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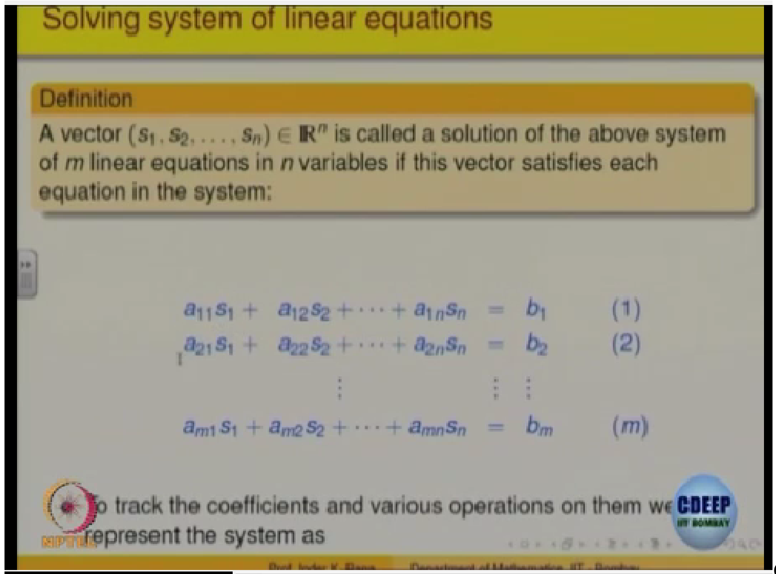

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So let us write, so before that let us just say a vector s1, s2, sn in Rn, everybody is familiar with R1, R2, R3, Rn, n components, right. So a vector we call it as, or a point s1, s2, sn in Rn is called a solution of a system. So when we say it is a solution, if I replace x1 by s1, x2 by s2, xn by sn in all the equations, then the left hand side is equal to the right hand side, right. So that is called the solution. Though the idea is how do we solve such a system, that is what we want to analyze?

So to track the coefficients and the various operations on them, we represent this system, we do not want s1, we do not want x1, x2 when coming in picture.

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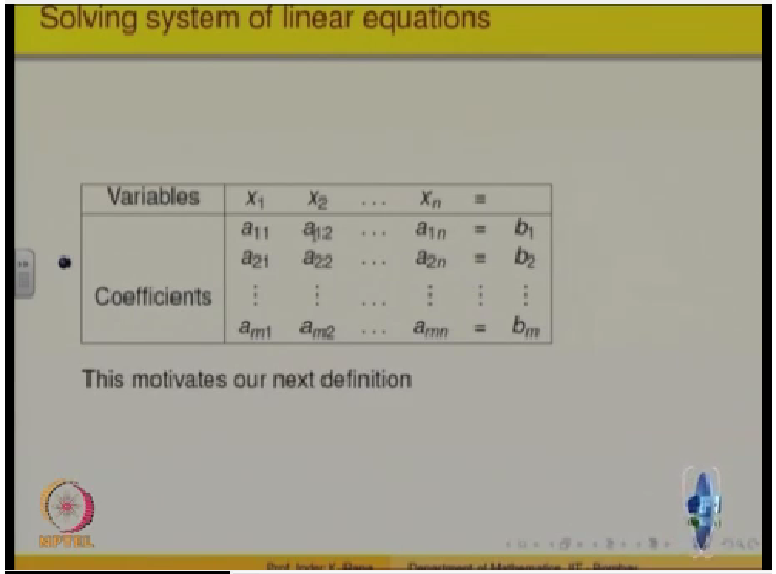

So let us write that. Variables are x1, x2, xn equality and the right hand side something appears, right. Their coefficients are for x1 coefficient is a11, right. For x2, in the first duration a12. So what we have done, we have fixed the positions of the coefficients. When we go in a row, will be representing the coefficients of that equation, x1, x2, xn. And when we are vertically down, then we will be representing for the equation, first equation, second equation, third equation and so on.

So return this way, all the data about the system is captured. We have fixed the position of the variables. This is captured.

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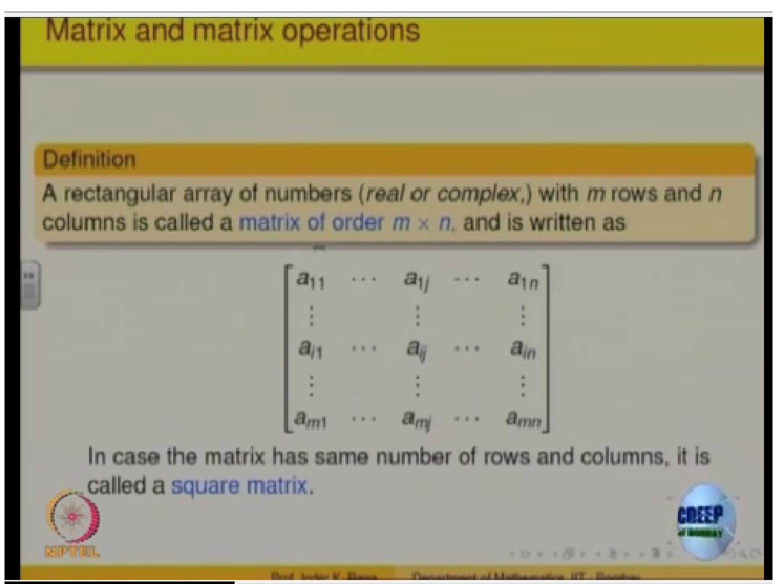

So this way (()) (16:26) a rectangular array of n equations in n variables. So basically what we are saying is to represent a system of a linear equations, a rectangular array of numbers becomes important, right. So we start defining, because this is going to come again and again. It motivates a definition in mathematics, namely a rectangular array of numbers, they could be real or complex.

In fact, they could be more general but we will be concerned only with constants which are real or complex, okay. So a rectangular array in which there are m rows, first row, second row, third row, m rows are there. And each row has got a n column, right. So this is called a matrix, okay. Why matrix is important you can ask? One of course the reason is because we are not going to bother about the variables x1 and x2, that is the real number system, right.

The important thing is when we do computations, you are not going to do it humanly. We are going to put it on a computer, all the computations, right. All multiplications, divisions, everything and a computer or a machine cannot store variables. We can only store scalars, right. So we can ask the machine to store the data in this format, right. Even machine will not know what is the row 1 or row 2 or row 3 or row n, right or number. So what you do it is write all if there is a rectangular array of m numbers in rectangular array, m*n, m rows n, n columns.

How many total numbers are there? m*n, right. So the computer will store it as m*n numbers but we will put some kind of a mark somewhere that this first n are for the first row, next n are for the second row and so on, right, for the computational purposes. So the basic idea is matrices written this way capture the computational aspect of linear equations, okay. So this is called a matrix of order m*n. When m=n, we will call this as a square matrix, number of rows=number of columns.

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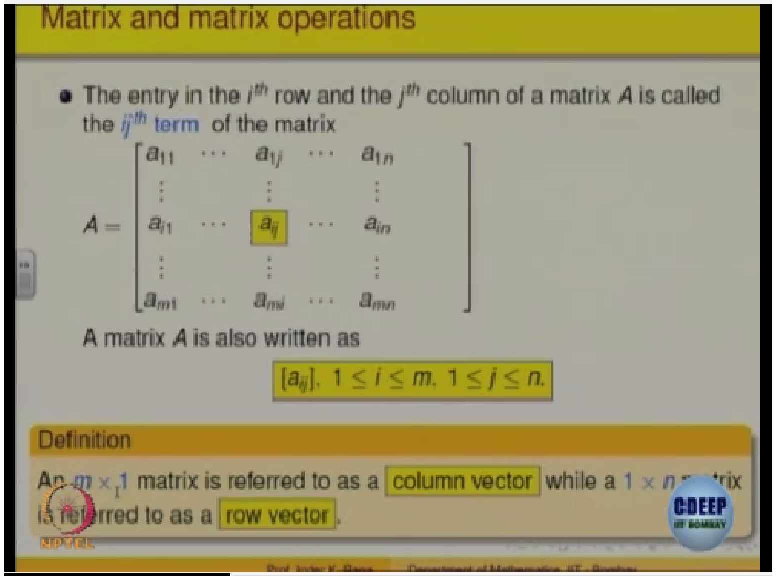

So if you are in the ith row and jth column, so that position is called the ijth term of the matrix or ijth entry of the matrix. So we will start using this term. We got a matrix whose m rows, n columns, ijth entry is so and so, right. So we will start using that language. So sometimes in short you write this whole, if you are not really interested in numbers.

But you are interested in the matrix as such as a quantity, so you write as aij where i and j vary according to the number of, i is number of rows, so m and j is the number of columns, n, okay. Sometimes this is important and useful also. If the matrix has got m rows and 1 column, so this is a column, right. And the matrix looks like a vertical thing. So it will call the column vector, right. And similarly, if it is 1*n, then it is called a row vector, okay.

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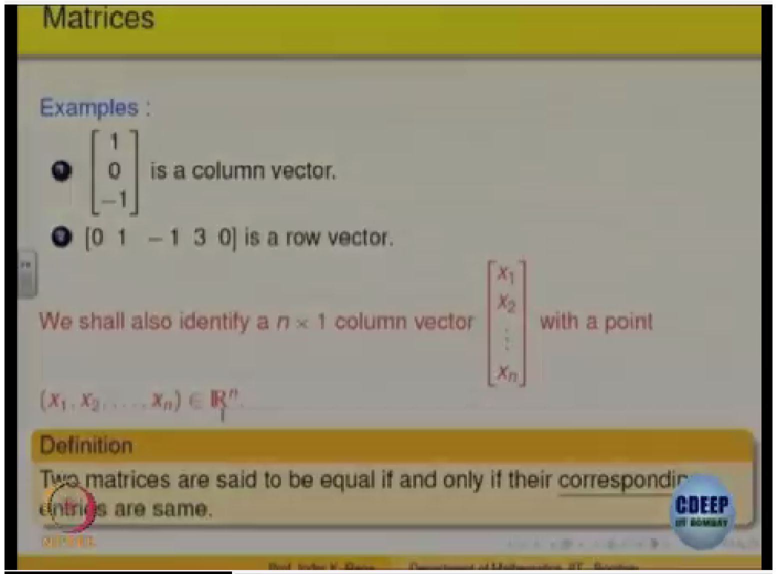

So for example, this is a column vector and this is a row vector. Later we will also do one thing that a n*1 column vector, there are n rows, okay, n*1, we also will identify sometimes as a point in Rn. So the column vectors will also be identified as points in the corresponding R to the power, right. So if it is a n*1, n rows 1 column, then it will be identified as a point in Rn. So this kind of identification we will also do when we want to simplify things, okay.

So we have defined a new quantity which involves numbers. So what are the possibilities? What kind of operations we can do? So this is very common in mathematics. You define a quantity. You collect all such quantities and try to do operations on these quantities. For example, you must have studied functions, right, real value functions. Then you can think a real value functions, defined on the interval.

We can add functions. We can multiply functions. We can scale or multiply a function. You can compose functions. So in the class of all functions, you can do these various operations. You can add, you can subtract, you can scale or multiply, you can multiply functions or you can compose functions. Similarly, we have got collection of matrices, right. So what do we want to do?

First of all, given 2 matrices of the same order, we say they are equal so we are defining equality of matrices. The order is same. So natural thing to say whether they are equal means what, each entry should be equal. So ijth entry of 1 should be equal to ijth entry of other, right, the corresponding entries must be equal. So that is equality of matrices.