Basic Linear Algebra
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology- Bombay

Lecture – 08
Reduced Row Echelon Form and Rank II

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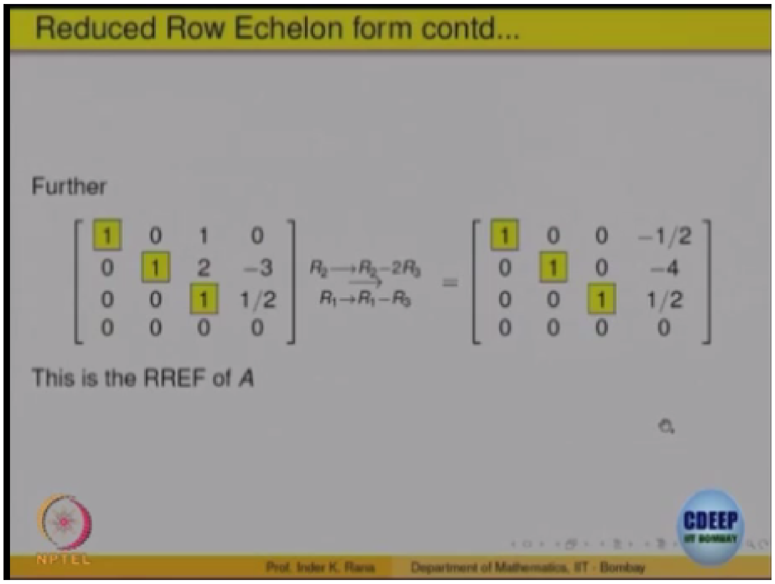

How to get row Echelon form, how to get reduced echelon form, right and we said that these row operations can be achieved by elementary matrices, pick up a suitable order elementary matrix and pre multiply A with that matrix, we will get that operation, okay, right.

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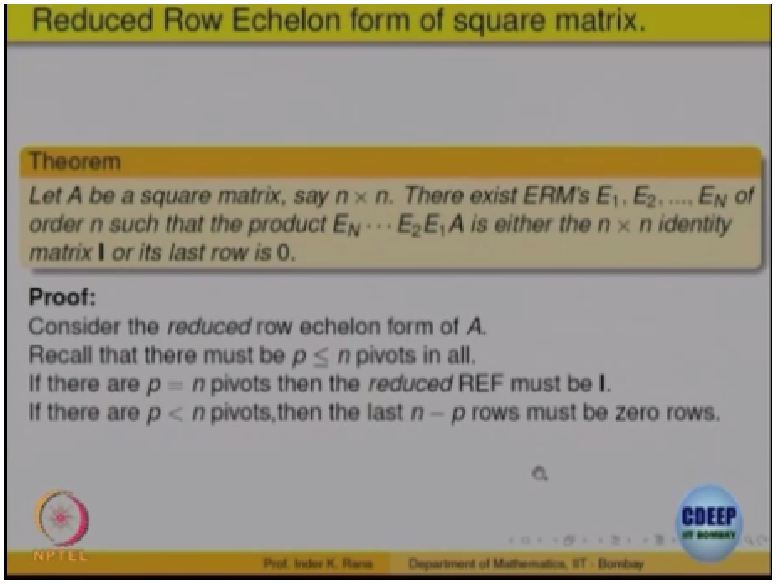

So, we can write another theorem that, let us specialises this to a square matrix, suppose you have got a square matrix n cross n, okay and you look at its reduced low Echelon form, what are the possibilities; a number of columns = number of rows, so one possibility is the number of pivots that r = n, right that means, every column will have a pivot, right; 2 possibilities, for a n cross n matrix, if you look at the reduced low echelon form of that matrix, then what is a possibility?

The number of pivots = number of rows = number of columns that means each row and each column will have a pivot, right and every other entry in that pivot column is 0, so what does that matrix look like; that is precisely an identity matrix, so one possibility for a square matrix, when it is reduced to row echelon form; reduced row echelon form, when it is transformed to reduced roe echelon form that form is identity matrix that is 1.

Or, what is the possibility; number of pivots is < n, right, one was = n, other is < n that means what; that means at least one row at the bottom must be 0, it cannot be more than, it cannot be = n, it can be at the most n -1, so there are only n -1 pivots that means, at least one bottom row should be = 0, right, so there are 2 possibilities for a square matrix in the reduced echelon form either the reduced echelon form is identity or at least at one row at the bottom which is all 0, right, okay.

The proof is either obvious because once we do those operations, right, it only depends on the number of; so if the number of pivots is let p < n, then n – p rows at the bottom will be all 0, right and you can see that on that portion of the matrix where pivots coming that will be identity that portion will be identity matrix, okay, right, p cross p on the left side will be identity.

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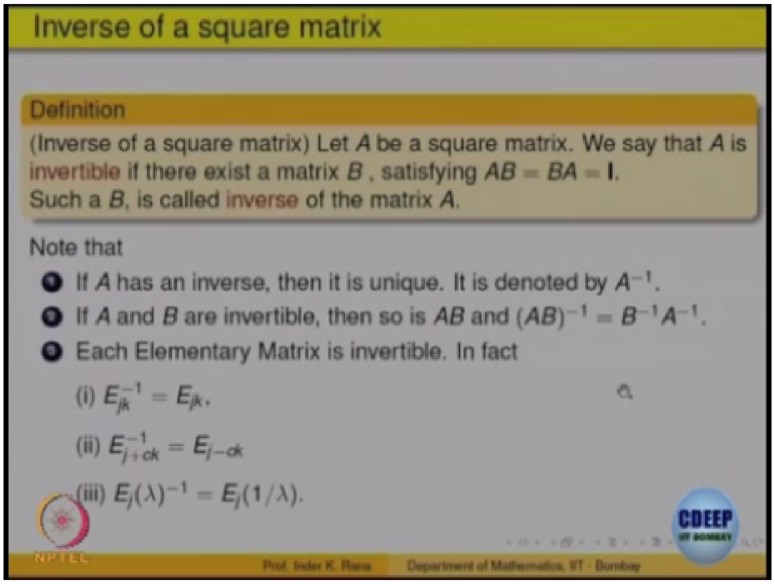

So, let us, now how is that useful; we are going to give you an application of it, so let us start defining what is called the invertibility of a matrix, you all must be knowing that already. Let; we have; invertibility is defined only for square matrices, it is something to do with multiplication like for number; every number which is non-zero, right say alpha has a multiplicative inverse 1 over alpha; product gives you 1, right.

So, in matrices for addition, we already have identify, right, if m cross n matrix A, you add a matrix m cross n which is all 0's that do not change the matrix, right, so that is additive identity for the matrices of order m cross n. When you want to multiply, so you can multiply on the left as well as on the right, so for example, for number 1 over alpha * alpha is same as alpha * 1 over alpha, multiplication is commutative.

But in matrix multiplication is not commutative, A * B need not be = B * A, first of all A * B may be defined and B * A may not be defined for a rectangular matrices, even if it is defined they may not be equal, right, so to say it is something like inverse exist, we have to put a condition that A is a square matrix of order; any order, we say A is invertible; if there is a matrix B, say A * B and B * A, both are defined.

They will be defined if B is also of the same order as the matrix A, right are defined and equal to identity matrix, A * B multiplied, that means multiplying A on the right side by B or multiplying A on the left side by B, both products should be equal and equal to identity matrix. In that case, we will say the matrix, this matrix A is invertible, right and this matrix B, we call as universe of A, B is called A universe, I am saying A inverse.

But one show very easily that if one such exist, it has to be only, it is unique, that means if there exist a matrix B with this property, right then there is only one, there cannot be some other matrix also, so if A universe exists, it is unique, so you call it the universe of the matrix. So, this I will leave it for you to verify, right that it is unique that means, if A * B = B * A, also = A * C and C * A = identity, then B must be = C.

Next, if A and B are invertible, so these are the properties of invertible matrices, if A and B are invertible matrices and you are able to multiply them, right that means what; they are of the same order, then A * B is also invaluable, product of invertible matrices is again invertible and the inverse of the product is product of the inverses. So, AB inverse is same as order interchanged; B inverse A inverse, so that helps you to compute inverses of products for invertible matrices.

So, these also you should verify, now let us look at elementary matrices, these 3 elementary matrices, what all the elementary matrices; we took an identity matrix, right, example; we interchanged ith row with the jth row, we got an elementary matrix. Is that matrix invertible that you want that you get, is that invertible? Intuitively, it should be because I can reverse that operation, right and what is that operation?

Multiplying on the left by some matrix, right, so I invert that; if I add it, I have interchanged, I can again interchanged, what I will get back? Identity, right so, elementary matrices; they are all reversible operations, right, interchanging ith with j is same as reverse, j with I, adding one jth row to i right, subtracting back, okay, multiplying by a scalar non zero that is why we said non zero.

The 2 reasons; 1; in an equation that equation may becomes 0, data will be lost, another thing now comes there, if you have multiplying a row by a nonzero scalar to get that elementary matrix, then you can multiply by 1 over of that we will get back the identity matrix, right, so all these elementary; 3 elementary matrices; 3 types of elementary matrices are invertible matrices, they are special matrices, they are all invertible, right.

So, for example, we have written there, if you take Ejk and what is the inverse of that? Right, so inverse of that is; it should be kj here, there is a typo, it should be kj, interchanged again kj. So, for example Ej lambda that means, you are multiplying jth row by lambda, so divide jth row by lambda that is I inverse, so inverse operations, so all elementary matrices are invertible matrices and keep in mind what is our definition of invertibility?

It is says there is a matrix B, so that A * B is same as B * A = identity, later on we will see other definitions of invertibility also, okay but we are starting with a definition that like a numbers, right for every nonzero number, alpha, there is an inverse 1 over alpha, multiply alpha times 1 over alpha = 1 over alpha times alpha = 1. Similarly, so a matrices, if given a matrix A, if there is a matrix B, so that AB = BA is identity, then we say A is invertible.

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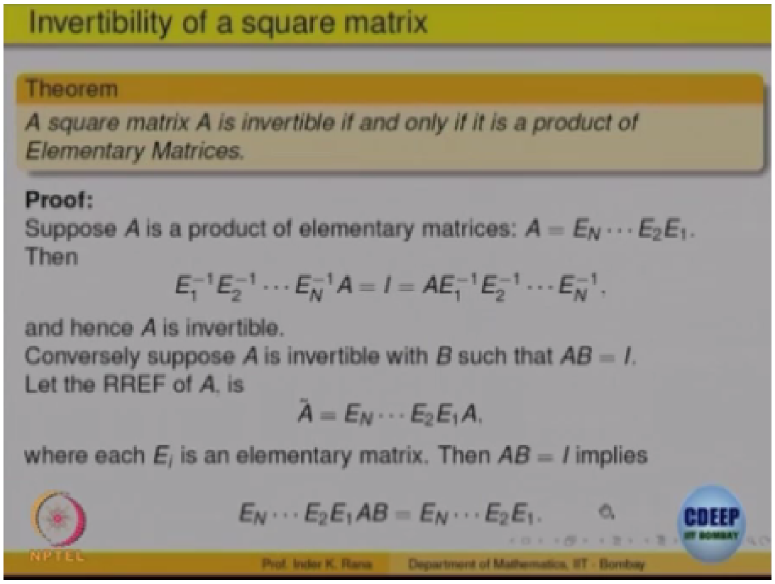

And B is called the inverse of A, it is unique, clear, okay, so now let us look at; now start relating invertibility with elementary matrices. So, theorem says a square matrix A is invertible if and only if, it is a necessary and sufficient condition says it is a product of elementary matrices, if a matrix can be split at a product of elementary matrices but obviously, each elementary matrices is invertible, right, product of invertible is invertible.

So one way is obvious, if a matrix is the product of invertible matrices, it is invertible that is of obvious, right because if a matrix A is a product of invertible, right, each invertible; each is; if A is a product of elementary matrices, each elementary matrices invertible, product of invertible is invertible, so A is invertible, so one way is obvious, so let us look at that proof. So, suppose, A is a product of elementary matrices, A is; this is the product, then what is the inverse?

So, I can multiply, okay both will be = identity, okay, because if this is A, I multiply on the left by EN inverse, this EN will be gone, I bring EN -1 inverse on the left side, what will be left is; identity, so that is equal to this and similarly, the other one, okay on the left side or right side, you can go on bringing inverse 1 by 1 because they are all invertible matrices, so that says, if A is the product of these elementary matrices, then what is its inverse?

It is precisely this matrix, you can see that right, not only it tells you what; it is invertible, it even tells you what is an inverse, for why did; you can write the elementary matrix as; you can write the matrix A as the product of elementary matrices, so it gives 2 things, if A is a product of elementary matrices, then this equation not only tells you that A is invertible because A * this matrix, you can call this matrix as B, right.

So, B * A is same as A * B = identity, so what should be this matrix? That should be the inverse, right, so that should be the inverse of A, it tells you how to compute the inverse provided, you can write down the matrix as product of elementary matrices, right so one way is obvious. Let us look at the other way round that means, what is the other way round conversely; suppose, A is invertible, what we want to prove?

That A is a product of elementary matrices, what is the definition of invertibility? It says A is invertible, if there is a matrix B, so that A * B = B * A = identity, so okay. Now, let us take this matrix A, which is given to us and look at its reduced row echelon form, how do you get that reduced row echelon form; by pre-multiplying by some elementary matrices, so we have to noted this A tilde to be the reduced row echelon form of the matrix A, which is obtained from A by pre-multiplying by some elementary row operations, okay.

So, each Ei is an elementary row operation, now when AB = identity, what does this give you; AB is identity, so I multiply by some matrix, I will get the same matrix, right, so I will get this equation that A is okay, these are the elementary row operations that we have been right doing to get A tilde, so those row operations multiplied by AB = those row; product of elementary, is that okay by using the fact A * B is identity.

Now, let us, so now what we will do is; we look at the left hand side and the right hand side and look at their row echelon forms, okay, so let us look at.

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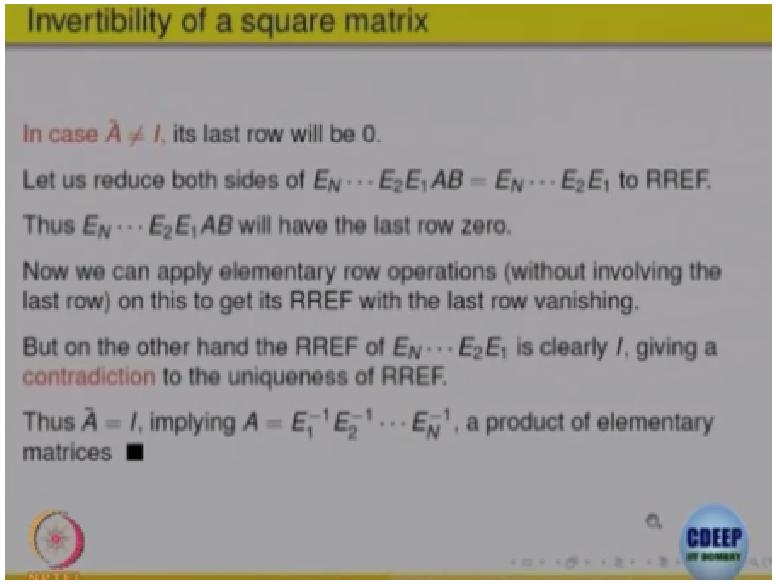

There are 2 possibilities that the row echelon form of A != identity, that is one possibility that will mean what; that means, at least the bottom row = 0, right there are not as many pivots as the order, right, there is one row, which should be 0, right, so bottom row = 0, so let us reduce both sides to row; reduced row echelon form, what will happen to reduced row echelon form of this product?

Last row is of this matrix EN, E1 that is A tilde right that is = 0, so that will stay as 0, right, if a matrices got a bottom row 0 and the reduced row echelon form that 0 row cannot become non zero suddenly, right because nonzero rows has to be at the top, so will have the last row as 0, because last row was 0 in this left side is that okay, this product EN, E2E1 * AB, the first this part of the matrix that is A tilde that has a zero row, when it reduced this product that will stay as 0 row.

Now, look at the right hand side, EN, E2 E1 that is the right hand side of the same thing, what will be the reduced row echelon form of that, can it have a 0 row, can that have a 0 row or not; it cannot, why, what is the reason? because if this is the matrix, EN E2 to the product, I want to reduced it to row echelon form, so I multiply on the left by EN inverse, N is invertible, right, I can; EN inverse; EN- inverse, E1 inverse, what I will get?

I will get an identity matrix and that should be reduced row echelon form, I am only multiplied on the left, so the reduced row echelon form of the right hand side is identity, on the left hand side, bottom row is 0, so you get a contradiction that means, this cannot happen, the reduced row echelon form of A cannot be !=, so that means, it is = identity matrix that means what, okay so if A tilde is, so then implying what is A = then?

If the reduced row echelon form of matrix is identity, that means what; A pre multiplied by E1, E2, EN = identity, right and what is A = then; that is a; on the right; take it on the right hand side, so A must be = E1 inverse, E2 inverse, EN inverse, clear but if E1 E2 EN are elementary, their inverse is also elementary matrices, so here is the fact which we should observed earlier. If you take an elementary matrix, which is invertible, example, interchanged I with j.

What is an inverse; j with i that is again the elementary matrix, again a same, right, row operation only, so all these are invertible matrices, their all elementary matrices, so A is the product of elementary matrices, so we have proved a theorem, so namely that if A square matrix n cross n is invertible if and only if, it is a product of elementary matrices, so this gives us a method of not only saying something actually proving also, okay.

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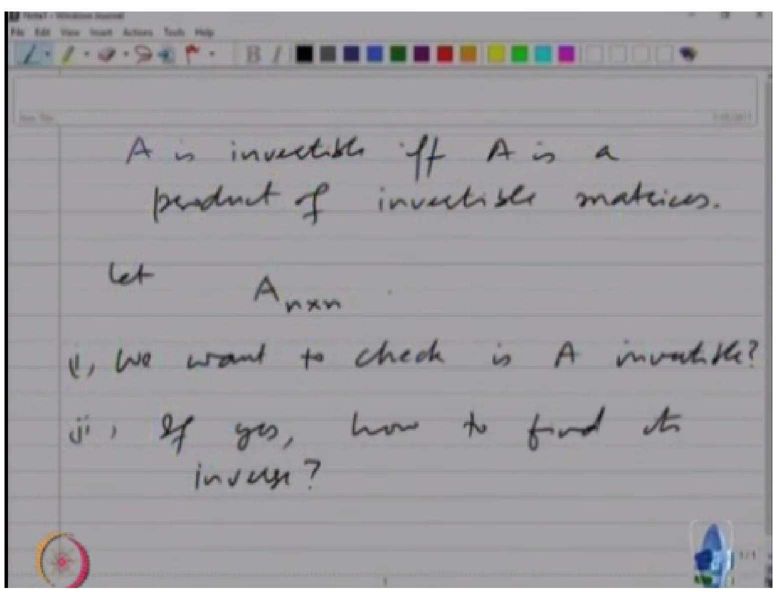

So, let us go ahead and look at some more examples, so we are going to look at, so we proved that A is invertible, if and only if A is a product of invertible matrices, right, so let us start with some matrix, A which is n cross n, so what want to do; we want to check, is A invertible, that is one question we want to answer and second; if yes, how to find its inverse? Right, so, a given a matrix A, we want to check whether it is invertible or not, right.

And whether we can find out the inverse of that all, so what should be the strategy; it says A is invertible if and only if it is the product of invertible matrices, right and in that case, its row echelon form we saw it should be = identity, right, in that case, row echelon form of A should be = identity matrix, so the idea should be take the matrix A and find its reduced row echelon form, if that is the identity, it will be invertible that is one, right.

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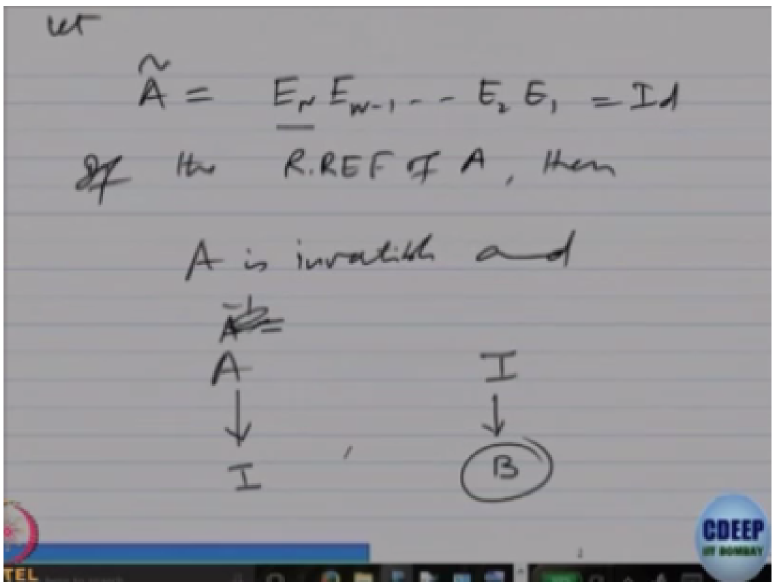

Now, I also want to compute A inverse, right, so how will I compute A inverse? It just said that if you take a matrix A, right, if so; if A is the matrix, right and A tilde is EN, EN – 1 E2 and E1, so let, I am just revising again what we proved in the theorem, if this is so, right is the reduced row echelon form of A that should be = identity, right, so let if that is okay, then what should happen A is invertible that we already said.

And right, this is identity, so what is A inverse = then; we just now saw; A inverse = inverse of this multiplied together, right that means, how do I get that? So, I want EN inverse, so let us apply that see, if I apply the operations again, I will get back the A, right, so this is I want take it to the other side, I will multiplied by EN inverse, right that means, what I should be doing; I should be keeping A and identity on side by side, go on doing operations; row operations on; go on doing row operations on A.

Same row operations you keep on doing on identity matrix, right so this will change, A will change to identity and this will change to some matrix, B, then B is the inverse of A that is what it tells us, right, clear, so let us try to do one example of this.

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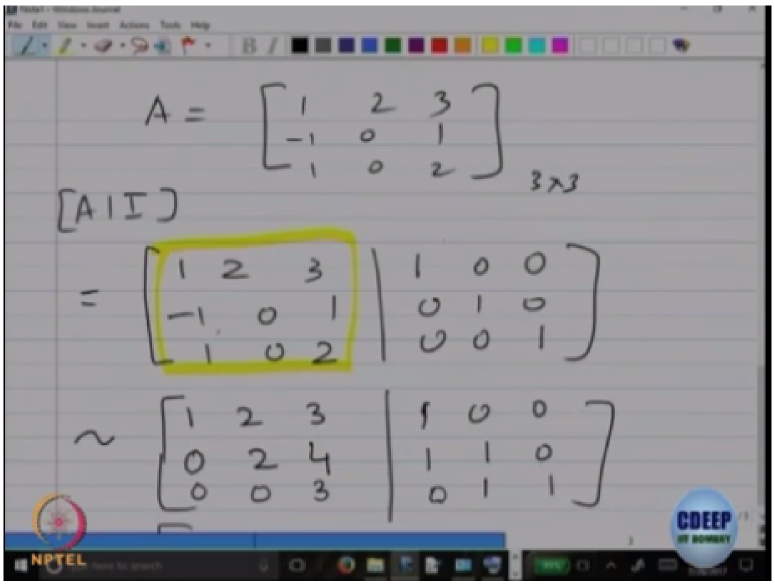

So, let us take the matrix A, I am just writing randomly some example, 1 2 3, -1 0 1, 1 0 2, let us write this matrix, okay, it has to be a square matrix, if I want to analyse its whether it is invertible or not, so it is 3 cross 3, this is the matrix which is 3 cross 3, I want to check whether it is invertible or not and can I find its inverse, right, so what we do is; because the same operations I will like to do it on identity also.

So, we write a new matrix which is A along with identity, so we write 1 2 3, -1 0 1, 1 0 2 and what is the identity matrix of the same order? 1 0 0, 0 1, 0 0 1, right, so that is this matrix, what my aim; look at the part which is in A, okay, so this is the part which is in A, okay on that I should do row operations and reduce it to reduced row echelon form but same operation I should be keep doing on the identity also on the bigger matrix also, so that I keep track of what is happening.

So, let us do that so this is equivalent to; let us do those operations, okay, so 1 2 and 3, 1 0 0, anything required in the first row was entry itself is 1, pivot is 1, so we are happy, so below that everything should be 0, so I should add r1 to r2, so this become 0, this will become 2 and this will become 4, right, so add, this will become 1, this will be 0, this will be 0, ah that is 1, right, so you should change it to that is 1 and that is 0.

Last one as it is for not anything to know, I want to make the last one also 0, so what I can do; either I can add these 2; r2 and r3 or –r1 with r3, I can add that whichever I like, so let us add because minus is 0, this is r2 to r3, 0, no sorry, so add 0 1 and 1, is okay, now what should be the next step, I have gotten the first pivot in column 1, everything below that 0, go to the next column now, I should be looking at that part of the matrix, right.

I should be looking at only which part of the matrix now; only this part of the matrix, right, non-zero entry is 2, I should make that entry as 1, okay, let us do that.

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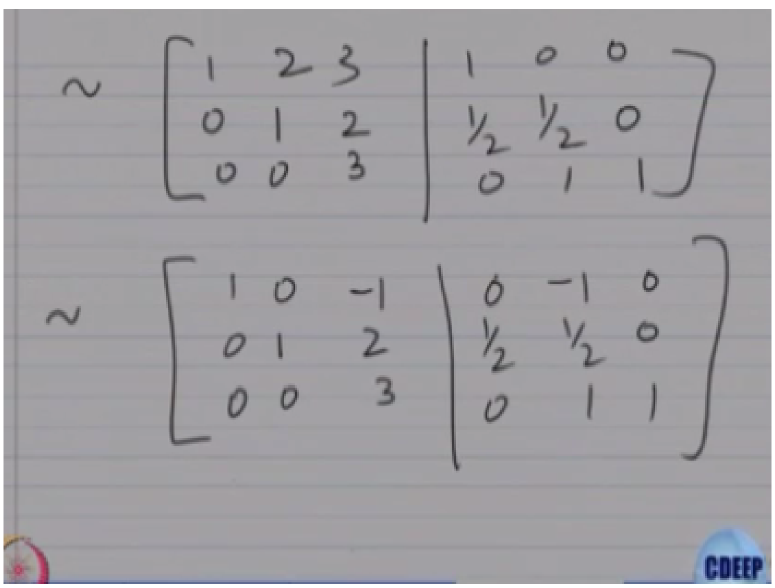

So, this is equivalent to 1 2 3, 1 0 0, this is 0, to make this 1, what should I do; divide this row by 2, so 1 2 1/2 1/2 0 okay and the last one is 0 0 3, 0 1 1, right, so I have got a pivot, in the second row to be 1, next step I should make everything in that column to be = 0, so let us make that, so that is equivalent to 1 2 3, 1 0 0 okay, so what shall I will be doing? Multiply the second row by -2 and add it to the first row, so that will become 0, if I do that, we remove everything.

So, multiply and add, this become 0, multiply this, -2 and add that become -1 multiply with this with -2 and add, so that becomes, is that okay, -2 and add, so -1 and 0, is that okay, right, so second one; 1 2 1/2 1/2 0 remains as it is, in the third one; 0 0 3, 0 1 and 1, right, so I have achieved second pivot which was coming in column 2 made 1, everything else in that column has 0.

I should be writing on the side actually, right those operations to keep track what I am doing, okay so when you do it, you should write that so that you can track back what is happening.

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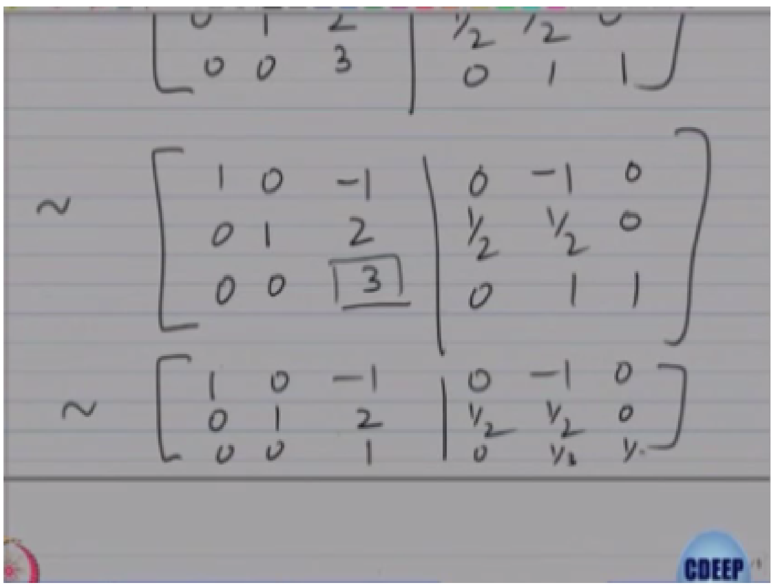

So, let us write now equivalent to; what should I do in next step, I should be looking at this part of the matrix, right up to here, I have column 2 I have done it, so for that only 3 is there, which is nonzero entry, so that is a pivot, I should make that pivot =1, so 1 0 -1 0 -1 0 and 0 1 2 1/2 1/2 0, so that is 0 0, what should we doing; dividing this last row by nonzero scalar that is 3, so 1 0 1/3 1/3 so that will be form here,

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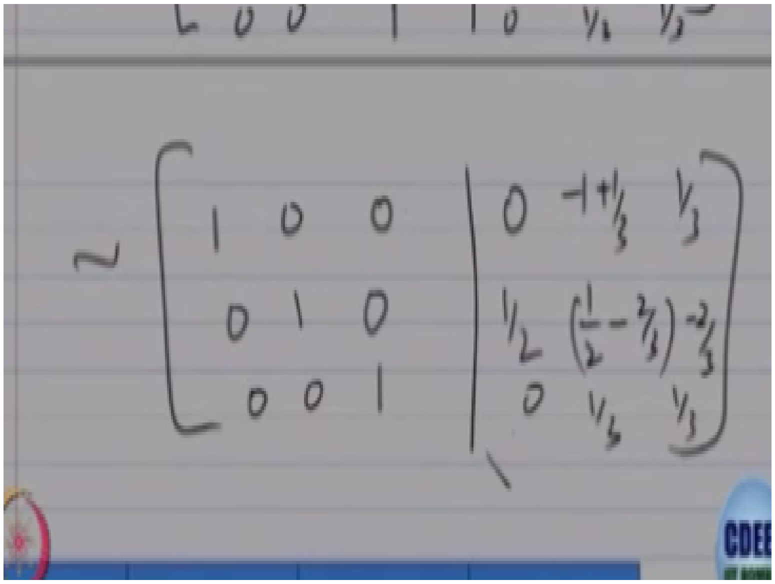

Still, I am not complete because pivot is 1, in the last column, everything else should be made as 0, so what shall I do; add it to the first one, so 1 0 0, when I add, so this is 0, so -1+1/3 and 1/3 so that is 0 1, okay, I want to make this also, so let us do simultaneously now, so this 2 also I want to make it 0, so I should multiply r3 with -2 and add, so, okay -2 and add, so this will not change, so 1/2 -2/3 this part and that will be -2/3 okay, so the last row is 0 0 1, 0 1/3 1/3, so what is my reduced row echelon form of a given matrix?

That is identity, right, so that means the matrix is invertible and that should be the inverse of the matrix, so you can write from here A.

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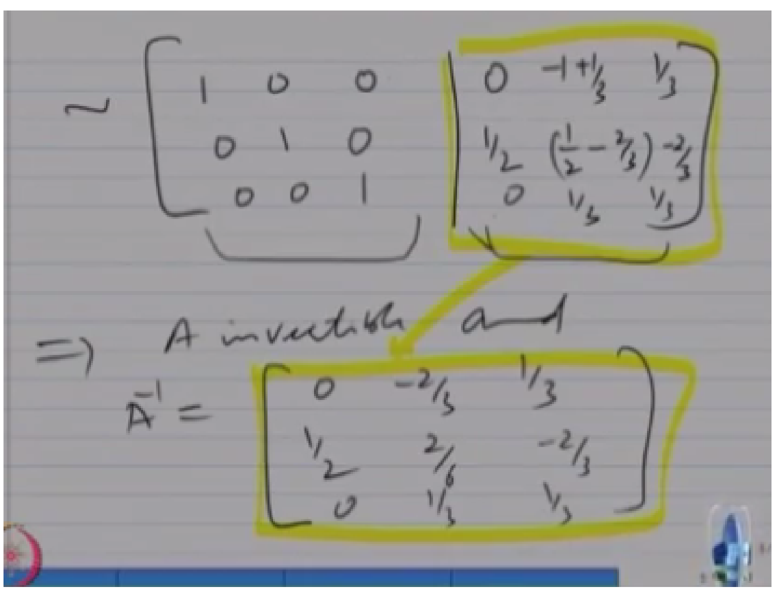

So, implies A invertible and A inverse = 0 -2/3 1/3 1/2 3 – 4; 2/6 anyway, never matter, simplify 1/3 1/3, right, so this part, so we are just saying that this part is precisely A inverse, right, so this method, so this theorem which we proved A is invertible if and only if it is a product of elementary matrices that not only tells us to check whether a matrix is invertible or not, at the same time it gives us an algorithm or finding the inverse also, right check and finds both, right.

So, what you do; take the matrix A along with it, write identity matrix on the same order, right, so we will get a bigger matrix, those number of rows will be same but number of columns will be double right, now go on doing elementary row operations on A part of the matrix, idea should be to reduce it to reduce row echelon form but do the same on the bigger matrix itself right not only on part A, on the part of identity also, so that will transform identity matrix into something.

If your transform matrix, the reduce row echelon form of A is identity, then in the last step, you can say that that is invertible, if it is not then A is non-invertible, right if it is identity, then the transform matrix identity which has changed to that would be the inverse of the matrix, right so this helps us writing elementary row operations as matrix multiplications gives us a benefit that we can put it on a computer and asked the computer to check whether it is invertible or not.

And find the inverse for me, right, everything is in matrix multiplication, whatever elementary row operation I am doing, you can writing on the matrix multiplication and eventually, identity matrix will changed to something will give me the inverse of the matrix, so checking and finding both at the same time, right.