Basic Linear Algebra
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Lecture – 38
Abstract Vector Spaces II

Let us look at R3 for example or Rd.

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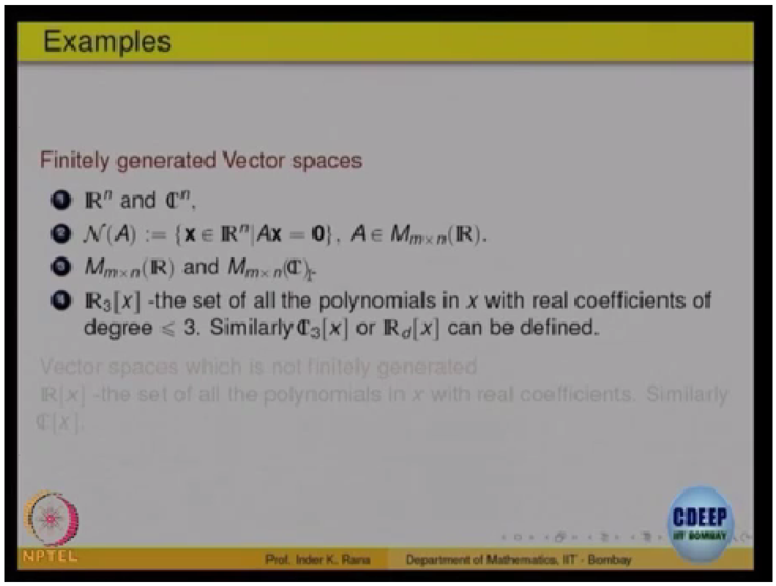

Polynomials of degree less than or equal to 3, is it finitely related? So let us look at it, okay. So let us look at this example. So we are looking at R3x.

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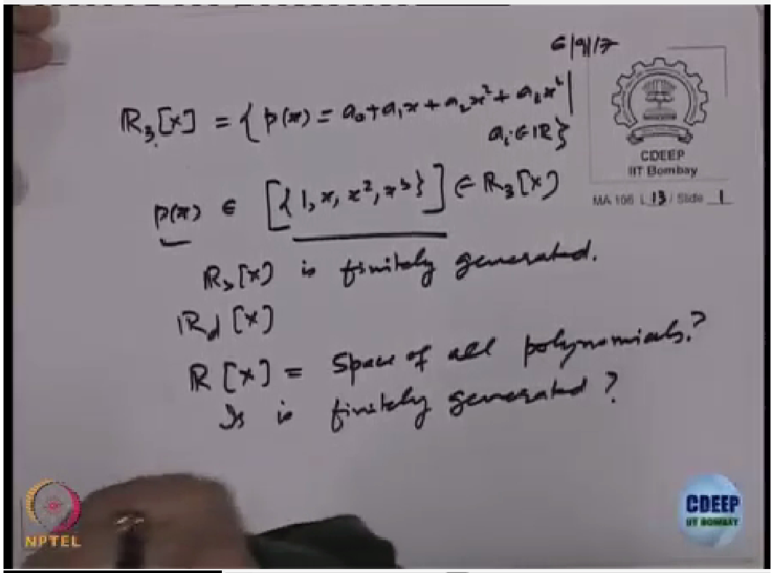

What are elements of it? They are polynomials px. And what is the polynomial look like? So that is a0+a1x+a2x square+a3x cube, where these ai's are, let us say we are taking real, so they are real entries. Is this okay? That is what polynomials look like. So clearly px is already a linear combination, of what? Constant 1, a0*1. a1*x, so if you take x as a polynomial, right, a polynomial degree 1, so this belongs to the set, okay, 1 x x, px belongs to the span of x x square x cube, right.

Because that is the way we write polynomials, right, okay. So px belongs to this. So that means what? Every element of, so that belongs to Rx. So every element is a linear combination of 1 2 3 and these 4 elements, right. So that means what? R3x is finitely generated. Is this okay? Finite number of them. If I take these 4 of them, every element is a linear combination of them. So this is finitely generated.

Same will be say c3 or any Rd of x. Why 3? I can take any degree, right, any degree d. This again finitely generated. What about Rx? That is space of all polynomials. Is it finitely dimensioned? It is finitely generated. Is it finitely generated? Yes? Can finite number of polynomials give you all polynomials.

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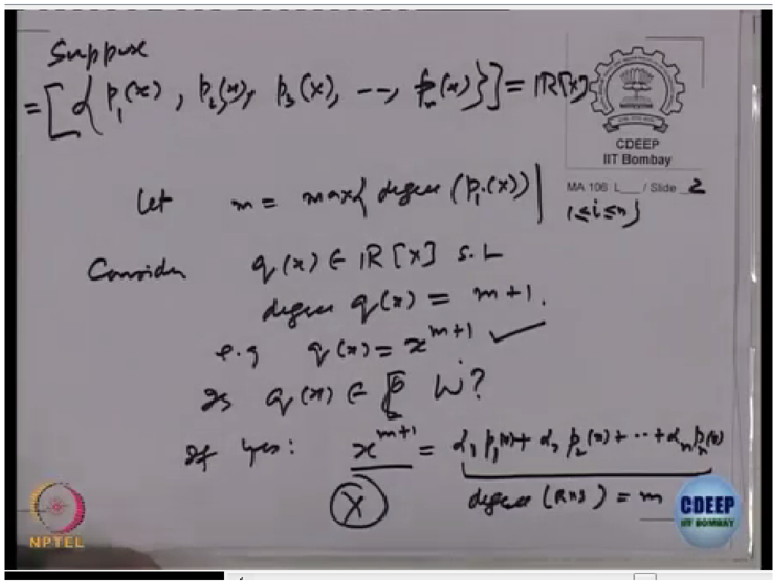

So let us take px. Let us say p1x p2x. So suppose p3x, some finite number of them. So pnx, they generate all. So let us assume finite number of polynomials generate everything, right. We are trying to test whether this statement is true or not. Let us say this is true, right. Now p1 has a degree, p2 has a degree, pn has a degree. So let us take m to be equal to the maximum of degree pi of x, i=1 to, you have taken n.

Take the maximum of these degrees. These are finite number on the right. This m will exist. So consider say qx a polynomial such that degree of, =m+1, right. Rx is all polynomials. So pickup any polynomial of degree. So for example I can take qx to be equal to x to the power m+1 itself, right. So the question is, is qx an element of, let us call this span of, this span will be something. So this is w.

The span of p1 p2 pn, is w, pn was w=Rx, right. So this qx belongs to, right, qx belongs to Rx, right. So implies, so question is, does this belong to w? If it belongs to Rx and if Rx is same as w, then it should belong to w, right. So if yes, that means what? That means x to the power m, say let us say, +1, if you have taken this 1, should be equal to some alpha 1p1, right. This should be a linear combination of this, +alpha 2, I am just spreading, okay, let me write x also, +alpha npnx, right.

That should be true? Now on this side, what is the degree of the right hand side? Is the polynomial. In the right hand side, what is the degree? m is the maximum, right. So degree of this right hand side=m. What is the degree of left hand side? m+1. So this is a contradiction, right. The polynomial degree m+1 cannot be equal to a polynomial of degree. See the basic idea is if you take finite number of elements in Rx, there will be maximum degree of all of them.

So what they will generate is? At the most, they can generate only polynomials of degree, maximum degree, that is m. They cannot generate something bigger than that. Clear? Because when you add linear combination of polynomials of degree less than or equal to m, will be again a polynomial of degree less than or equal to m.

It cannot be bigger than m. So no finite number of polynomials in Rx will generate the whole of Rx, right. That means Rx is not finitely related. Each Rnx, each Rdx is finitely generated. But when you take the union of them, that becomes a vector space which is not finitely generated. Is this okay? Clear? Actually you can write something in fact.

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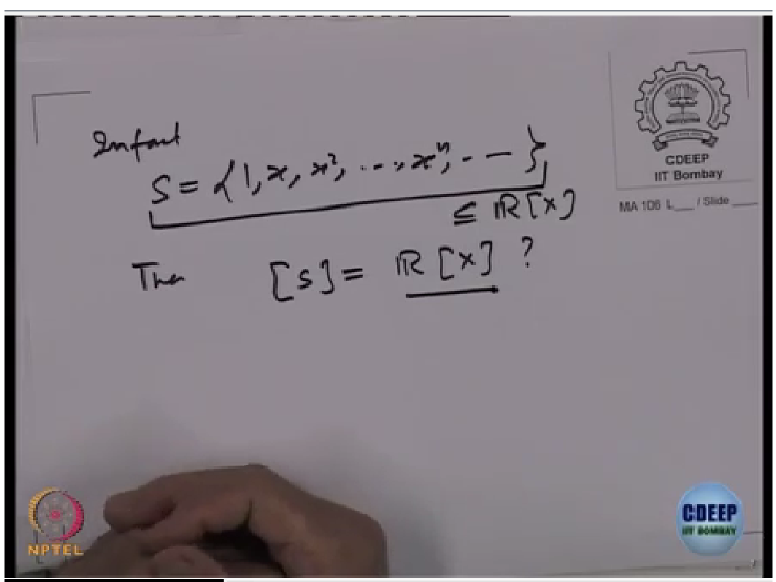

So let us write in in fact. If I write S which is 1 x x square xn. If I take this x which is a subset of Rx, it is not a finite set, this is a infinite set, right. Then the space span by S=Rx. They will generate Rx. If I stop somewhere, that will not. That is what just now we showed, right. Because if I stop somewhere, it mean power will be some m, right, higher degree will not be generated. But this will generate now. Why? Because if I take any polynomial in Rx, give polynomial of some degree, right, say n.

Then 1 up to xn will a linear combination will give me that polynomial. Is that okay? Right? So actually we have got a set of generators which is infinite. We have got a set of generators and no subset of that will generate it, that also is clear, right. No finite subset of Rx will generate Rx, okay. So and we have got a set of generators which is infinite and (()) (09:16). So Rx is a vector space which is non-finitely generated. You can call it if you like is an infinitely generated, right. So not finitely generated, this is one example.

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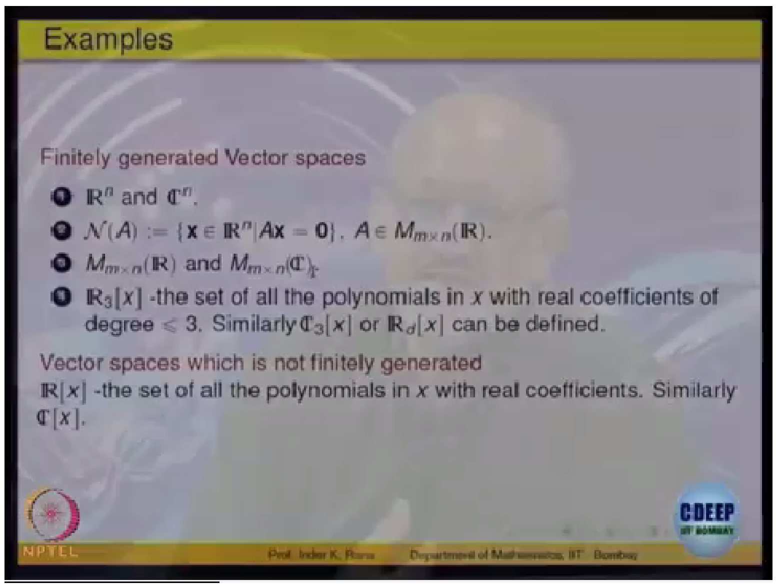

Can you think of some other example of a space, vector space which is not finitely generated? The idea should be, see if you take a vector space which is finitely generated, I will take a subspace of it, that also is going to be finitely generated, right. Now if you take a vector space which is infinitely generated and put it inside a bigger thing, that bigger cannot be finitely generated, right.

So let us think of polynomials as functions, right. Each polynomial px, you can think x varying in, x is a variable. You can think it varying in 0 to 1, right, interval 0 to 1. So here is something which you will find useful namely.

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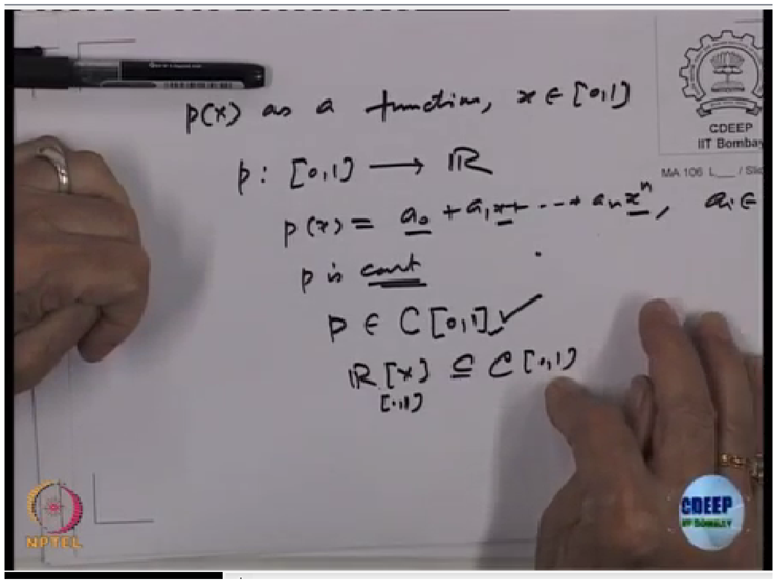

Let us think of px as a function or x belonging to 0 to 1, right. That means what? p is a function from 0,1 to R, right. So let us just look at, so what is p of x? Is a polynomial of some degree, right? So let us take, let us say it is a0+a1x+anx to the power n, where ai belong to R, right. So instead of taking px as a polynomial, a quantity which is less, a linear combination of this, right, you treat as a function?

So do you think this p is continuous? Now a bit of calculus is coming. Is this polynomial as a function on 0 1 to R, is it a continuous function? Yes, it is. Because constant is a continuous function. This is a continuous function, x to the power n is a continuous, sum of continuous functions is continuous. So this p which is a polynomial on with x varying in 0,1. I can treat it as a continuous function in 0,1. It is an element of C0,1.

Is set of all continuous functions, right. Now, this itself is a vector space. Given 2 functions on 0,1 taking real values, I can add them, right, fx+gx, that is f+g. Scalar multiple of a function, alpha*f is, defined that x is alpha*alpha of x. So this itself becomes a vector space. If I treat addition of functions as scalar multiple of all functions, as operations, right. So then polynomials, so then Rx sits inside C of 0,1.

Here I am taking this polynomials in 0,1 only. x varying in 0,1, right. So space of all continuous functions is again a vector space in which sits inside it a subspace of infinite generation, right. So

C0,1 cannot be finitely generated. So that is again infinitely generated, right. So that is another example. So what I am trying to say is that vector spaces which are not finitely generated are of importance. That is why one should be studying what are called abstract vector spaces, right. So let us look at some more, then what did we do?

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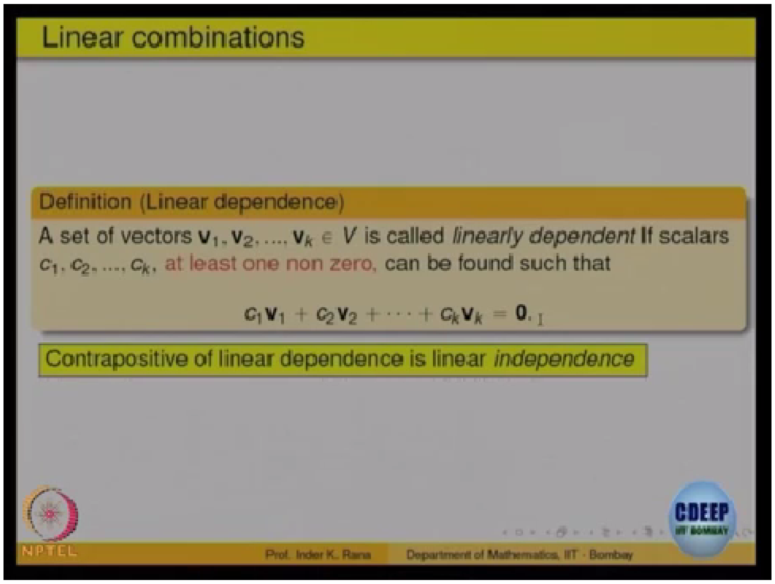

We looked at what is called linear independence and dependence. Same you can define, you can say a finite, given a set of vectors or finite set, you can say they are linearly dependent if a linear combination is equal to 0 where all the scalars are not equal to 0, that is one of them should be non-0. So there is a linear combination which is 0 with one of the scalars appearing non-0.

Then you say it is linearly dependence set, contrapositive of that something which is not dependent, it is independent, right. So what is linear independence? If a linear combination=0, all scalars must be equal to 0. Now here for a given set, it is not necessarily that set should be finite. So keep in mind. So what I am saying is the following. If S is a subset of a vector space v, S any set, of course non-empty.

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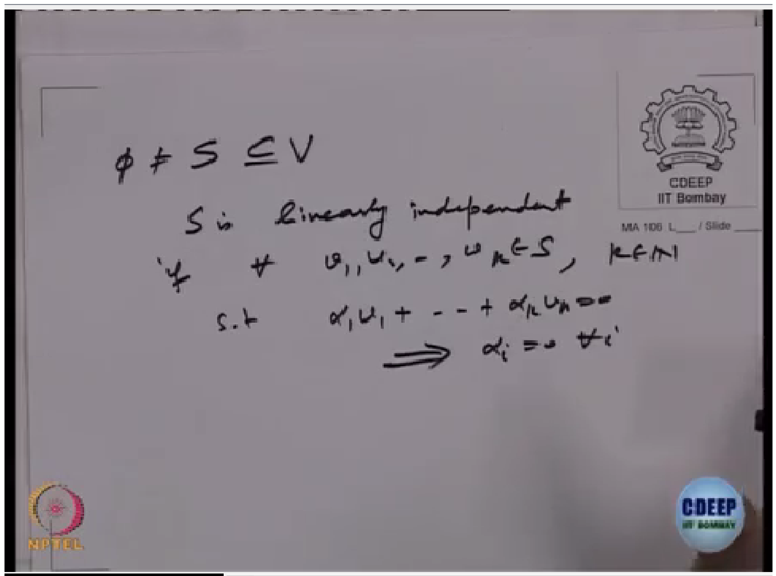

So S is linearly independent. This set is called linearly independent if for every v1, v2, vk belonging to S. K could be any number. You choose any finite number of elements of S, right, such that alpha 1v1+alpha kvk=0 should imply alpha=0 for every i, right. We are not saying S itself is a finite set. S could be any set.

Any set of vectors will be called linearly independent, is called linearly independent if, you pickup any finite number of elements of that set S, take a linear combination of the finite number, because linear combination is defined only for finite. If that is equal to 0, implies its scalar is 0, then that set S is called linearly independent. Is it clear? So let us, before going further, let us look at some example, okay.

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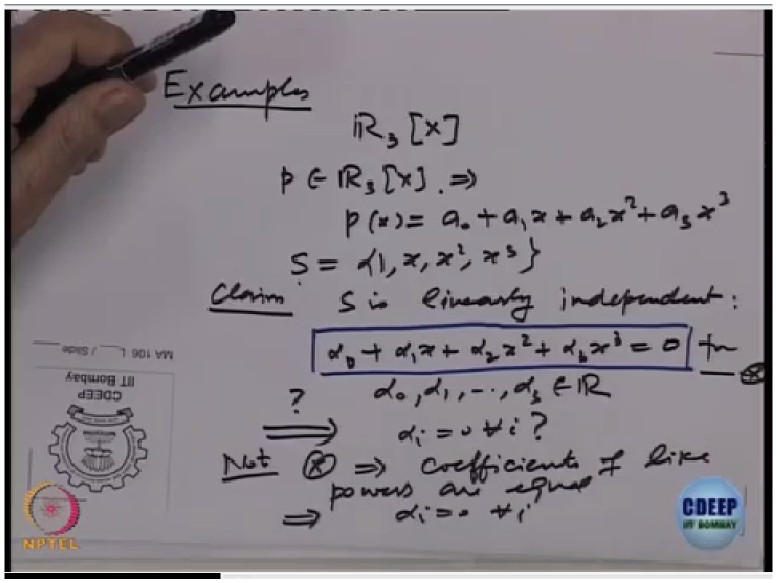

So let us come back to that friend which you have been giving us many examples, let us, right. So this is a vector space. What does px look like? p belonging to R3x implies p of x=a0+a2x square+a3x cube, right. That is what it looks like. Now, so let us write S=1,x, x square and x cube. So claim that means S is linearly independent. This is linearly independent set. So for that what were to check? Take any 1 of them, 2 of them, 3 of them, all of them, does not matter what you take.

So let us take all of them, right. Then cover everything. If alpha 0*1+alpha 1*x, that means what? Alpha 1x+alpha 2x square+alpha 3x cube=0 or alpha 0, alpha 1, alpha 3 belonging to R should imply alpha i=0 for every i, right. That should happen. Now if this equation is satisfied, okay, that means what? What is the meaning of this? Let us look at, that is what is given to us. What does this mean that this is equal to 0?

Left hand side is a polynomial, right. What is the right hand side? What is 0 on the right hand side. You cannot have a polynomial equal to 0 on one side a polynomial, other side a number. Equality of 2 same things, similar things make sense. So what is written here is which is what normally mentioned or understood is that 0 is a polynomial, 0 polynomial, right. So it is 0*+0*x+*xx square + whatever you want to write it.

So this right hand side is a 0 polynomial. If that is a 0 polynomial, that is equal to this polynomial. What does equality of polynomials mean? When to say 2 polynomials are equal? If the like power coefficients are equal, right. So this star, so note, * implies coefficients of like power are equal and that means that each alpha i=0 for every i. That is by definition itself of equality of polynomials, right. That itself implies this is 0. So this set 1, x, x square, x cube, right is a linearly independent set in the space of all polynomials of degree <= 3. Okay, clear?