Basic Linear Algebra
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Lecture - 40
Inner Product Spaces - I

So last lecture we had seen how abstract vector spaces can be defined and now will look at the notion of perpendicularity and the inner product on the abstract vector spaces.

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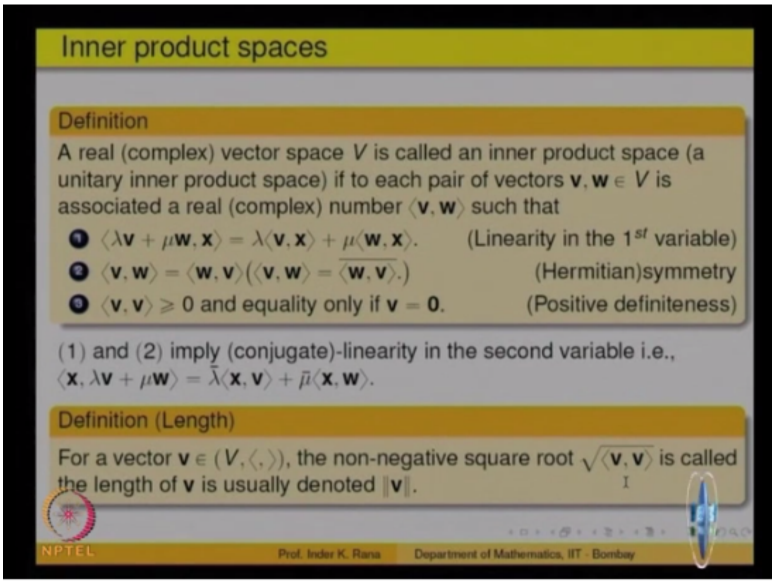

So basically the idea is that the properties of the inner product the dot product in Rn are taken over as definition of an inner product. So given any vector space which is either real or complex depending on whether reals or complex, for every pair of vectors v and w, we associate a number which is real in case it is over the reals, is the complex number if it is over the complex and the number is denoted by that square bracket this corner bracket v, w.

So it is a kind of a function right on v x v2 the underlying field. So what are the properties of this? Basically, that of inner product so it is linear in the first variable, so if we have a linear combination lambda v+mu w inner product with x is same as lambda times inner product of vx+mu times inner product of w, x. So in the first variable, it is linear right. Scalars come out and addition becomes image of the addition is addition of the images.

So that is linearity in the first variable and it says that if you interchange the two, so inner product of v and w is same as inner product of w and v if it is over the real numbers right if it is a real number. If not, then it will be a complex number, so then it is the bar of that okay. So inner product of v, w, so in general you can write it as v, w bar depending on real there is no conjugation, so that will be same as this.

And lastly third property namely that inner product of v with v it is bigger than 0 and 0 only if and only if v is=0, that is the dot product of a dot a is right in Rn so that property. So these 3 properties of the dot product I have taken of Rn and put an abstract definition of the inner product on abstract vector space. So such a space will called as inner product space, so it is a vector pace either over real or complex.

And for every pair of vectors that is the notion of a inner product or dot product you can call it with these 3 properties. Now this property, we said it is linear in the first variable right. What about the second variable right? So because of this reason, so if you have something on the second variable coming out such as scalar multiple here, then it will be equal to lambda w, v bar.

So that lambda in the first variable will come out as lambda bar right. So in the second variable, it is conjugate linear if it is over complex otherwise linear okay. So consequence of the first property and the second property that is conjugate linear in the second variable. So x lambda v+mu w, so addition remains addition but the scalar from the second variable comes out as lambda bar and mu bar right.

So that is because of this property 2. So once we have this property v dot v is bigger than 0, we can define the notion of length, magnitude of the vector. So magnitude of a vector is nothing but square root of the inner product of v with v right. Norm a square is a dot a right, so same property holds here because of this. So we define, so this gives a definition of the magnitude of a vector in an inner product space but to be equal to square root of inner product of v with v.

So basically the properties which were true for inner product in Rn are taken as definitions in an inner product space, nothing more than that okay.

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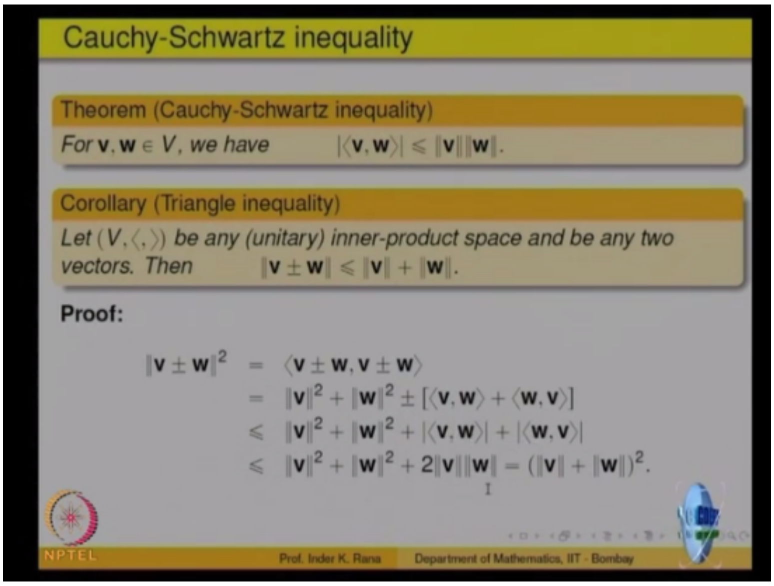

We have Cauchy-Schwartz inequality for the Rn. Same property is true for any inner product space, so will not give a proof of this so will assume it. So inner product of v, w absolute value is a real or complex number, it is absolute value is less than or equal to norm of right a dot b in the real line absolute value is less than or equal to norm a norm b right magnitude same property is true.

So that is what we call as Cauchy-Schwartz inequality and using Cauchy-Schwartz inequality one can prove the triangle inequality that the length of v+w or v-w is less than or equal to this right. That proof is simple and straightforward because if I look at the square of this left hand side okay, so I can write it as inner product of the same vector by itself and now explained using your properties of the inner product.

So v and v will give you norm v square, w and w will give you norm w square+-cross product terms right and now this is less than or equal to norm v+norm v, so put that property here okay. So it is less than or equal to each one of them is less than norm v norm w right. So this is less than norm v norm w, this is less than norm v norm w, so two times and this is the square of that.

So nothing more than that using Cauchy-Schwartz inequality once right and using that v+w square is dot product of it with itself inner product, expand and use here Cauchy-Schwartz inequality that each term here is less than or equal to this, so that proves triangle inequality. Once you have the notion of dot product, you can define the notion of perpendicularity right in inner product space.

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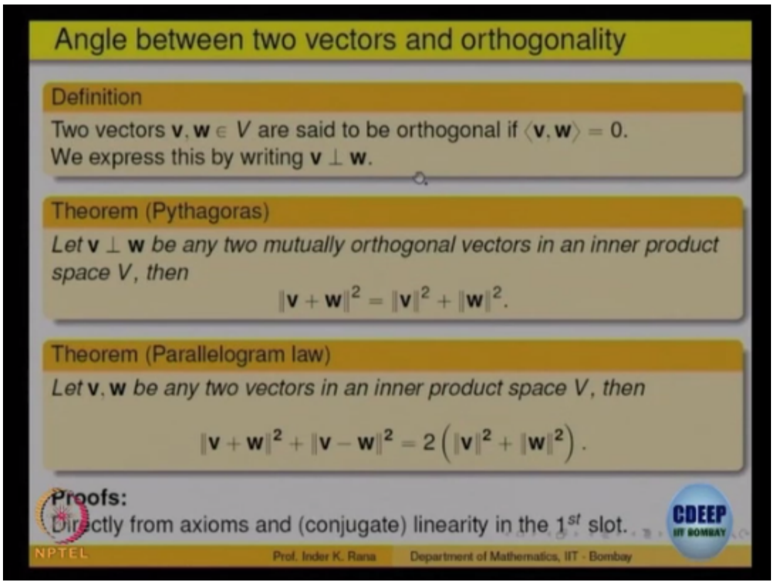

So two vectors are said to be orthogonal if their inner product is=0 right. Once the perpendicularity is defined, we can prove the standard Pythagoras theorem right. Norm of v+w square is norm of v square+w square if v and w are perpendicular to each other and the proof is actually straightforward, you write this norm square, so expand it as dot product of this with itself, open it out and because of perpendicularity the cross product terms will be 0 right.

So norm of v, w and w, v will be 0. So you will get only this much. So that is standard proof and similarly there is Parallelogram law which is again same, write both of these as dot products, expand and you will get that, term will cancel out and you will get the thing. So proofs are straightforward, what you do, write norm square as dot product of that vector with itself and expand right, so nothing more than that.

So Pythagoras theorem is true, Parallelogram law is true for abstract inner product spaces once we have the notion of the perpendicularity right. Same proof so that nothing, no change comes okay.

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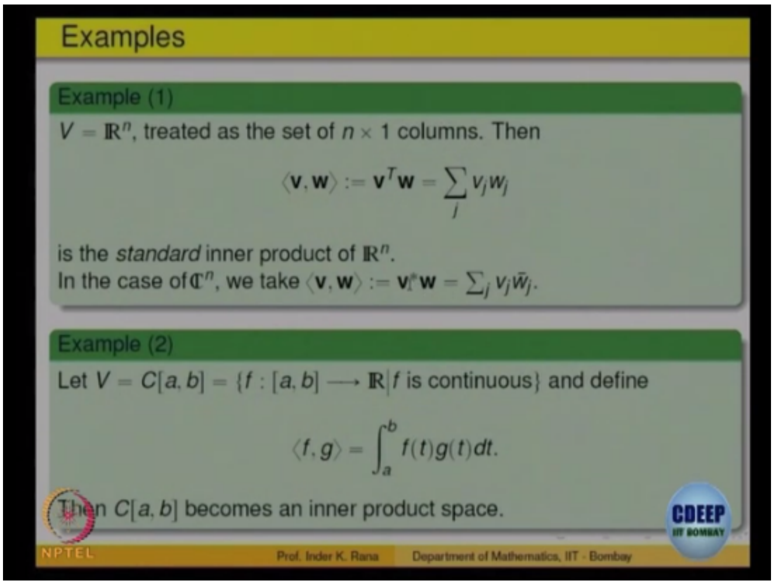

So some examples. This is a prototype of inner product space in Rn right. If you write v as a column vector then v transpose w, this is the product of matrices, so dot product is ai bi right, v is a vector with components v1 v2 vn, w is a vector with components w1 w2 wn. So what is this? vj wj right. The in between term is if you want to write it as in terms of matrices if you write a vector in Rn as a column vector, then this transpose will be a row vector 1 x n, 1 x n n x 1 product will be a scalar right.

So this is in terms of matrices and for complex the only difference comes is you put a bar here. So vj wj bar. We had seen that in when we did. Now let us look at example of an inner product space which is different from all these. Why this is required? So that is this example. Look at the space of all continuous functions C a, b. So they are real valued, continuous functions defined on an interval a, b to R right.

And f is given to be continuous, so all continuous real valued functions defined on the interval a, b. You can add them right, any two functions can be added f+g defined at a point axis fx+gx yeah **"Professor - student conversation starts."** No, actually this is a typo here. This should have been star there actually. Yeah, so forget the middle term, actually there is a typo. This star should have been here actually in w okay.

So that is why this is wj bar coming here. Yeah, v, w star okay right **"Professor - student conversation ends."** So on C a, b so let us just look at slightly more okay.

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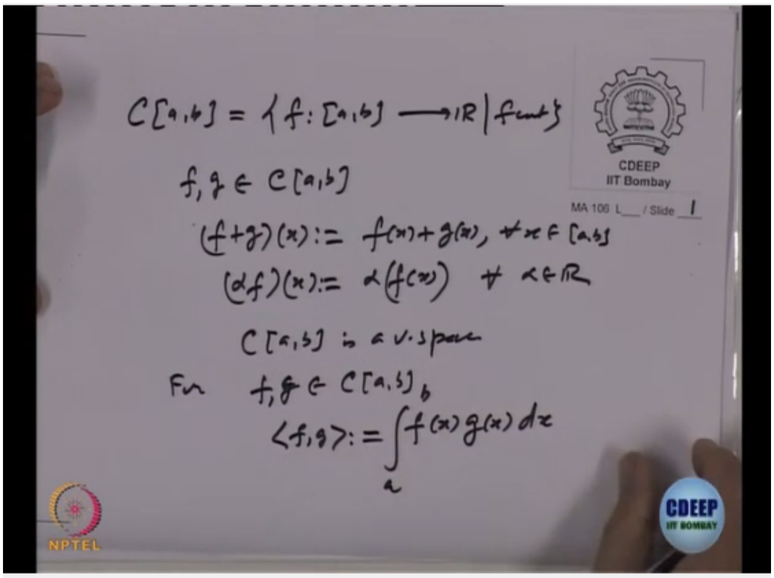

So we have got C a, b so that is space of all functions which I defined on a, b to R f continuous right. So for two functions f and g belonging to C a, b, if you recall so this is a definition of a sum, so that is fx+gx for every x belonging to a, b right and what is alpha times f, so definition is alpha times f of x for every alpha belonging to R. So then C a, b under this addition and scalar multiplication C a, b is a vector space.

It belongs to vector space because addition of function is associative, commutative right, zero function is there, so all those properties hold. So it becomes a vector space, so this is addition and scalar multiplication, make it a vector space. Now on this vector space, we are defining an inner product so for two functions f and g belonging to C a, b continuous functions. Look at the function fx the product right.

They are integral from over a to b dx, this integral is defined. If f is integrable, g is integrable then calculus tells that f+g also is integrable. So this number is defined and this number we denote it as an inner product right. So there is a definition of dot product for the vector space C a, b, want to check this has all the properties, the dot product has because if I write f+g right, so checking that this is a vector space.

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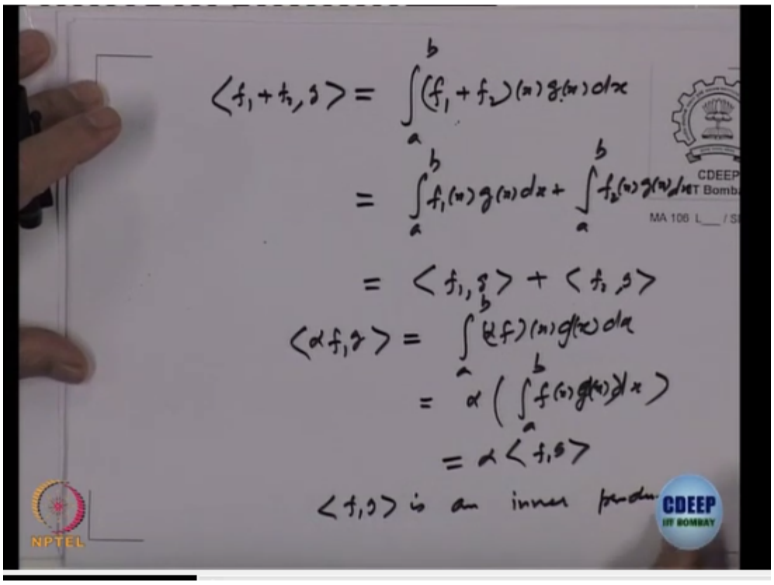

So let us look at f1+f2, g so what will be that, so that is integral of f1+f2 x gx dx. So that by the properties of integration right, the integration is a linear process so this is nothing but a to b f1 x gx+a to b f2x gx dx right. So we are using the property that integral of sum is equal to sum of the integrals and that is equal to integral of f1, g. So that is a dot product of f2 of g right, so linearity.

And same way if you put alpha here so alpha f of g so by definition it will be alpha f. Again property of integration, integration is a linear operation, so this alpha comes out. So that is a to b fx dx. So that is alpha sorry there is a g here so gx dx. So that is alpha times f of g right. So that says that this is an inner product. We are not checking the other property, so what is the second property we had, f, f should be>or=0, so let us check that also.

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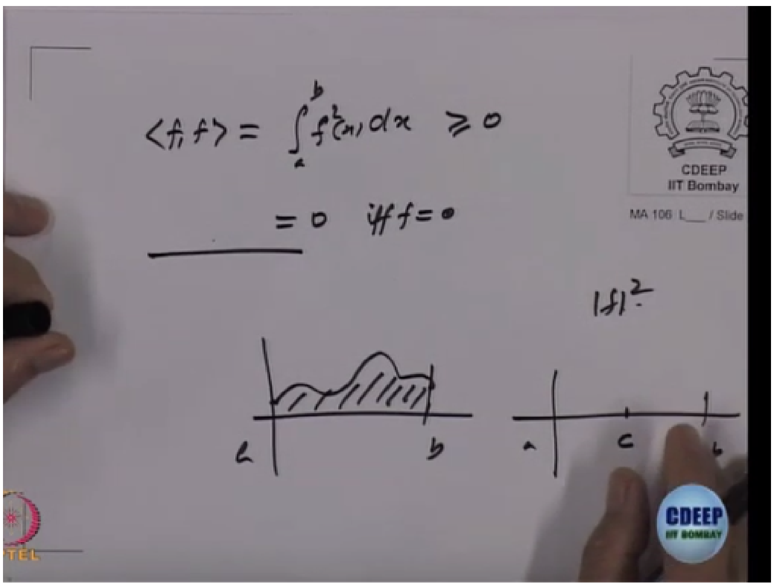

That is a straightforward again. So if I look at f, f that is=integral f is same as g so it is f square x dx a to b right, f square is a non-negative function, so this will be always bigger than or equal to 0, integral of a non-negative function is non-negative and it is equal to 0, when it is equal to 0?

When the function is identically 0 f square=0. So if and only if f is=0 right. Integral of the continuous function non-negative continuous function is 0 if and only if the function is 0 right. So f square=0 so that means f=0 right okay. So that property, so that is the inner product. So that is what we are saying that this becomes an inner product space, f square=0 means what? It is mod f square.

Now is it clear? Yes, f square I can write as mod f square. Is it okay? So if mod f should be equal to 0, the mod f of sin that is not equal to 0. So we will have a problem there okay. See that calculates the theorem is quite nice actually. If it is a and this is b right, if the function is continuous and non-negative right, so the graph will be above the x axis right. So that means what somewhere it starts non-negative and it has to where is the graph on function right.

So if it is continuous function, then the area below the graph is=0. If it is a continuous function and it is integral is 0 that means what? The function is non-negative and you take its integral right then what is the integral and it is area below the graph of the function geometrically right. If it is non-negative, then it will be bigger than okay. So this and if this area is 0 that means what?

Our claim is if this area is 0, function is non-negative; the function has to be 0 everywhere right. So let us see in the picture why is that happening? This is a to b, suppose at some point c the function is 0, then what will happen? There is a continuous function then what should happen? So go back and look at the calculus thing okay. For a non-negative function if the integral is 0 and the function is continuous that is important.

See if the function is not continuous then you can give many kinds of examples.

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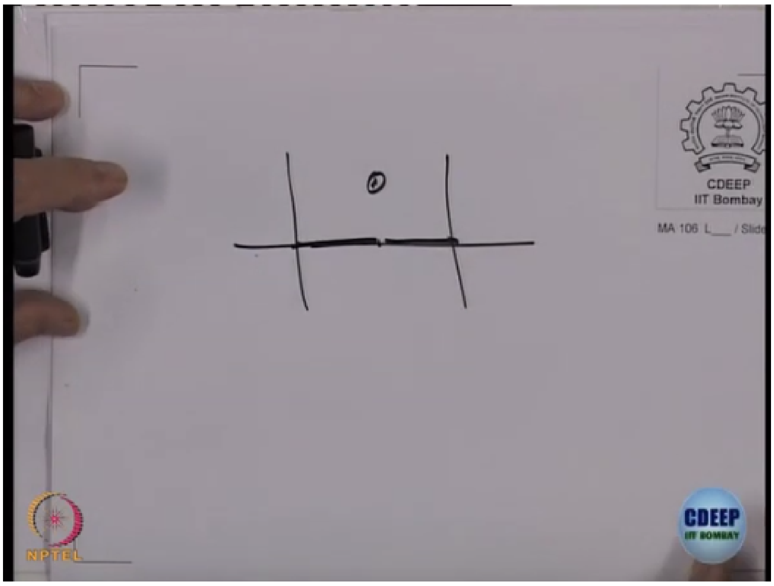

So example if the function is not continuous, I can give value 0 here and 0 here and here I can give the value 1 right. Then, what is the integral of this function? That is 0 right, whatever the function maybe, whatever the value is 0 here, 0 here and it is only 1. So integral is 0 but the function is not 0 everywhere because there is a discontinuity. For a continuous function that cannot happen okay.

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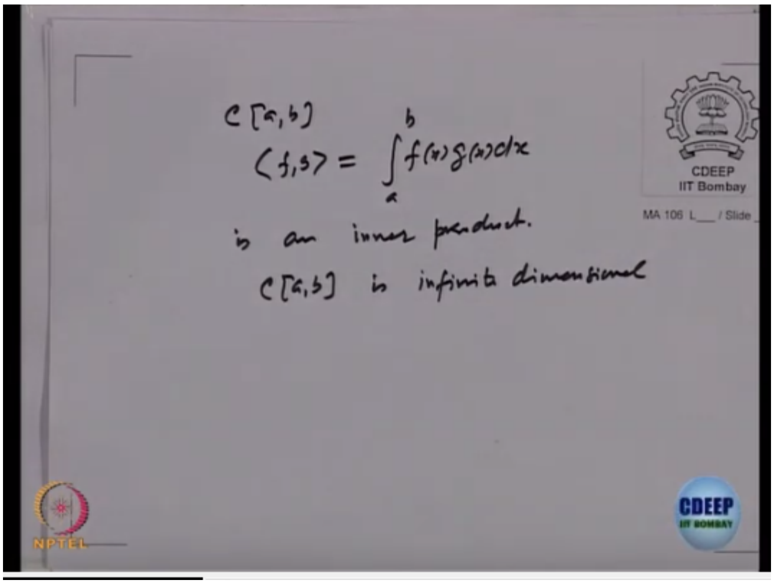

So this C a, b so what we are saying is this space C a, b with f, g=integral fx gx dx a to b is an inner product right okay. You can just for the sake of more understanding this is a vector space right over the reals, so and we have shown that it is infinite dimensional right. This was example of a vector space which is infinite dimensional. So you got an infinite dimensional vector space on which a notion of inner product is also defined right, so keep that much in mind okay.