Basic Linear Algebra
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Lecture - 29
Isometries, Eigenvalues and Eigenvectors-II

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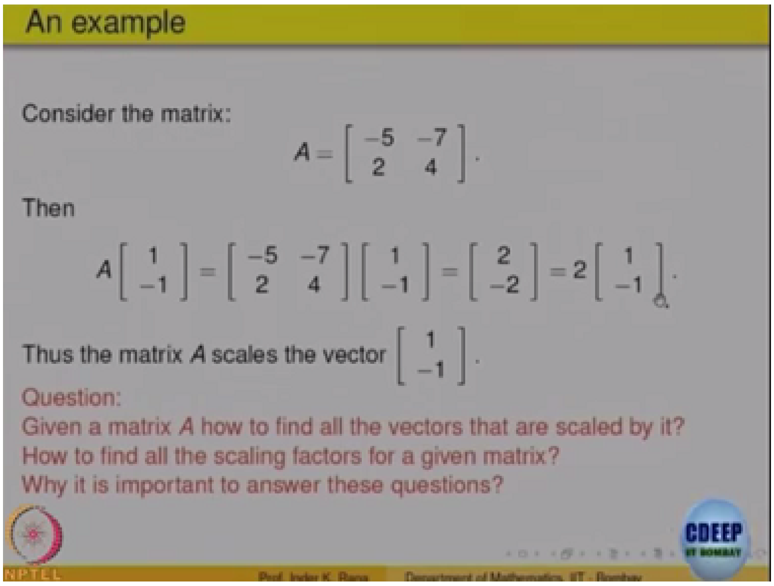

Let us start with an example to motivate everything that we want to do. Let us look at a matrix 2/2 simple matrix right -5, 2, -7, 4. Let us compute what happens when you multiply okay multiply A with this vector 1 -1 why I have chosen 1 -1 I will come and bit later I will tell you why this special matrix this special vector I have chosen, but let us just for the time being compute this product.

So A is 1 this vector is so what is a product -5, 1, right -7 when you multiply by this so what we will get +-5+7 that is 2 so similarly other one is -2. So ordinary products we are doing matrices and that is I can write 2 times I can take out 2. So this matrix right with respect to this vector 1 -1 has a special property. A apply to this vector is 2 times the same vector. So what is it doing?

So this matrix A is just changing the vector 1 -1 to scale we are just stretching it kind of thing right to twice. It is not changing it to anything it just stretching it that is all right. It lies on that line -1, 1 right 1 -1. So it was this vector now it is stretching twice of that nothing more than that okay. So here is the question given a matrix say how to find all vectors that are scaled by it.

For this matrix we have one I have given you how I got it I will tell you soon so that is one. How to find the scaling factor of a given matrix. Given a matrix does it have some property that for some scalars there will be some vectors say that A apply to that vector X it will be lambda times applied to that vector. So what are those scalars right, how to find those scalars and how to find corresponding vectors which tells that it scales it.

So this is and why is this important, why these all exercise we are worried about it and why it is important. So these 3 questions we like to answer okay. So let us look at one by one. I will keep this example in mind for a time being we will work out with this example only so that we understand everything right. So the question is given this matrix say I have selected one particular vector 1 -1 and which has a property that matrix says scales it.

It scales by a scalar 2 I know the scalar I know the vector which is getting scaled right. Actually if I take any other any vector with this say if I take 5 -5 that also will be scaled right, but the scaling factor will be different then that is all, but if I keep 2 then it will be same 1 -1 it will be scaled by some other factors, but if I take 5 -5 then it will be again 2 times 5 -5. So along that line every vector is getting scaled that is all whatever vector we choose it will be scaled by twice of that right. So the direction is lot of important.

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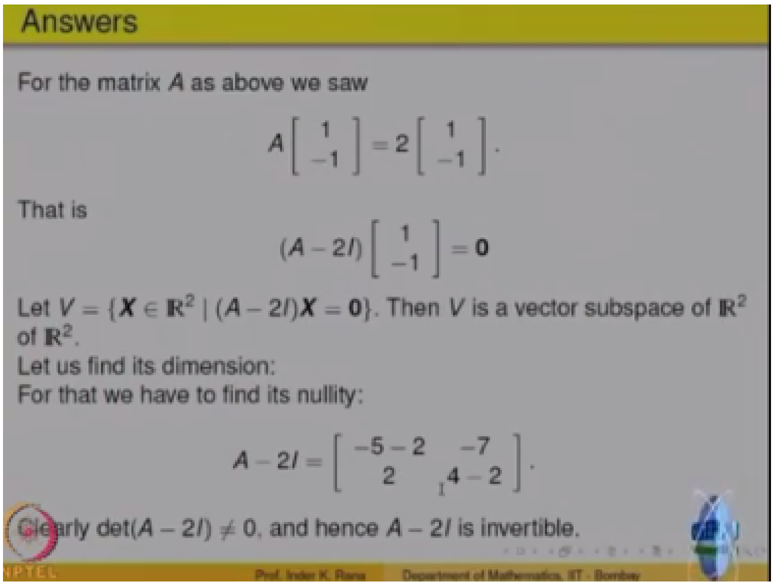

So let us look at it. So I am rewriting it A apply to that vector is 2 times that so let me take it on this side as a matrix equation. So A so when I take it on the other side what I get A- 2 times identity matrix apply to 1 -1 is a zero vector right. I just brought it on the one side I want to make it look like a homogenous equation now kind of thing right. So what he is saying is so let us 1 -1 is a vector which lies in the null space of A- 2 I.

So let us put the null space of A- 2 I as a set let us call it as V. So V is all vectors which are killed by A-2 I. So I have got one element in that right 1 -1 is in this right is it okay. If A – 2 I apply to this was zero. So 1- 1 that vector belong to this set V right. Now and we have already seen that we a homogenous system this is a vector space right. Any matrix apply to x=0 all of such x gives you a vector space.

So this is a vector space in which I have got in one element and this is subset of R2 right. So what could be the possible dimensions of this subspace already there is non-zero vector so it cannot be zero it has to be at least 1 it could be= whole R2 right. So whether it is= to that or not we want to know that right. So what is the dimension of that means what I am looking at what is the nullity of this linear transformation right in a earlier language A-2 I is a matrix.

I am looking at the dimension of the null space of it. I already have one non-zero vector so it has to be at least one. The question is it more than 1 or not right. So let us look at one way of looking at this when we look at this matrix of A -2 I right. I am looking at all possible solutions right. Now if this matrix if I have a matrix if AX=0 okay and the matrix is invertible then what are the solutions.

For a homogenous system AX= not this A some matrix some matrix apply to X=0 null space of a matrix. If the matrix is invertible what is the null space? No, for the matrix what is null space? Null space is a space I am asking you space not the dimension. If it is invertible rank is full so what is nullity 0. So that means this null space should be 0 for that right. If a matrix is invertible this null space should be 0.

So let us look at the matrix of this one A-2 I that is this matrix right and what is the determinant of this is 2/2 to find invertibility. I can just find out what is a determinant, what is a determinant is it 0, it is 0 or not 0, 0 okay. So if it is 0 then what should happen? It should not be invertible right. So that means there is some solution that is what we have gotten already one vector is already there right so this is a typo here.

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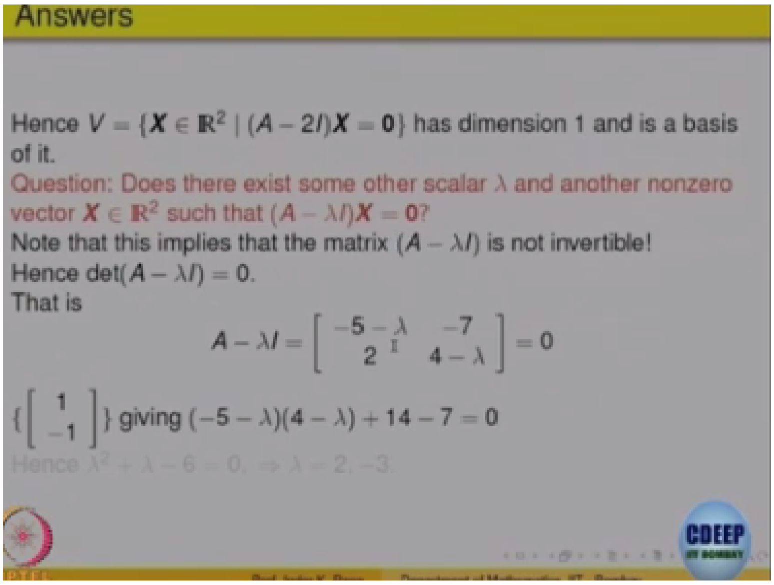

So let us find out so what will be the dimension of this how will do I find the dimension of that null space. I have got a one vector so what shall I do if I want to find the dimension of null space what shall I do let us look at. I want to find the dimension of A – 2 I.

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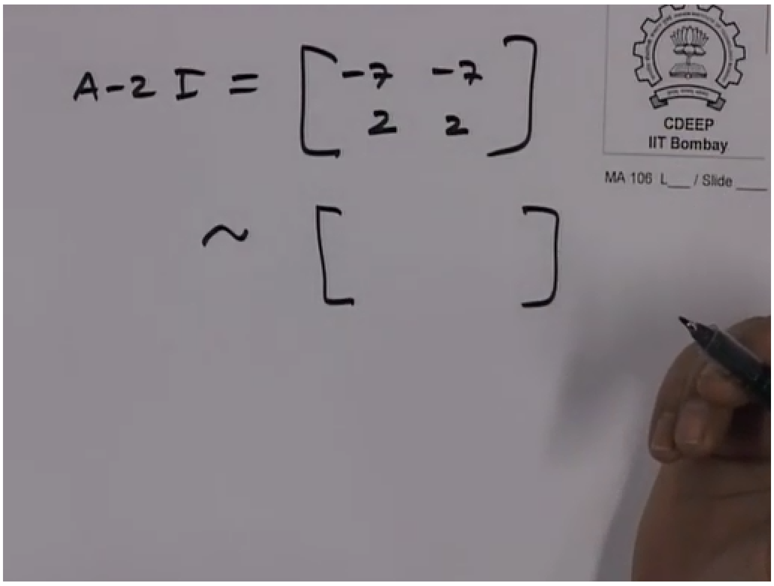

So what is A- 2 I that= - so that is -7, -7, 2 and 2 right is it okay that is A -2 i. So what is this equivalent to, how do we find the solution space? See I will reduce it to row it will be formed right. So anyway 2 columns are coming equivalent right so what is going to be rank already we know at least it is 1 it is =1 so only one solution is possible for the null space right is it okay. If I look at the column, then look at the column also we have shown that column rank is same as the row rank right.

So for the null space for so far this dimension has to be=1 it cannot be 2 right because we found the rank is=1 right so nullity is= 1 dimension is 1. And we already have one non-zero vector 1 -1 right for which we have shown that the solution side, so dimension is=1. So the question comes does there some is there some other scalar for this matrix right. We have got the scalar 2 and we got a vector 1 -1 so that A apply to 1 -1=2 times 1-1 question is there some other scalar also and some other vector which has a similar property right.

So how does one find out that scalar. So what do we want to do? We want to find out a scalar lambda such that A- lambda I apply to X= and 0 and one should have a non-zero vector X. Earlier I just gave you as a sort of without any motivation why I should take 1 -1 I said 1-1 has that property and scalar comes at 2. So here if this is=0 for a non-zero scalar X that means what that means this matrix A-lambda I for that scalar cannot be invertible because already a non-zero solution is there right.

It cannot be invertible right because if it is invertible only solution possible is X=0 right. For a homogenous system the only solution possible is 0 if and only if the matrix is invertible right. So if a non-zero solution is obtained some matrix apply to X=0 x is non-zero that means this matrix cannot be invertible it has to be singular. If it is singular what should happen determinant of that should be=0 right.

So if this is true then A- lambda I cannot be invertible first determinant must be=0 right. So that means what so let us look at A- lambda I okay determinant of that okay=0 what does that give you. Determinant of this is=0 so giving that -5, -lambda right 4- lambda multiplied --+14 okay. What is this 7 I think there is a mistake here right typo. What is the determinant of this let us just compute? I think while typing some mistakes have come. So let us look at the matrix is 5- lambda -7, 2 and 4- lambda.

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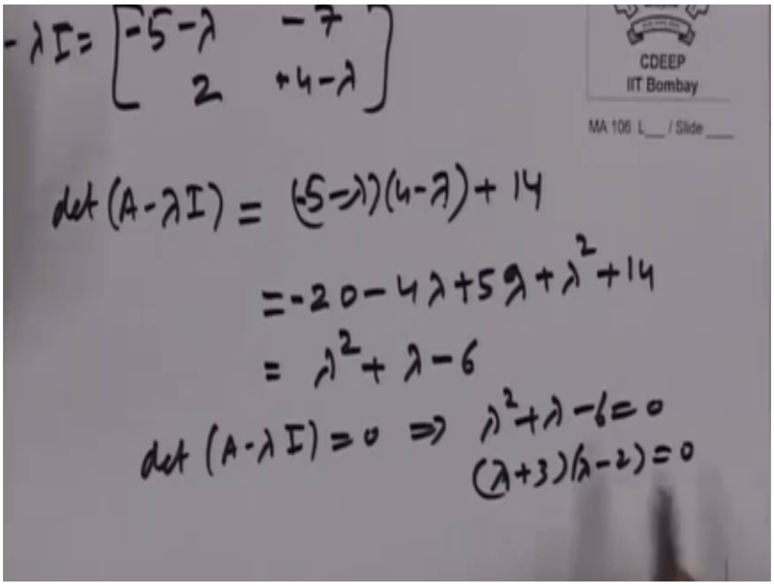

So that is A- lambda I right. This is a matrix so what is the determinant of this? So that is 5- lambda, 4- lambda+ 14 right. So what is that= let us compute 20-4 lambda-5 lambda that is I think that this is -4 here in the matrix yes -5, -7 so there is a -here that is +4 here okay. So that is -5 – lambda 4- lambda and this is +14 that is okay right so that gives me -20, -4 lambda+ 5 lambda+ lambda square+14 right. So what is that= lambda square + lambda -6 right.

So determinant of this=0 implies lambda square + lambda-6=0 so what is it that lambda+3, lambda-2=0 is it okay. Going back to school factorization 6 factors 3*2 you want + here. So that means lambda= -3 and 2 that is how this scalar 2 has come in the beginning right.

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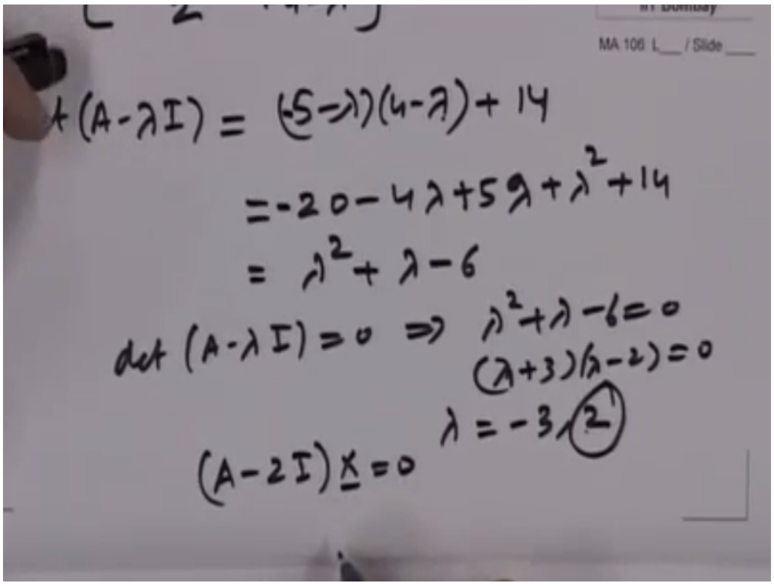

And what I did for this scalar we solve the system A-2 I X=0 that is homogenous system I can get a solution of that, that is how it gave me 1 and -1 right. So let us do it the same thing for the other one.

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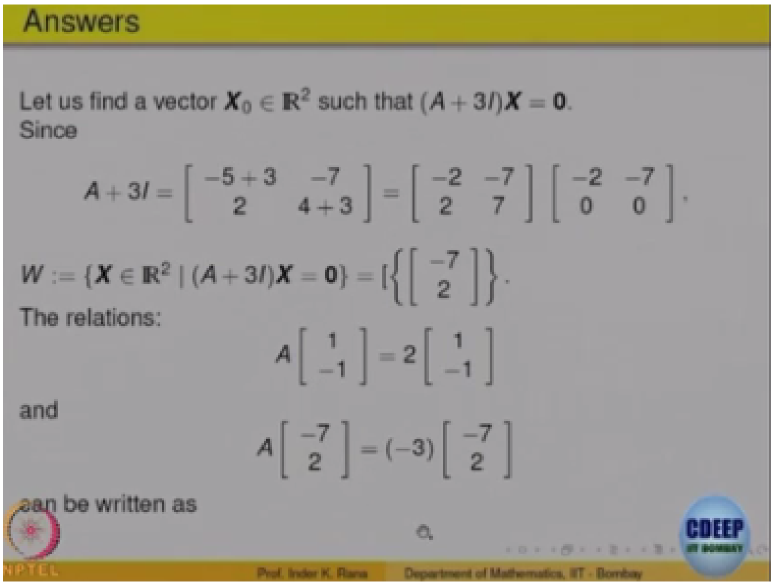

So let us do it for okay I think it is better to do the calculation by their maybe typo let us just see. So lambda is -3 so A+3I so -5+3-7, 2, 4+3 is that okay. So that gives me -5, -2, -7, 2, 4+3, 7 right.

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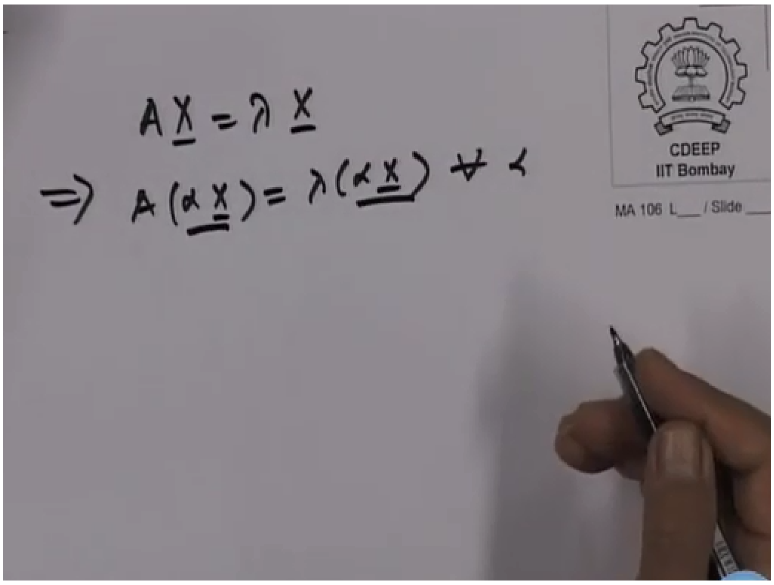

So let me clarify what I mean and this is supposing A X= lambda X suppose for some lambda there is an X right. If I multiply it by alpha X will that be= lambda alpha X for any alpha. If this implies this right. So for the same scalar lambda this also is a solution so and has the vector of a same line is alpha times that vector which we have 1-1 chosen and so on that is all right. We will take any vector on that is okay that is what I meant that clear okay.

So let us come back so I am looking at the solution for lambda=-3 A-3I okay lambda is -3 so A+ 3I=0 I was trying to solve this system. So this is what it comes out. So how do you solve the system by reducing the matrix to the AX=0 reduce the matrix to row echelon form. So it has 2 rows identical so that means I can subtract and I will get 0, 0 right. So this is row echelon form. So what is a solution -2 X1- 7 X2=0 right.

So you can put X to any value and get the value for X1 right or X1 is determined in terms of X2 right. So if you do that so that is what you get -7 and 2 right the vector is -7 and 2 or any scalar multiple of that is that okay how do you get this one by solving -2 X1-7 X2=0 right so solve that. So what I get is this I got 2 vectors and 2 scalars with the same property for this matrix that A apply to 1-1 is 2 times if scales it by 2 and this vector is scaled by -3 right.

So let us try to combine these 2 in one equation how do I combine that in one equation. I want to write one equation where both these are captured. So let us call this as C1 call that as C2 and form a new matrix right then what will happen?

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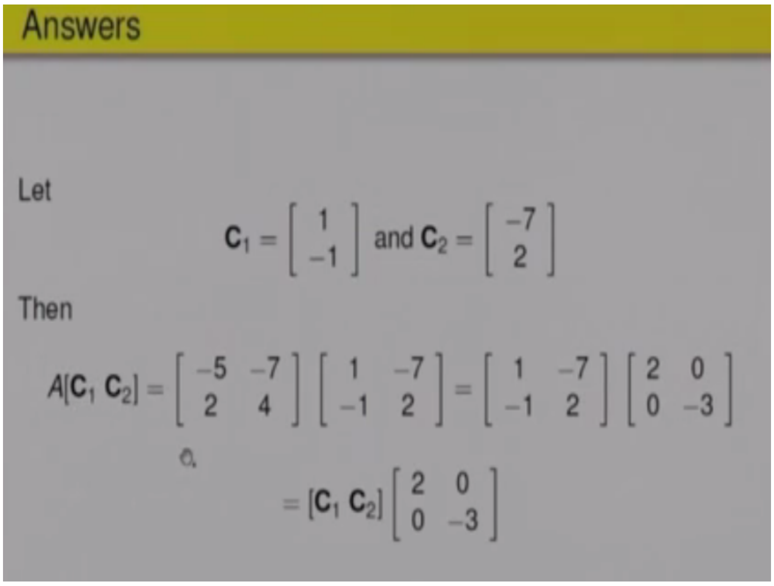

Let us call C1= this and C2= this then what is A of A apply to this matrix it will be A C1 right+ A C2, but A C1 is what is A C1 that is the first lambda times that vector second is second scaling second thing right. It is just first vector, second vector and scaling is 2 and -3. So I can write it as a matrix multiplication and that is same as this matrix is same as C1, C2 and that is a diagonal matrix. So the 2 questions have been put together into this equation now.

And now look into C1, C2 this matrix with this first column and this as the second column let us call that matrix as something.

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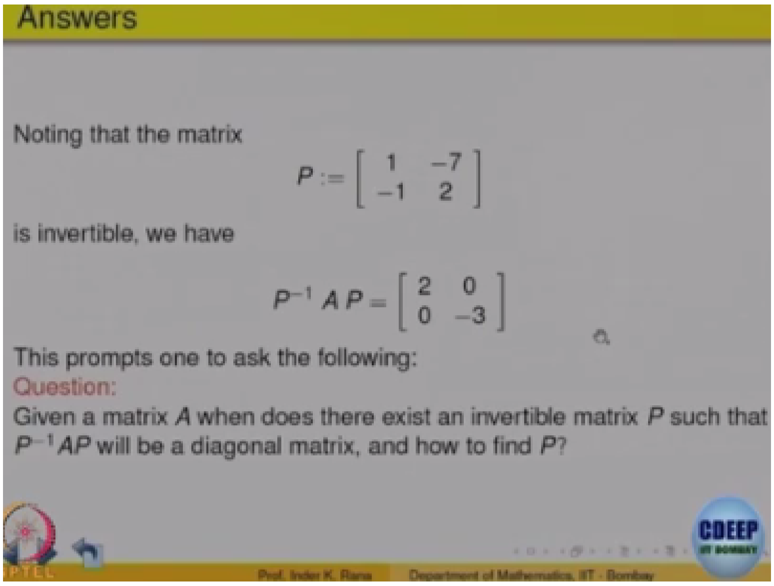

Let us call that matrix as P 1-1 second column this will be invertible matrix right is it invertible determinant not=0. So this is invertible so that equation AP=P times this diagonal matrix P is invertible so this is invertible I can take it on the other side by multiplying by inverse of that. Yes, I can multiply on both sides on the left by the inverse of this matrix this is called P.

So P inverse AP will be= this diagonal matrix yes is it clear. So I am saying P inverse AP= diagonal matrix. So what we have done. What we have done is for this given matrix that we had I have found 2 scalars namely 2 and -3 what were these 2 scalars they were the solutions of determinant A-lambda I =0 for each one of them I solved the system of homogenous system and found out right.

Here I was able to get one vector 1 -1 other one -7 and 2 *such that the matrix found by them is invertible and as a result I got P inverse AP= diagonal right. So this is a process I have done by which I am able to find the diagonal entries for that given matrix A I am able to find the matrix P either this property holds. Is it okay?