Basic Linear Algebra
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Lecture – 30
Isometries, Eigenvalues, Eigenvectors-III

Now what is the advantage of doing all these kind of things. Why I am bothering and doing such kind of a thing you can ask a question. It is nice it looks nice mathematically, but why one should bother about it.

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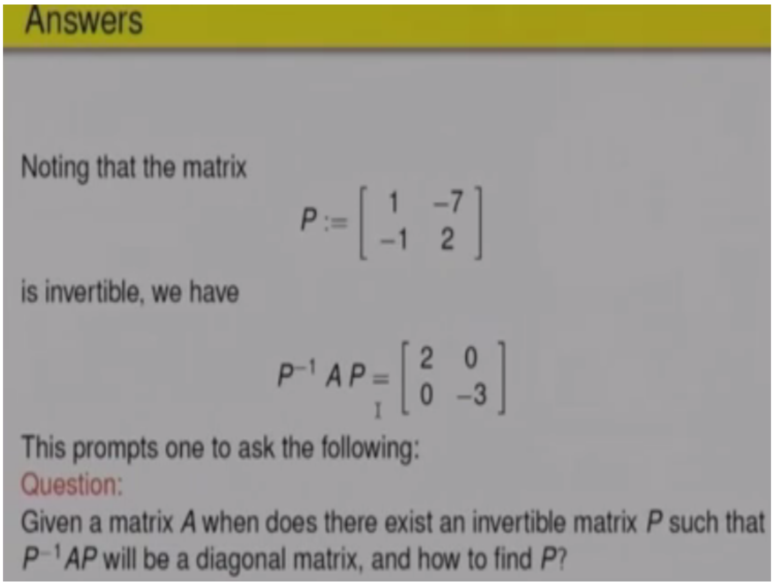

See look supposing A is a matrix for that matrix that we had taken -5 right -5, -7 and 2 and 4. Suppose I want to compute the 100 power of that matrix right that was the matrix. What was the matrix we have.

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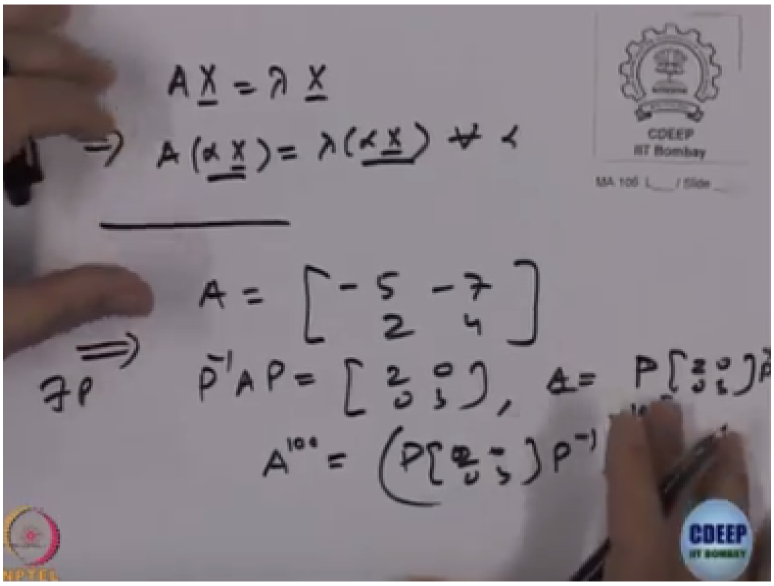

A was the matrix which we had -5, -7, 2 and 4 right. These are the (()) (01:14) and for this matrix A have got P inverse AP= 2, 0, 0, 3 there exist P such that this one is right. I want to compute what is A raise power 100 right. So from here what is A= I can write A= from this equation so that is A= 2, 0, 0, 3 on the right I multiply it by inverse so P inverse and here I multiply by P is it okay.

So A=this so what is A 100 that is P this matrix 2, 0,0, 3 P inverse whole thing raise to the power 100 what is this= looks surprising what it is= let us compute it for square first of all and then we will see what is happening.

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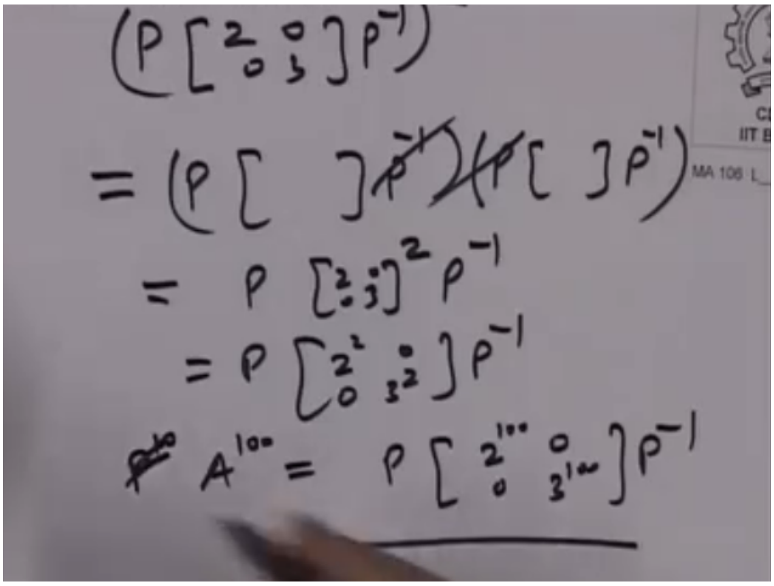

So let us look at P 2, 0, 0, 3 P inverse square so what is that= P this matrix P inverse multiply by itself right. So what is it P inverse what is that. So what I will get P this square P inverse this is 2, 0, 0, 3 what is a square of this it is 2 square, 3 square 0, 0 right. So what will be P to the power 100 what will be A to the power 100 that will be P 2 to the power 100, 0, 0, 3 to the 100 P inverse. See the computation of the powers of a matrix become very easy because that is a diagonal matrix now on the right hand side.

And for which powers are very easy to compute and multiplying by I have to only find out what is P and what is P inverse only 2 matrices. I see only one matrix and it is inverse right. So such a result is what is called diagonalizability of a matrix A. So what we have done is given a matrix A which was our special one we have said we can find a matrix P which is invertible such that P inverse AP is a diagonal matrix.

A diagonal matrix is known a matrix P is also known right. So the question comes for what kind of matrices one can do that. For this particular one we are able to do it, but we may not be that lucky so we have to find a condition which ensures if this matrix satisfies this property then it can be diagonalized not only it can be diagonalized that means I can find a matrix P such that P inverse AP will be diagonal and I know what is a diagonal matrix also right.

So those are the scalars which are going to appear on the diagonal are nothing, but determinant of A- lambda I=0 solution of that. So it becomes important to solve determinant of A- lambda I right so let us write that as a definition.

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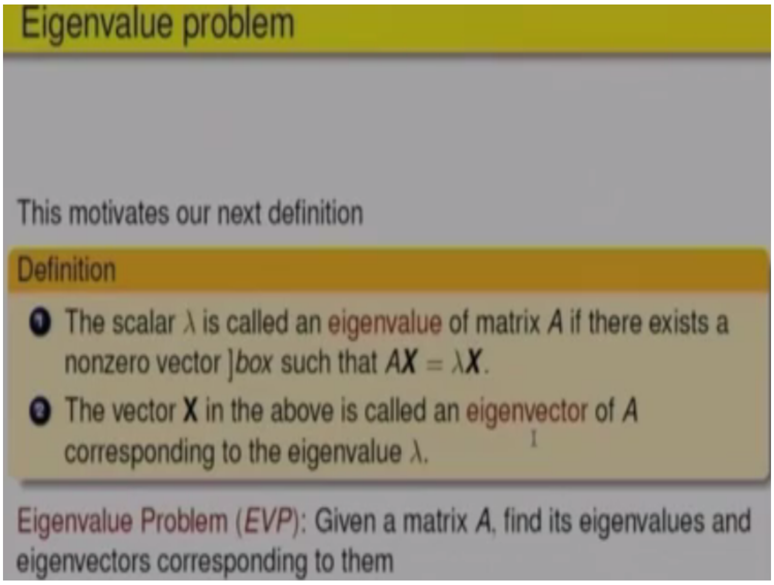

So these are the question so let us write lambda is called an eigenvalue. So such a scalar is called an eigenvalue if there exist a non-zero vector oh gosh it is a typo. There is a vector X such that AX= lambda X. So if there is a non-zero vector which is scaled by that vector by the matrix A then that is called an eigenvalue for and this vector is called an eigenvector for that eigenvalue it is called an eigenvector.

And just now I said for a given Eigenvalue in this example the previous example Eigenvalue is there too, but there are many eigenvectors possible for you if one is a eigenvector scalar multiple of that also is a eigenvector right the same eigenvalue. So there could be more than one actually we can put them all together in one box you can call it as Eigen subspace of with respect that Eigenvalue so we will come to that slowly.

So that is called an Eigenvector. So the problem is this is what you called the Eigenvalue problem we call it and given a matrix A find its Eigenvalues and find corresponding Eigenvectors right at least the process is quite clear. There is nothing we have already done one example of that. So what is a process take the matrix A it is a square matrix look at for a scalar lambda look at A- lambda I because you want AX= lambda X.

So look at A-lambda I that matrix you want a non-zero solution for that right with that matrix as a coefficient matrix A- lambda I apply to X=0 for a non-zero X you want a non-zero X that means you are looking for a non-zero solution for the homogenous system that is true if and only if the matrix is not invertible right. Nullity should be at least > or=1 right the rank should not be full that means the matrix should not be invertible.

And in terms of determinant that says determinant of A- lambda I should be=0 right. So let us give that determinant of A- lambda I also a name okay.

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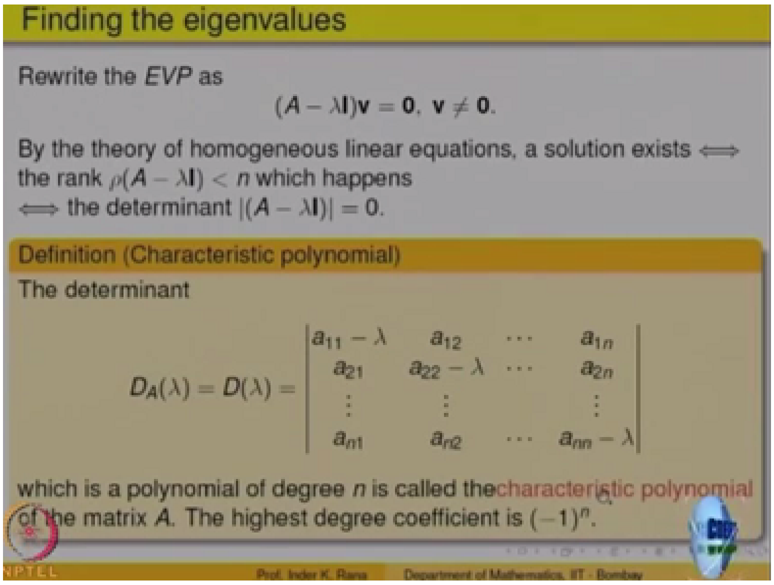

So that is called the characteristics polynomial of the matrix. Given a matrix A look at determinant of A- so what will be A-lambda along the diagonal you subtract lambda right everything else remains as it is. So determinant of this what it will look like try to (()) (08:07) expand. See in the example 2/2 it was a quadratic in 2/2 it came out as a quadratic lambda square –right whatever that was.

So here let us says suppose you expand by this right you expand by first column I am saying so this will be gone and this will be gone. So we will have a11- lambda into determinant of the remaining right. So when you write determinant the remaining again we will have to expand by that. Let us again expand by this column then it will be a22- lambda coming out as a factor into determinant of the remaining one.

So what will be the highest power of lambda that will come lambda to the power n right+ some lower powers+ some lower. So essentially we are not exactly proving it, but giving you a hint that intuitive we are saying that this determinant is a polynomial in lambda of degree n. So the characteristics polynomial is a polynomial that is why it is called a characteristics polynomial because it actually turns out to be polynomial and it is characteristics of that matrix is a property of the matrix is a determinant A is a polynomial of degree n.

And what will be the coefficient of the highest term lambda to the power n it will be- lambda or – lambda – lambda it will be –1 to the power n that factor will come out anyway right. So it will be lambda 1 to the power n* lambda n+ something, something it will be=0 right. If you want the problem is how to solve that polynomial find the roots of that polynomial. These are the values which are going to be called as the Eigenvalues of the matrix right.

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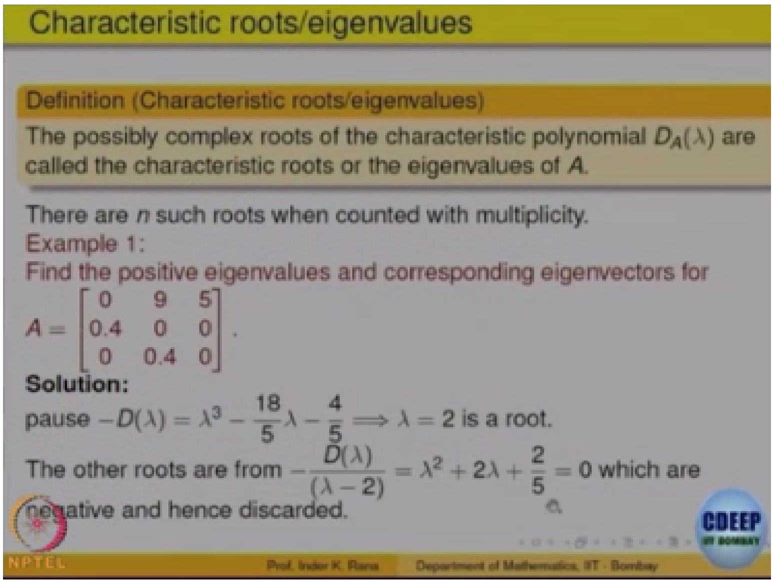

So let us look at this so the roots of this are the eigenvalues right. Roots of the characteristics polynomial of the Eigenvalue. Now here is a theorem in algebra called fundamental theorem of algebra which says here the coefficient are all going to be real right real matrices, but the roots of a real polynomial may not be always real they can be complex right. So the characteristics polynomial will be a polynomial with real coefficient.

It may not have any real root right all the roots maybe complex. So Eigenvalue need not exist that is a possibility. So for a given matrix the Eigenvalue no Eigenvalue may exist. So gone case for that you can do nothing you cannot diagonalize that matrix at all, you are gone at the first step itself. It may have Eigenvalues right some of the roots may be real, some of the roots maybe complex how many all will be there total is a polynomial degree n.

So a fundamental theorem of algebra says there will be n roots for this. Some of them maybe real some maybe complex, but the complex roots always occur in pairs right. So if it is a polynomial if the matrix n is of odd order then at least one real root will exist right. So then a hope is there you start with that at least. So that is coming from algebra basically okay. So let us we will slowly come to so a root there is something called the multiplicity.

If a root is repeated right then how many times it is repeated that is called the multiplicity of that root. So Eigenvalue may or may not exist it may exist with some multiplicity that are possibilities. Now let us look at this matrix looks slightly. If you calculate it I think this example just to show you computation determinant of this you get lambda=2 okay. How do you get for a cubic how do you solve a cubic even for a 3/3 matrix finding roots is not a easy job. There is no definite algorithm like for a quadratic.

For a quadratic you know (()) (12:38) that gives you all the roots (()) (12:42) or complex whatever it is for a quadratic for a cubic one does not know. So the hope is you find one by hook or crook divide by that and get a quadratic and try to solve it kind of thing. For 4th degree there is no hope at all right. It is very difficult to find roots of a polynomial okay, but here anyway for this by just looking at it you can see that there is a possibility of saying that there is a root lambda=2.

Sometimes if I can look at 1-1 2-2 and some (()) (13:21) this is for just the sake of examples nothing more right. In life in mathematics life is not easy to find roots of a cubic even it can be tricky right. So here you find 2 is root and all other root are negative so probably one should. Let us look at only for 2 what is a Eigenvalue eigenvalue 2 what is a.

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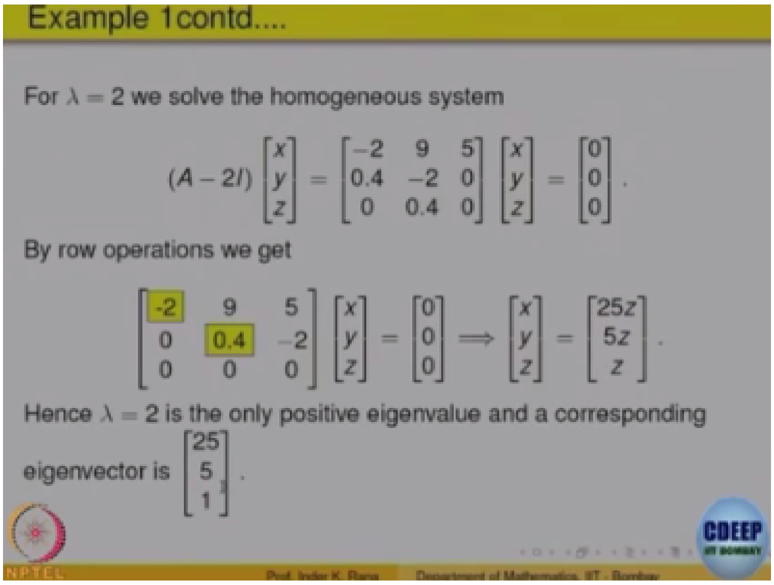

So I am just giving a process for lambda=2 to find an Eigenvector. This you will do for all Eigenvalue A-2I apply to a vector X=0 so you apply that you get the homogenous vector right homogenous system. So how do you solve reduce it to the row echelon form right. So you get this right. You see all along in all our course till now the only one idea which is important what is the single most important idea.

Given a matrix find its row echelon form find it reduced row echelon form everything comes back to that right. Everything gets related in that because that is a way you will compute things on machines okay. So find this so that means what if this is=C so what you get x, y, z you solve it right. This decimal you can remove so you will get 25 z, z is arbitrary there is a one row which is right rank is less one row is 0.

So what is our variable you will get arbitrary value. The third one z right you give z arbitrary value you find xyz so find y in terms of z from this equation put back the value you get the value of x so comes out= to this. So you can take z out okay so that is a scalar okay times 25, 5 and 1 so that is a eigenvector for eigenvalue. So the Eigenvalue for lambda=2 is this is an eigenvector so is clear.

How to find Eigenvalues and Eigenvectors corresponding to that solve the characteristics polynomial and for each root solve the corresponding homogenous system okay.

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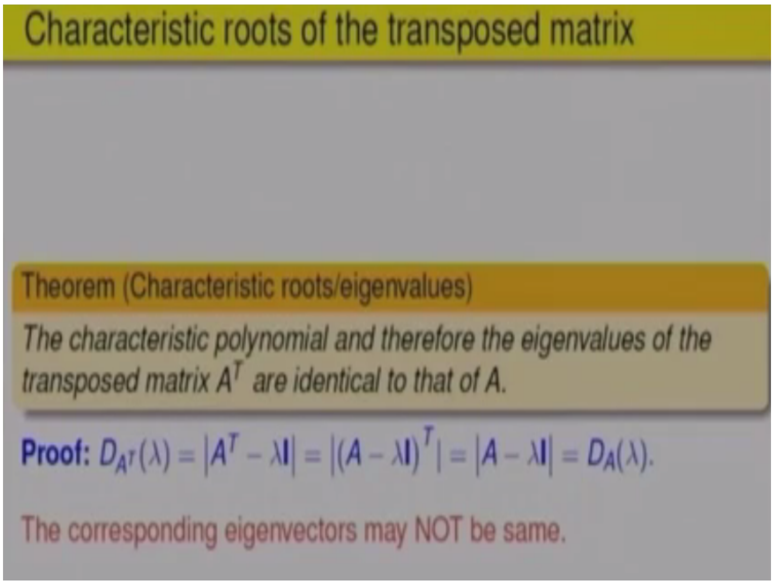

Here is a roots of see what is a determinant of A it is same as determinant of A transpose right. So finding the characteristics polynomial for A is same as finding characteristics polynomial for A transpose Eigenvalues will be same, but Eigenvectors may not be the same for both because you will be taking A- lambda I or A transpose –lambda that will change. So for computation of Eigenvalues you can replace A by A transpose if you want.

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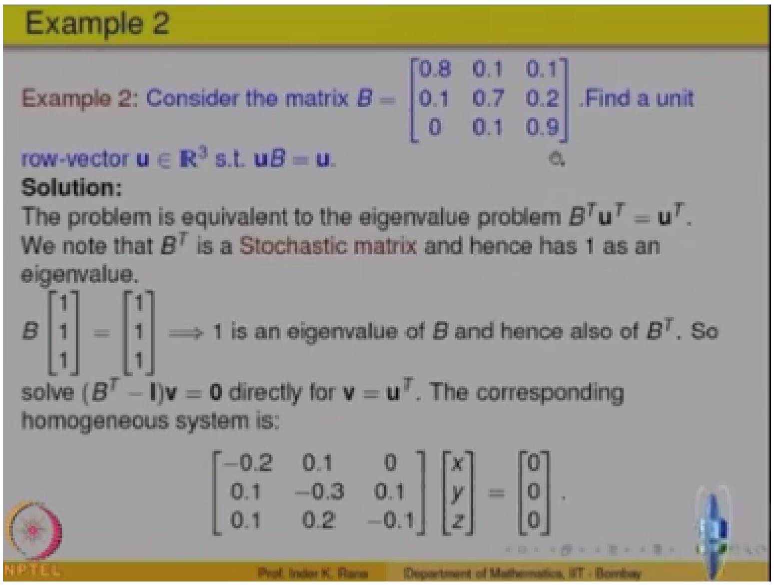

So here is one example of that so look at this matrix okay some funny looking matrix to look. We want to find a vector u so that u times B=u what does that mean? It does not seem to be related with Eigenvalue right, but supposing you take transpose of this equation what you will get B transpose of light to u transpose= u transpose that means for B transpose you are saying that for an eigenvalue=1 find an Eigenvector that will be u transpose is it okay.

Once you have found u transpose you can find u is it okay so that is the idea of application of that. So B apply to here is some observation which is not really important, but for this matrix if we look at the transpose of this so what is the sum of the rows of this matrix 0.8, 0.1, 0.1 that is 1 some of the rows=1. So if we take transpose some of the columns=1 such matrices are called stochastic matrices whose column each column add up to 1.

They basically arise in probability theory, statistical analysis because the probabilities are distributed so total=1 essentially is something like their probability of something happening kind of thing and you want to this is called the stochastic matrix because it describes the system in probability and there you will like to know if you apply that process again and again what happens.

So you will be required to compute the powers of that matrix and that is why for such a matrix diagonalizability becomes important when can you diagonalize and one shows a theorem that for a stochastic matrix number one is always an Eigenvalues that can be proved as a theorem. We are not bothered about that. So here one is an Eigenvalue so we look at B transpose so B transpose-1 I v=0 solve that system right.

So that will give you the vector v and what is that vector v that is the transpose of the vector u that you are looking for right. So this was basically just brought into say that for determinant of A is same as determinant of A transpose that can be used in finding Eigenvalues of a vector.

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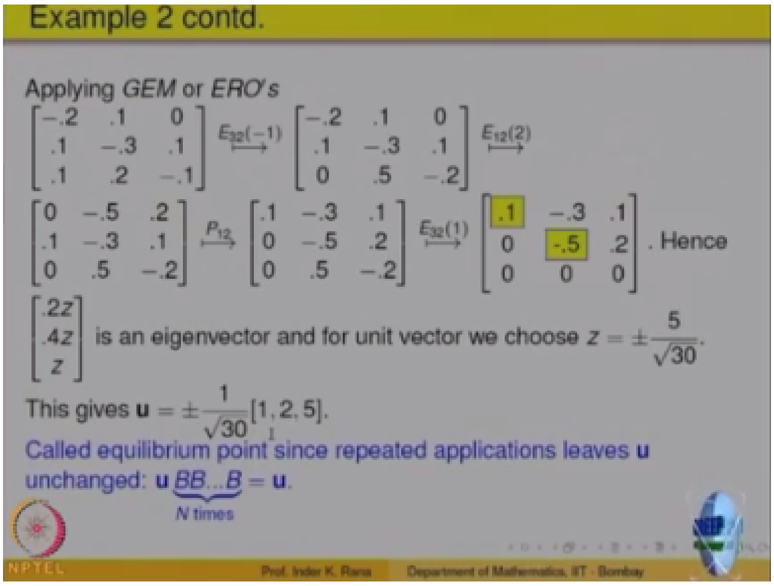

So that is what you find solve that computation and you find you can choose if you want a unit vector. These are eigenvector right 0.2, 0.4, 0.1 okay. If you do not want decimal you can multiply everything by 5 that also is an Eigenvector so that is 1 to 5 you can normalize that If you want. If you have a normalized Eigenvector which is non-zero anyway divided by norm you will get a Eigenvector of length 1 right.

And this is what is important you go and applying there is a something called equilibrium of a stochastic system and after some stage it does not change the system at all you have reached equilibrium in probably theory so let us do not bother this line okay. So important thing is for the Eigenvalue lambda=2 and Eigenvalue lambda=1 for a stochastic matrix you will find the Eigenvector so this has importance in other subjects.

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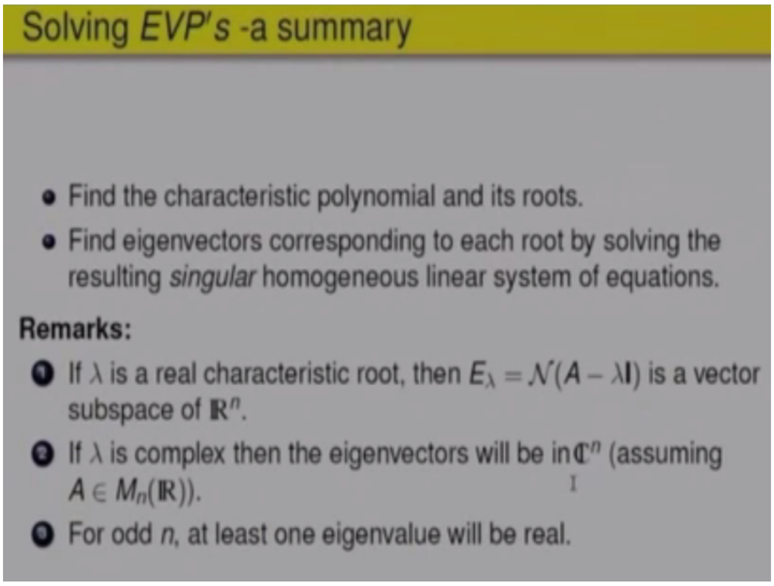

So basically let me summarize finding characteristic polynomial and its roots Eigenvalue problem is basically. Find a characteristics polynomial, find its roots, find its eigenvector by solving the resulting singular homogenous system right that is so as I said you can put them altogether in one box the null space of A-lambda I that depends on lambda Eigenvalue that is null space of this matrix.

You can also call it as Eigen subspace right vector space subspace corresponding to the Eigenvalue lambda. Now there is something which probably will bother next time. So given even if a lambda an Eigenvalue exist there is a Eigenvalue that means you have to find a Eigenvector for that, but sometimes one allows even complex Eigenvalues the complex roots which are coming for the characteristics polynomial.

Even one takes into account that there are then the corresponding matrix that you will get the solution A- lambda I then you do not restrict yourself to real entries your entries will change to complex also right. So you may get a vector X which is a complex vector. Complex vector means what entries are complex number right. So instead of looking at matrices over real one start looking at matrices over complex numbers also because it is forced by that problem of Eigenvalue problem right.

So we will start looking at this next time more. Let us stop here I think.