Basically Real Analysis

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Lecture No 01

Real Numbers and Sequences-Part I

So, let me welcome you to this course on Basic Analysis and this is called basic real analysis. So, what is real and what is analysis? And let us understand the name of the course at least. So, real because it concerns with the real numbers and analysis on the set of real numbers. So, what does this mean? So, we will try to understand what is the set of real numbers, what are the properties of threats of real numbers. So, that is the object on which we will be doing our course.

And then analysis basically means analyzing various aspects of real numbers. And then functions on real numbers. You will find this is the sort of basic trend in most of the courses in mathematics as well as statistics, you have a basic object you will study that basic object properties at object and then you look at functions on that object, properties or the functions on that objects. So, that is a normal sort of way, mathematics progresses and different topics progress.

So, what are real numbers? If I asked you that question, probably you may find it difficult to answer that but think about what is the real number? Why did mathematicians invent real numbers? What was the need for that? When we start our journey in mathematics as childhood, we start counting, we get familiar with natural numbers 1234 so on. We come to next stage integers. That is because no 1 wants to look at solve equations n plus m is equal to some K and that may not be always on will for a given NLM okay.

So, from my evolution point of view that might have been because of writing debt kind of thing, you borrowed some money, you returned some money, how much is left kind of a thing, so, keeping account of those kinds of things. So, negative numbers must have origin, must have originated.

And then, 0 came much later historically because when you say I have nothing why should we 0 for that why there should be a symbol for nothing. You can just write nothing but it helps to have a symbol for that and, and gives us the courage that we discovered 0. Indians discovered 0 let us take credit for that.

Integers and then came evolution and then human beings evolved, 1 wanted to have fractions when you want to share things could be, could have been anything like an old man is about to die he has got five kids want us to distribute his property may be a piece of land. So, how will redistribute how much everybody gets kind a fractions came into picture.

So, for a long time fractions are good enough till historically around 300 years before Christ the Greek mathematicians, Greeks were the ones who are doing a lot of mathematics before Christ around 300 years Pythagoras, Euclid and so on. So, 1 of the persons they believed that any given any 2 magnitudes, so for them number it did not exist, it was only magnitudes like five, weight five kg of something, five did not exist for them five of something kind of magnitude. And they believe that given any 2 magnitudes, they are commensurable. What does commensurable mean?

Means, 1 is a multiple of the other given 2 magnitudes always 1 is a multiple of the other and that in the modern language if you write if A 1 magnitude B is another than A by B is a number. That means, what does that mean? A Equal to some Nb. So, A by B is a number that means it is a, what does it mean? It is a rational number.

So, they believe only in rational numbers, okay, they did not believe that there is something beyond rationals also, they thought every length should be a magnitude of something and they were fascinated by geometry. So, they wanted to represent numbers by geometric objects. So, their idea was that we should be able to represent numbers on a geometric object that is line.

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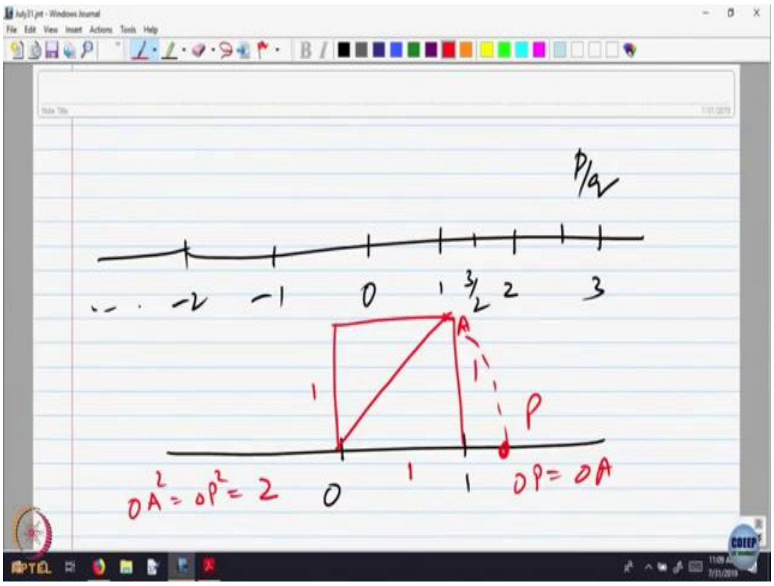

So, what they do, so, they took the horizontal line and what a point so, I was saying that they believe that every number is a geometric object. So, they took a horizontal line and marked a point called 0 and equal distance points and so on. So, all integers were, they were able to assign to every integer, a point, a position on the horizontal line. So, geometrically able to say that this point represents the magnitude 0, this represents one. And similarly they were able to assign, say, for example, this.

So, 3 by 2 for example half of midpoint of that. So, that way, every fraction P by q, they were able to put on the (num), on the line, horizontal line. And they were very happy and they thought that there is a 1 to 1 correspondence. I am slightly making it mathematical between the points on the line and the set of rational numbers meaning what, meaning that every rational number gets occupied by a point on the line so geometric object and every geometric object that is a point on the line is represented by a rational number.

So, that went on for quite some time till 1 person I forget his name (())(7:15) I think he discovered that this is not always the case. And there is a very simple example that he gave so let us take this line this is 0, this is 1 and let us construct a square on this of length 1 alright. And let us look at what is the length of the magnitude, that is a length. Geometric object. So, if you like you so, that gives you a point so, OP is equal to OA. So, he said that this point P cannot be represented by a fraction that was his claim.

Because by Pythagoras theorem, you know that this OA square that is OP square must be equal to 2, because this length is 1, this length is 1 so Pythagoras theorem, Pythagoras mathematics was available to them. So, they said this must be equal to 2.

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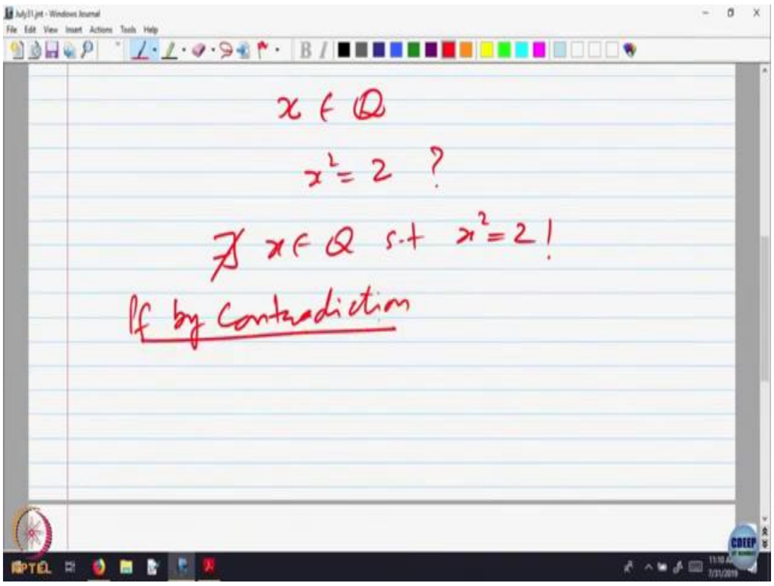

So, mathematically this leads to a question, it led to a question can I find a number x belonging to rationals say that x square is equal to 2. They only thought of geometric problems that were there this point on the horizontal line is representable by a fraction or not. So, now a mathematically we can frame this as a question does there exist a number x which is a rational number or square is equal to 2 and most of you have gone through a proof of that this is not solved.

So there does not exist, x belonging to Q such that x square is equal to 2. Probably. It is very interesting to go through a proof of this which is normally found in textbooks. So, the proof is by contradiction. Proof is by contradiction. So, that means what?

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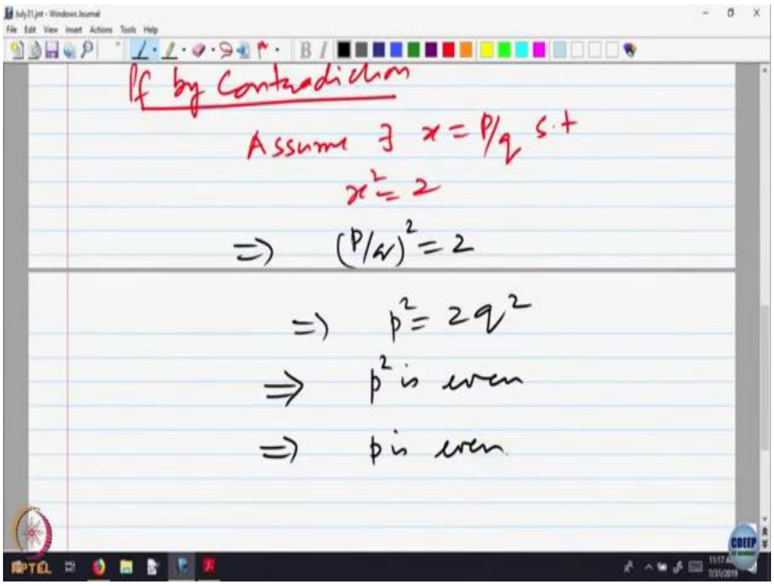

So, assume there exists x equal to P by q such that x square is equal to 2. So, this is I am just quoting the proof that most of the textbooks have I think the most famous book on analysis is mathematical analysis by "Rudin" principles of mathematical analysis. And you will find a proof of that are if you look at NCRT standard 10^{th} textbook you will find a proof there also school level. So, that means, so, this implies P by q square is equal to 2. So, that implies p square is equal to 2 of q square.

So, that implies, what? It implies that p square is even. Because there is a multiple. So, we assume proof is by contradiction that is another technique given by the Greek mathematicians proof by contradiction. They were the first 1 to, so, it says p square is even and that says P is even.

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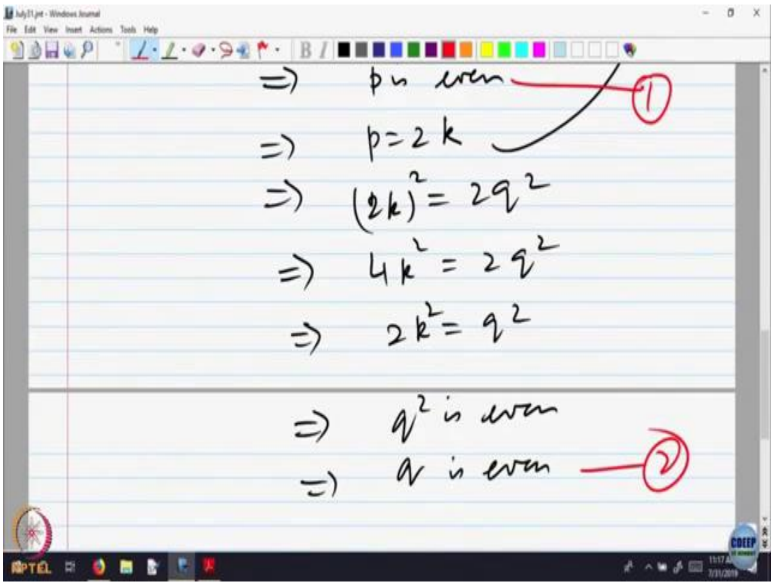

I am quoting the proof word by word either given Rudin or in textbooks of school level, so implies p is equal to 2 k implies, so put it back here, so 2 k square is equal to 2 q square. So, that implies 4 k square is equal to 2k square, and that implies 2 k square is equal to q square. So, that obviously implies q square is even and that implies q is even. So, here are 2 things, namely p is even. So, here is 1 and here is q is even. So, that means 2 must be a factor of p and q both.

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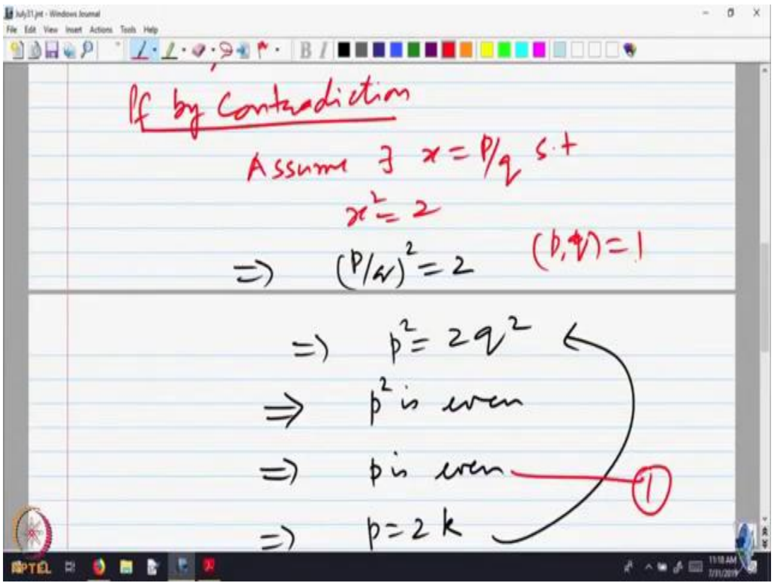

But we could have been a bit smarter. Could have started where P and q have nothing in common. Even common factors are there in P by q would have canceled it right in the beginning. So, that you get a equivalent form of a rational number. So, that would have, this will lead to a contradiction.

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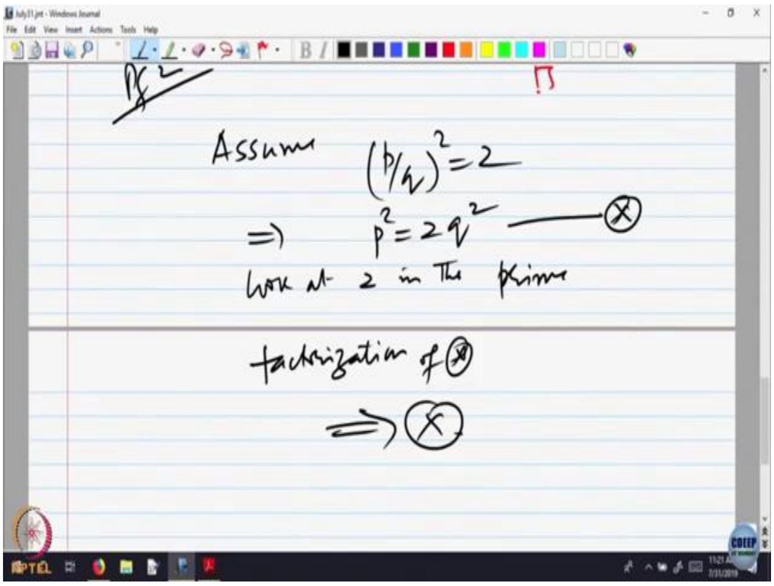

So, that proves that, that gives a proof. Now, what is my worry is, if so, how does this happen? If p square is even why should be p even that fact is not quoted in school books at all. I do not think that is quoted even in textbooks in mathematics even higher level. That means a proof and that is not proved in anyone of the classes. It can be proved very easily, right. One line says if you take a odd number its square is always odd. That is all one line has to be added in the bracket because square of an odd number is but nobody bothers to mention that fact.

And school kids and even teachers they assume, they remember this, everybody remembers this proof nobody tries to understand this proof. So, that is unfortunate thing. So, one could add a line and be happy about it okay. I want to give you an alternate proof of this proof is same essentially, but it is interesting. So, let us assume, so here is proof 2 assume P by q square is equal to 2 and that implies p square is equal to 2 q square. Now, what I am going to do is 2 is a prime.

And 2 occurs on the right hand side. So, if I look at the prime decomposition of the number 2 q square right hand side every number can be written as a product of primes. So, look at the prime factorization of the number 2 q square. How many times the number 2 can appear in that? 2 is already there for q square if a prime occurs in the prime factorization of q square, it must occur even number of times. So, total number of times 2 can appear as a prime in the prime factorization of right hand side is odd because 2 is already sitting there. We will look at the left hand side how many times 2 can appear? Contradiction, proof is over.

If we look at the prime factorization, the prime appearing in the prime factorization of left hand side it can appear even number of times, in the right hand side it should appear in odd number of times. So, that is a contradiction. That's all. So, look at 2 in the prime factorization of star that leads to a contradiction proof is over. You do not have to do anything.

And advantage of this that is how mathematics progresses. In this can I replace 2 by 3? Same proof. You may replace 2 by 3 and repeat the arguments what does it proof? It will prove 3 is not, there is no fraction, P by q whose square is equal to 3, why 3? I can put any prime number there is no rational whose square is a prime same improve works without any change at all only had to replace 2 by 3, 3 by P or even why prime I can put P by q square cannot be, whole square cannot be a number, which is a perfect square that will also work. So, that is advantage anyway. So, this is what?

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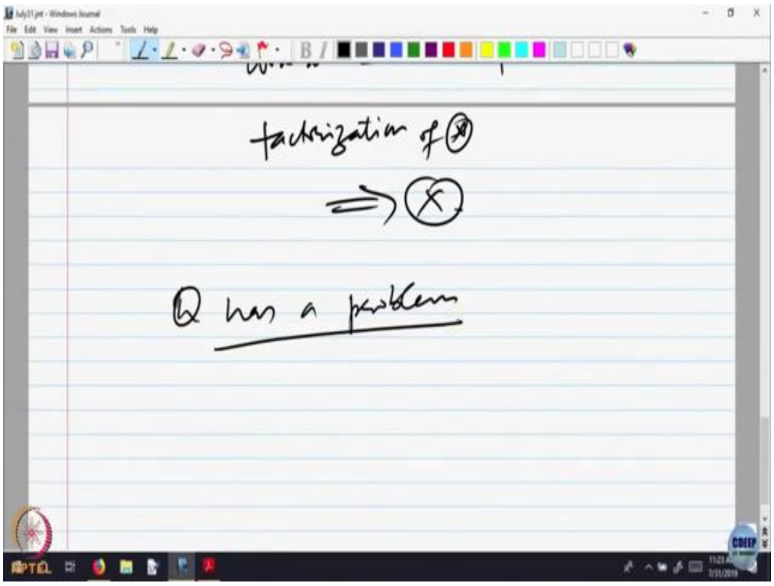

So, Q has a problem. The rational numbers and the interesting thing is the person who discovered this in the Greek mathematics around 200 years before Christ probably, you know what was the reward he got? He was put on a boat without food, without any help and left in the deep see to die. Because the Greek philosophers did not want to accept that they have a problem in their mathematics. They wanted to hush up this discovery. And as a result, discovery of this fact was delayed by 2000 years, approximately.

So, it was only in 1800 and 1758 and by a mathematician called Richard Dedekind, and a mathematician called George Cantor in 1871. They said rational numbers are not complete, in the sense that such kind of equations do not have solutions. So, we should discover, we should add more numbers to rational numbers, so that it becomes complete so, that construction is a non-trivial construction and both did independently both approaches are different, but both lead to a common object which is called a complete ordered field.

So, nowadays, either it is done in undergraduate course, the construction of real number system or it is just assumed. So, because we have a prescribe syllabus to finish, so, we will start with that we are given the object called real number system will say what are the properties given of it and go ahead with it. Right. So, let us start with the real number system. Okay. So, this is the first part of our course.

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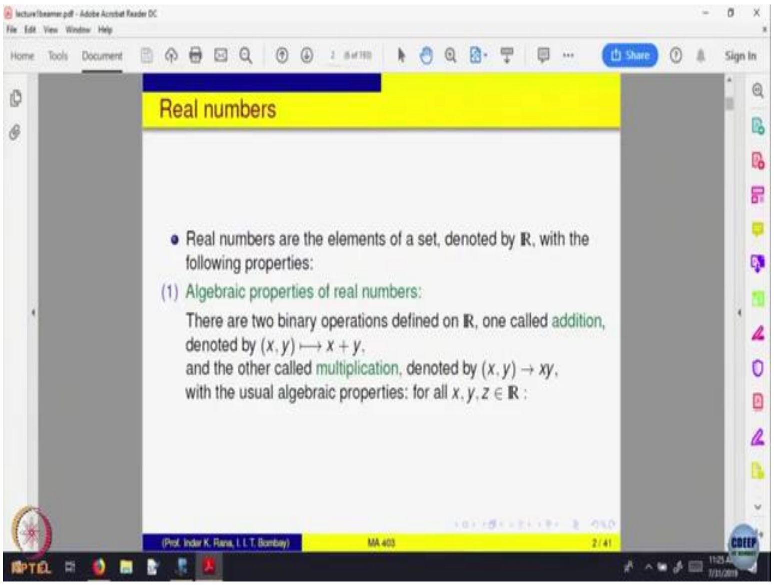

So, real number system, it is a set first of all. What is that set?

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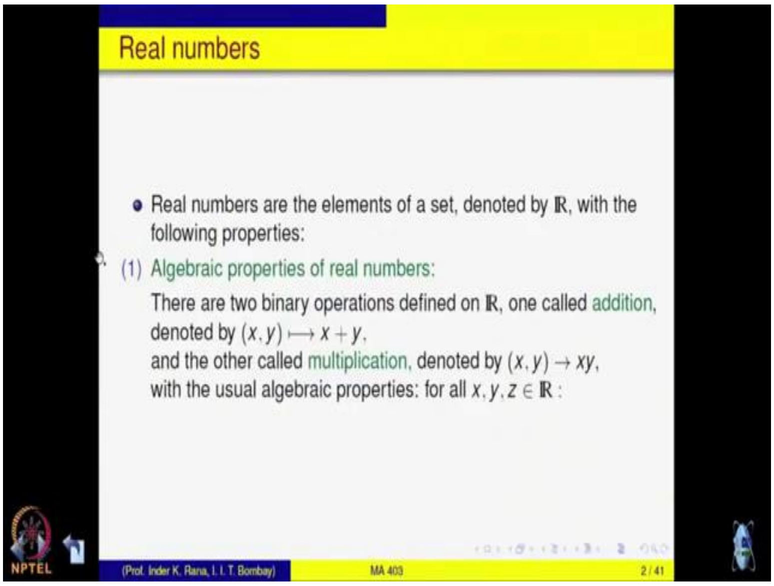

So, real number is set, what are the objects in that set we do not specify, because you actually have to construct them using rational numbers. And the process is a bit long. So, we assume that we have a set called the set of real numbers and is denoted by this funny symbol. Normally, you will find such symbols quite common this is called script R, okay. So, that is real numbers. It has two operations on it, it is set with two binary operations, one is addition, other is multiplication. So, here are the algebraic properties, there is operation of addition, there is the operation of multiplication with the usual properties and what are the usual properties?

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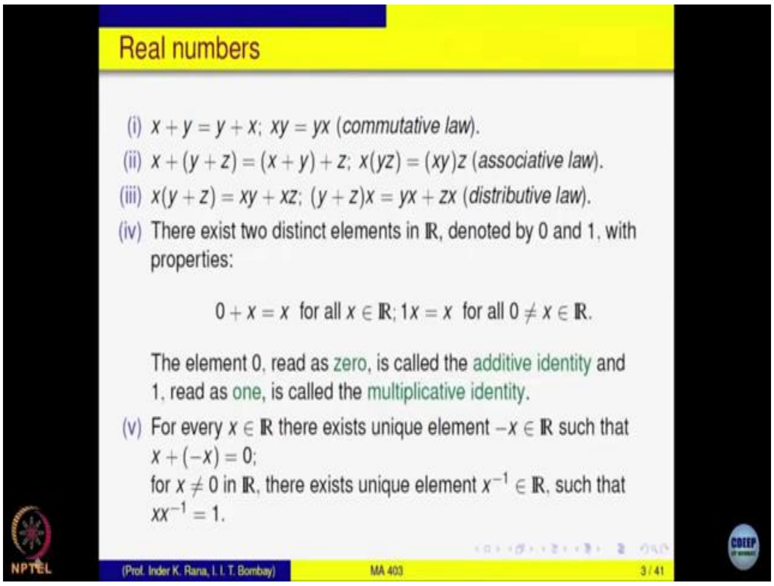

So, let me just, I think all have you know these properties basically saying addition is commutative, edition is associative, multiplication is commutative, multiplication is associative, how does addition and multiplication interact with each other? These are 2 operations on the same side. So, they interact in a way that is called the distributive property. It is like saying two, two human beings living in the same room. How do they should interact, what are the rules for interacting. Otherwise both are independent and everything right.

So, this is the interaction between them, then there is a unique element denoted by this symbol there is unique element denoted by this symbol, which has that this plus x is equal to x and this multiplied by x is equal to x for every x naught equal to 0. These are called additive identities and multiplicative identities. Okay. This one is read as 0 and this one is read as 1. So, multiplicative identities given the name 1, additive identity is given the name 0.

And more than that, for every x there is another real number in that set denoted by minus x. So, that when you add you got 0 and similarly for multiplication x naught equal to 0 x into there is a number called x minus 1, which gives you 1. So, essentially, it says that under addition and multiplication, the two form abelian group, or a group, if you know that word, if you do not know forget about it, does not matter, you should remember all these needs.

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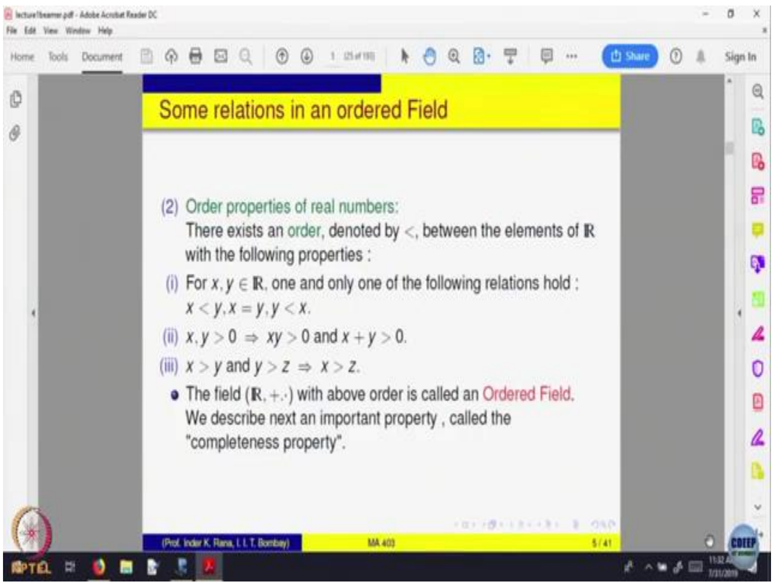

Then there is something called a order on real numbers very important that means given two in that set of real numbers and compare. And the comparison says that given any two, so this is the properties namely, given any two numbers, we should not be calling as numbers at present given any two objects in that set R you can compare given x and y, either x will be less than y or y will be less than x or the two will be same.

This is called the law of Trichotomy that given three at least one and only one of them will hold and this is very important the second one, if x and y now, there is the order. There is a 0. 0 is the additive identity, given any x if x is bigger than 0, y is bigger than 0, then their some and product should be bigger than 0. So, sum and product of what we call as positive, objects should be positive, this is the rule and once this is true, you say that you have got a set R with addition with multiplication with order with all these properties is called an ordered field.

Such an object is called a field in algebra. Where group is there, where and under addition it is abelian group, under multiplication it is a abelian group, the 2 interact resistivity property, that thing is called a group. Sorry that thing is called a field. On field there is an order. And that order respects addition and multiplication both. It interacts nicely in what way? If x and y are bigger than 0 then x plus y is bigger than 0 x into y is bigger than 0. So, this is called an ordered field. So, till now what I have said is it R is an ordered field.

There is one more crucial property of this, which is called the completeness property, which requires a bit of more discussion to state also. Basically, if you look at the rational numbers, you can add rational numbers, you can multiply rational numbers, you can also compare rational numbers, 1 rational number big and they all have all the properties that till now we have studied for reals.

So rationals also form a ordered field, what is the difference between the rationals and the reals. So, we want to describe that, because that is going to be crucial for our future discussions. But anyway, here are some things you can try to prove.

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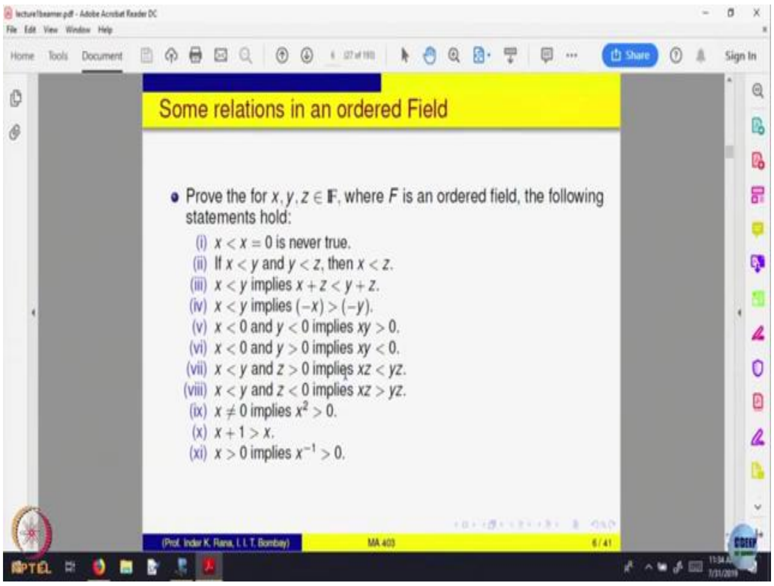

As far as order is concerned, these are some properties of various properties you can try to prove. x is less than y and z is bigger than 0 and xz should be. We all assume these kind of things, right in our arithmetic. But what I am saying is given that R is a ordered field, using only the axioms of an ordered field, you can prove all these properties. For example, you can prove 1 is bigger than 0 multiplicative identity, there is additive identity you can prove 1 should be bigger than 0, these are 2 objects in R and there is order.

So, 1 and 0 should be comparable with each other. We know that they cannot be equal, because multiplicative identity cannot be equal to additive. So, which is bigger than what? So, one can prove only using these axioms that 1 has to be bigger than 0. All this is nice, very nice to prove these things, so try to prove yourself we will not prove it. We will assume these things. Okay.

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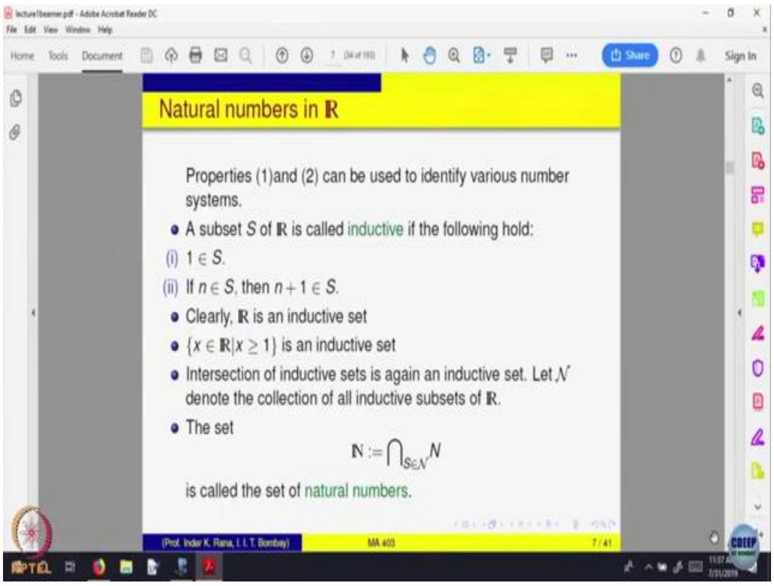

So, here is something you can identify what are natural numbers? If you, if is an, we are calling R as a set of real numbers. Then where is our familiar object natural numbers in this? So, here is the familiar object. Let us call a set to be inductive S is a subset of R we call it as a inductive set. What is a property we want? 1 should belong to that it is non empty set 1 should belong and whenever a number n belongs, whenever an object n belongs, then there is 1 additive identity I can add 1 to N.

So, x plus 1 or n plus 1 for every number, real number x look at x plus 1 that should also belong. So, a lot of you can call it as additive successor of a number x, x plus 1 as a additive successor of x. So, when our x belongs additive successor is also inside it, that set. So, that is second property. So, for example, can you give some examples of, for such sets?

For example, look at all x bigger than or equal to 1. We have got set of real numbers. It is a ordered field in that look at all x, such that x is bigger than or equal to 1. So, 1 belongs to it. x plus 1 is going to be bigger than 1 so it is going to belong. So, that is a inductive set. But what we want is, we want the smallest inductive subset of reals we have defined what is the inductive set 1 belongs successor belongs, I want to construct a set which is inductive, which is a subset of R and it should be smallest.

So, take the intersection of all of them. Given 2 inductive sets, if you intersect, that again is a inductive set because 1 belongs to both, so 1 belongs to intersection. If x belongs to intersection, x belongs to a, x belong, x plus 1 belongs to a, x plus 1 belongs to b, so it belongs to intersection also. So, intersection of inductive sets is again a inductive set, so if we look at intersection of all inductive subsets of R that is also an inductive set n is the smallest that we denote as by n. That is a set we denote by N.

and you can see clearly here what does inductive means, is the induction that is you are applying, what you call as mathematical induction, if 1 belongs, n belongs, n plus 1 belong and the smallest with that property, this is what is mathematical induction also. So, that also holds in our new setup. And this set n 1 belongs to it. So, its successor must belong, 1 plus 1 should belong. You can call that you can give a name, you can give a symbol to it called as 2. So, you can define 2 to be 1 plus 1 3 to be successor of 2.

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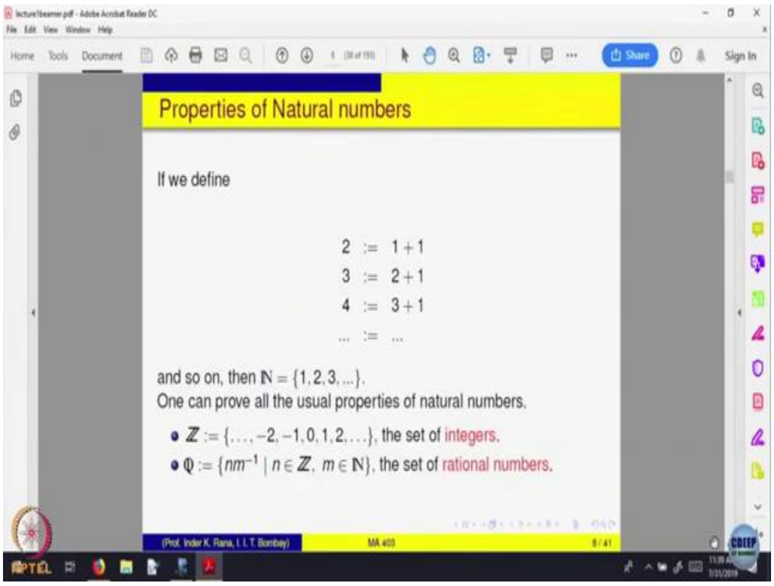

So, one can prove that n is nothing but 1 multiplicative identity its successor, its successor, its successor and so on, that is all. That intersection is nothing but this smaller set. And that is a definition. Normally we take of natural numbers. So, we identify natural numbers as part of our new setup of real numbers within abstract object. We do not know what that is, but we are trying to identify familiar things.

Once you have integers or once you have the natural numbers, you can define what are integers. We have addition. We have 0 additive identity for every x there is minus x. So, collect minus x for every x in n, along with 0 call that a set of integers. Once you have set of integers, we can have fractions, the rational numbers, Q. So, that is n multiplied by m inverse where n is a natural number and m is a integer and n is an natural number.

Normally, you will find in books it is written n by m, where n and m are integers and m is not equal to 0, there is no need to do that, you can just write n over m, where n is integer m is a natural number because you never right denominator to be a negative anywhere. Because fraction means what 1 by 2 means what? Divide 1 into 2 parts and look at one, each one of them that is 1 by 2 what is 1 by minus 2?

If you look at fractions as we understand it from our primary sections, what is minus 2 parts of 1? Can you divide? So it does not, so there is logical, a lot of difficulty in understanding. So, take at minus 1 as an object and divide it into 2 parts. So, and anyway when you do arithmetic of rational numbers, you always take signs on the numerator and then take denominators and LCM and whatever it is. So anyway, so, this is a better way of writing fractions as, so integers, natural numbers, integers, fractions, we have identified these objects as part of our set R.

Now, there are things which are not part of Q for example, those objects like x such that x square is equal to 2 there is no such number. So, those we call as a set theory compliment of Q in R. We already have a bigger set now, R so take compliment, so numbers, objects which are not rationals in R they are called irrational numbers. So, that is what definition of irrational and one can do many things 1 can go to decimal representations and prove all those things the decimal representation of rational either repeats or terminates and all such things we will not do that we will assume all this but this is, logically one can start with a ordered field and do everything.