Basic Real Analysis
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Lecture 16
Topology of Real Numbers: Compact Sets and Connected Sets Part 1

So, let us recall we had started looking at what are called compact subsets of the real line.

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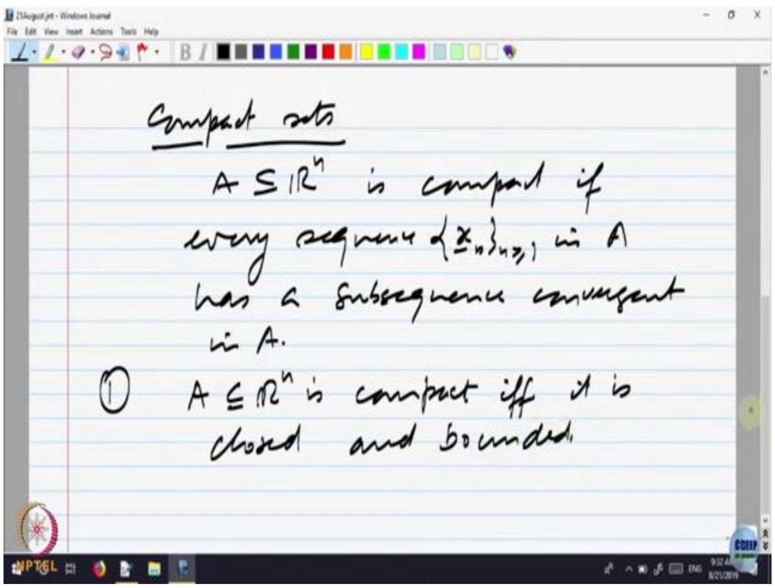

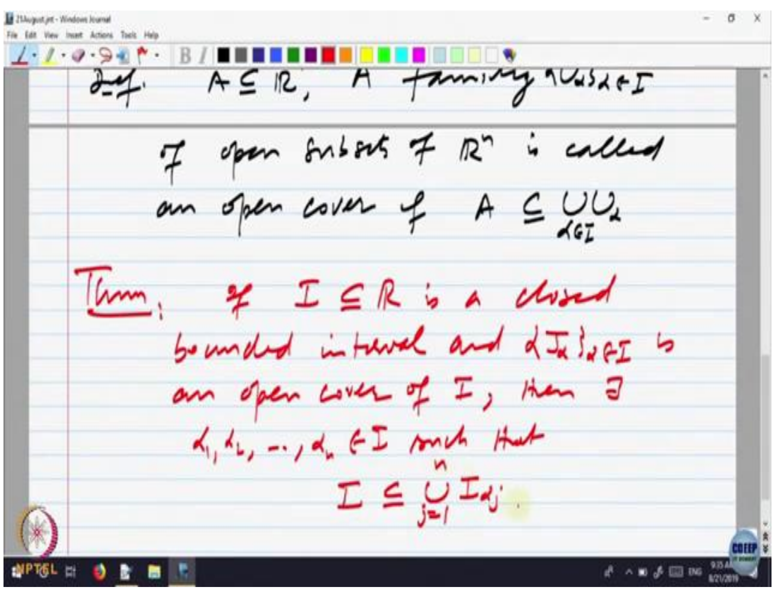

So compact sets. So a subset A of Rn, we said is compact if every sequence xn, in A has a subsequence convergent in A. So, then we proved one result namely, A contained in Rn is compact if only it is closed and bounded. We were looking at another way of describing compactness. So we defined A is a subset of Rn. Family of sets, say U alpha of open subsets of Rn is called an open cover, we say it is an open cover if A is contained in the union it covers it, in sense.

So, we were proofing a theorem that if I contained in real line is a closed bounded interval and let us say, J J alpha is an open cover, I then there exist alpha 1, alpha 2, alpha n belonging to I such that the interval is covered by this finitely many I alpha j, j equal to 1 to n.

So, what we are saying is that if you take a closed boundary interval in the real line, of course because it is closed bounded it is a compact set. So, we are saying if A is a compact set which is an interval, closed bounded interval then every open cover of the closed bounded interval I has a finite sub cover, that means given any open cover for the interval I there are finitely many of them open sets, which are only needed to cover it.

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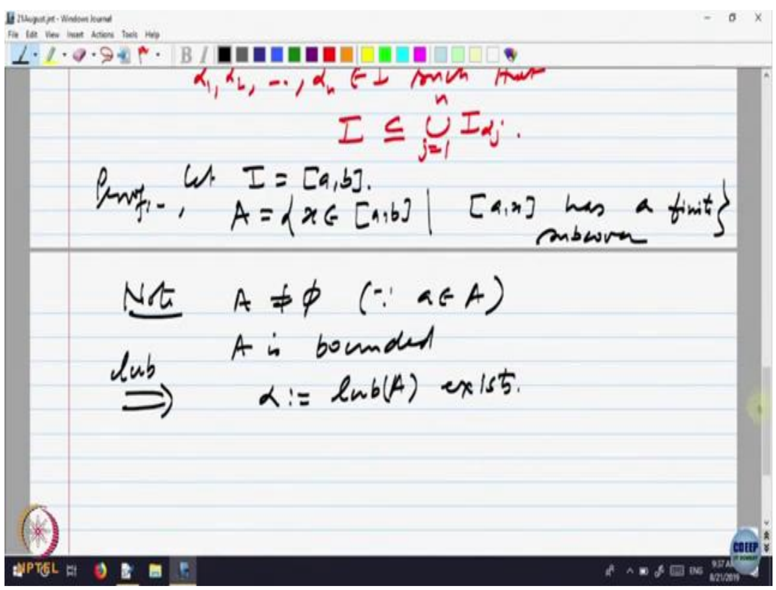

So, let us prove it and we had almost proved it last lecture. So proof, the idea was that let us look at the set A of all x belonging to ab. Such that the interval a to x has a finite sub cover. We are trying to show that the interval I has that property. We have not said what is I so let is probably, let I is closed bounded interval. So, let us say it is ab. I is interval which is closed bounded so let it be the closed bounded interval ab.

Then look at the all the points x in ab such that the interval a to x has a finite sub cover. The idea is to show that a to b has a finite cover. So, let us look at the note we proved last time, that the set A is not empty because the point a belongs to the A. A is bounded because it is inside the closed interval it is the subset of ab. So, implies by the lub property that alpha equal to least upper bound of A exist. So, least upper bound of the set A exists.

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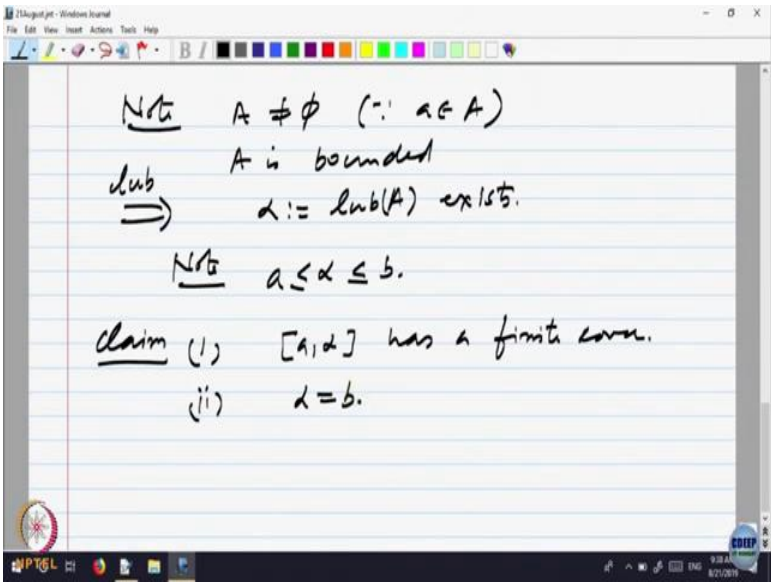

So, we observed that because the set A is inside ab, alpha is between a and b. It is a set between alpha is a number which is a least upper bound of A and A is a subset of ab so least upper bound has to be a part of the interval a to b. We claim first that the interval a to alpha has a finite subcover and the second part would be that alpha is equal to b. So, these two claims will prove that the interval A to B has got a finite sub cover.

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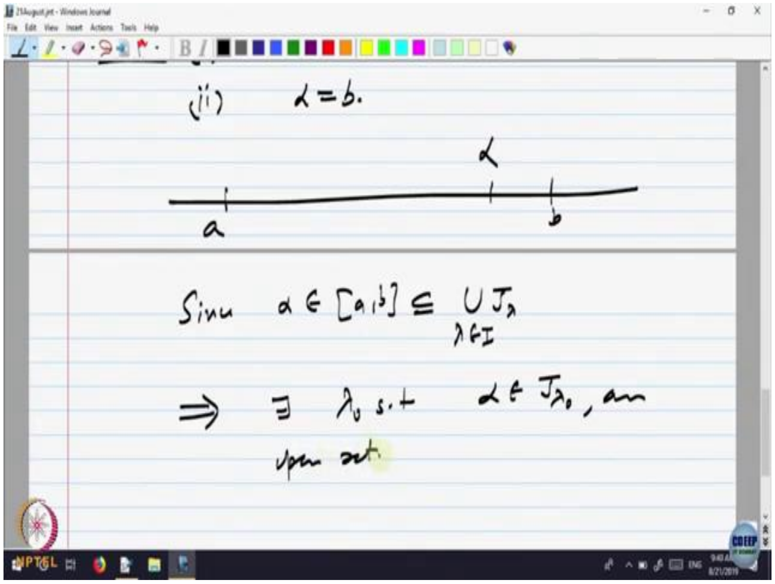

So, let us check both of them one by one. So, this is the idea of the proof, so let us check here is a and here is b and somewhere we have got alpha in between. We do not know alpha is equal to b or not but at least alpha is less than or equal to b. So first of all, let us observe that since alpha belongs to ab and ab is covered by, there is a covering given. So ab is covered in union of J alpha, alpha belonging to I. So, what does it imply so this implies, just a minute I think I am using same alpha here and same alpha there. So, let us change it one of them. Probably this is indexing set let me call it as Ja lambda.

So, does not matter what you call it. But so that no confusion comes, so lambda, the open cover is J lambda I just renamed it does not matter. Now alpha belongs to ab closed interval and that is covered by J lambda, so alpha must belong to one of them. So, implies there exist some lambda 0 such that alpha belongs to J lambda 0 and J lambda 0 an open set. So, what does it imply? What is the definition of open set? Every point is a interior point so there must be a open ball around the point x which is inside that open set. But we are in real line so there is an open interval around a point alpha which is inside J lambda 0.

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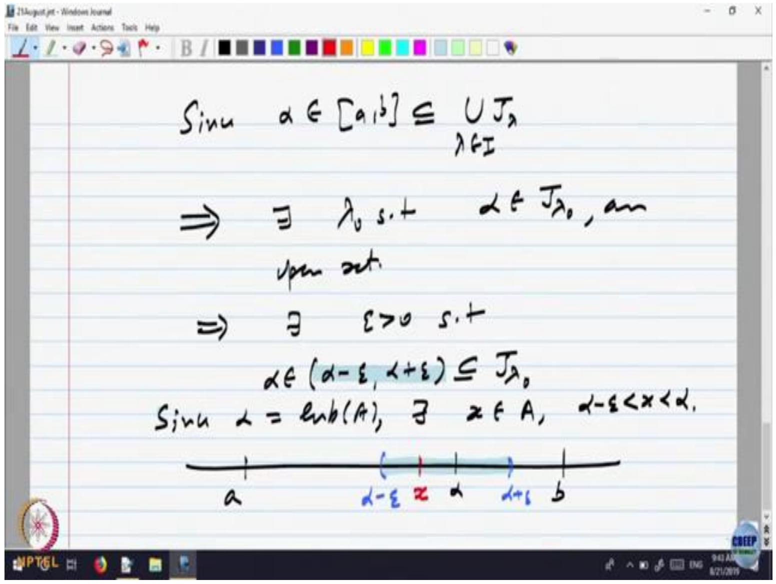

So, implies, so there is let us call it say something epsilon bigger than 0. Such that point is alpha so alpha minus epsilon alpha plus epsilon is a subset of alpha belongs to it and that is a subset of J lambda 0, because of openness. (()) (9:24) coming to intervals, so let we draw the picture what is happening. So here is a, here is b and here is alpha. So, there is some J lambda naught, so there is an open interval alpha minus. So, let us say this is alpha minus epsilon and this is alpha plus epsilon, so that this interval is inside. So this is this interval, that is inside J lambda naught.

Now this alpha is least upper bound, so alpha minus epsilon cannot be the least upper bound for the set A. That means what, there must be an element of a which is inside alpha minus epsilon and alpha. So, let us write since alpha is equal to lub of A, there exist some point let us call it as x belonging to A, such that alpha minus epsilon is less than x is less than alpha. So, here is the point x.

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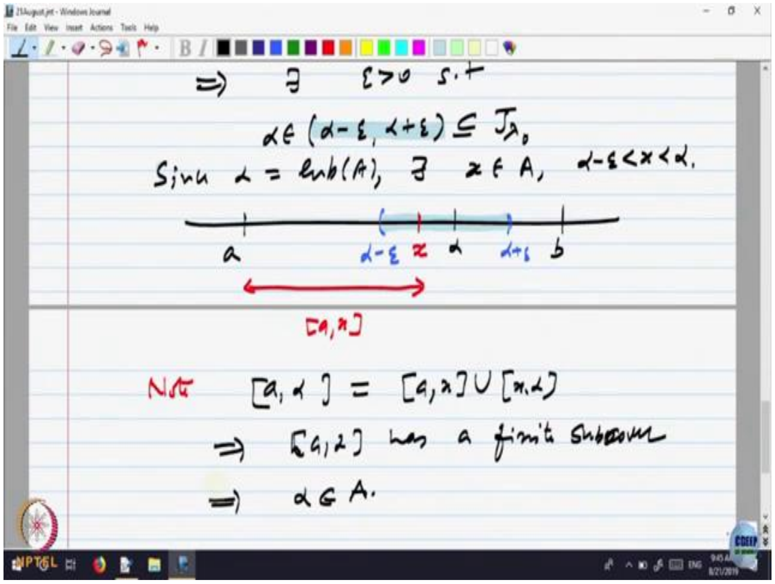

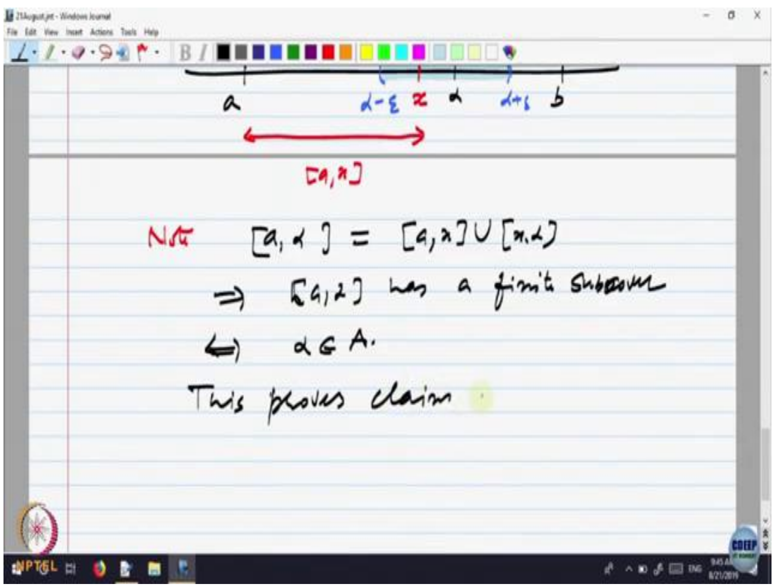

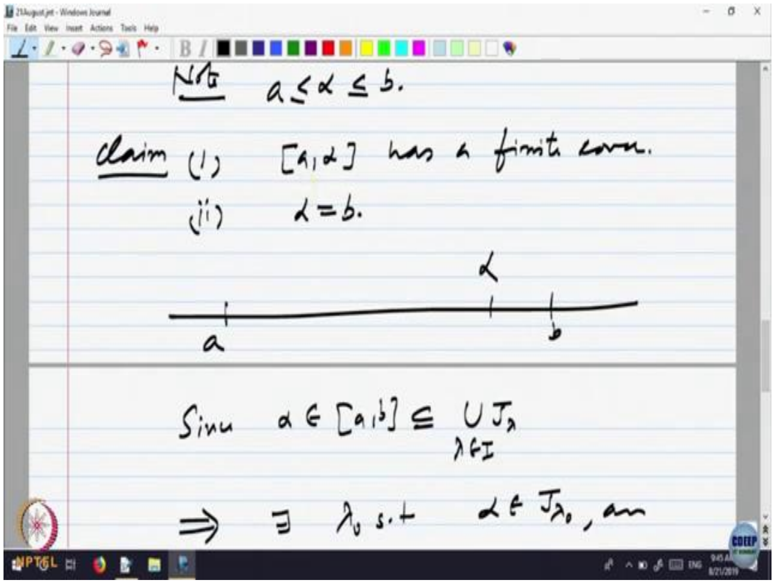

Now look at this interval a to x. So, this is the interval a to x, x belongs to a, so this must be covered by finitely many elements of that covering by the definition of a and alpha itself is inside this interval alpha minus epsilon to alpha plus epsilon which is contained in one element J lambda naught.

So, claim is that note, so this is a crucial thing to note that a to alpha is equal to a to x union x to alpha which is contained in a to x. x belongs to a so is cover by finitely many and alpha x to alpha is inside alpha minus epsilon to alpha plus epsilon which is inside J lambda naught. So, this is has a, implies a to alpha as a finite subcover.

Because a to x has a finite subcover by definition of x in a and the interval x to alpha is covered by one of those the G alpha naught that we have selected. So, put together it is a finite sub cover of a to alpha. So, that implies alpha belongs to a, that means a to alpha has a finite sub cover. That is what we wanted that is equivalent. So this proves this proves claim one. So, what was claim one we wanted to show that a to alpha has a finite sub cover. The claim alpha has to be equal to b, that will complete the proof.

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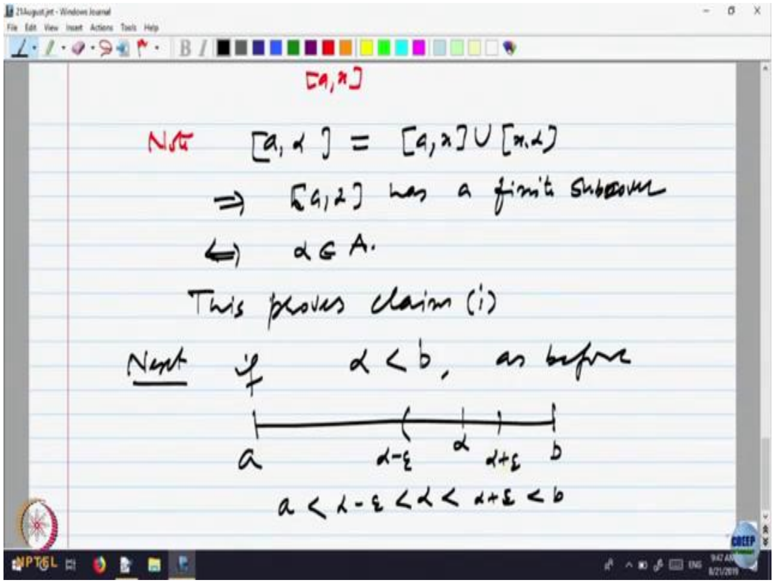

So let us see how it is happen. So next if alpha is less than b, if alpha is less than b then what does our earlier construction give us? This is a and this is b and here is alpha. Actually the picture drawn earlier was saying that it is less than b. But any way if alpha is less than b we had that there is alpha minus epsilon, alpha plus epsilon and this is contained in, if as before a less than alpha minus epsilon less than alpha less than alpha plus epsilon less than we can choose less than b, that earlier picture I am continuing. We know a to alpha is covered by finitely many but alpha to alpha plus epsilon there must be some element in it.

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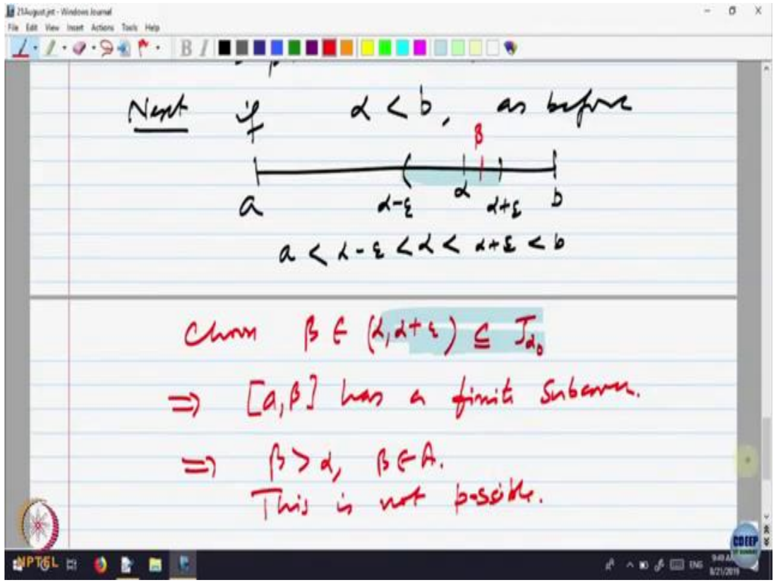

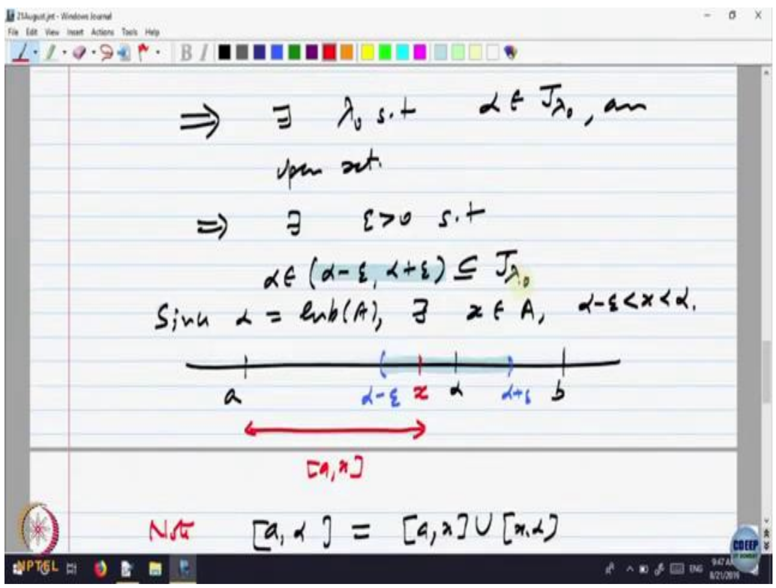

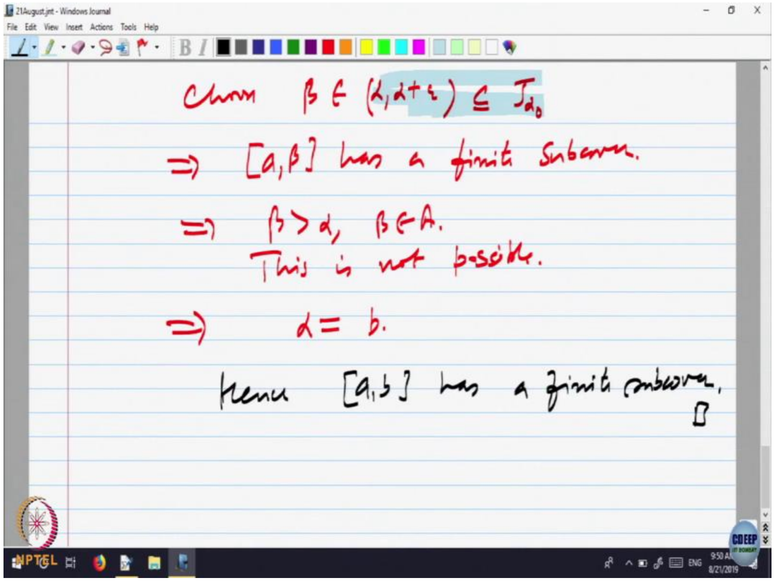

So, let us choose anything that you like. So, let us choose some, let us choose a beta so here. So, choose, so this is, so choose any beta belonging to, any beta belonging to alpha to alpha plus epsilon less than b. Alpha plus epsilon, anyway that is less than b that we have already. So, that is a crucial thing, choose any point beta, where does beta belong?

Beta belongs to alpha minus epsilon to alpha plus epsilon and that is containing G alpha naught. Alpha minus epsilon to alpha, because the point alpha was inside an open interval, a open set J alpha naught, so there must be an open interval that is how we have constructed. And now we saying on the right side of alpha pick up any point beta.

Now a to alpha is covered by finitely many and alpha to beta is inside this open interval which is inside J alpha naught. So, what does it say that a to beta is also covered by finitely many. So, implies a to beta has a finite sub cover, for everybody because a to alpha we have already shown as a finite subcover and beta belongs to this interval alpha minus epsilon to alpha plus epsilon. So, it belongs to this and that is inside J alpha naught, so that, so if I put together this one element J alpha naught in the covering and covering of a to alpha than I get a new covering which is finite for a to beta.

But what does it imply? Implies beta bigger than alpha and beta belongs to a. a to beta as a cover, finite subs cover and beta is strictly bigger but what is alpha? It is a least upper bound of a. So, there is nothing of the set can be bigger than alpha, so that is a contradiction. This is not possible.

So, what is our assumption, as our assumption was that alpha is less than b that is giving us the contradiction. We are able to find an element beta because alpha and b there is a distance.

There is some points, if alpha is equal to b I cannot find beta that is what precisely we wanted to say. So, implies alpha is equal to b. So hence a to b has a finite sub cover. So, essentially the idea is quite simple, start with a the single turn a has a infinite sub cover, gone stretching it and see how much you can stretch, so that a to x has a finite sub cover and try to saw that stretching goes up to b, by taking the set A least upper bound and showing it is equal to.

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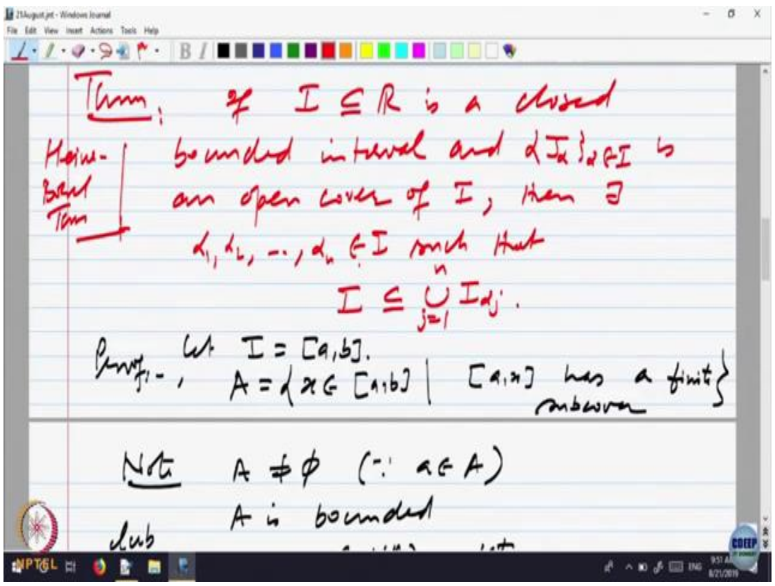

So, we have got alternate way of describing closed bounded intervals. The closed bounded intervals have that property, given any open cover there is a finite sub cover and we said this goes by a name, this goes by the name called Heine-Borel Theorem for intervals. One can actually extend it slightly further, so let us do that.

Say, here what we have shown is every closed bounded interval has this property. We want to show every compact set has got that property. Closed bounded interval are compact, by definition or by the property that a set is compact if and only if it is closed and bounded but there are closed bounded sets which are not intervals obviously. So, for example you can look at sets which are the union of two closed bounded intervals.

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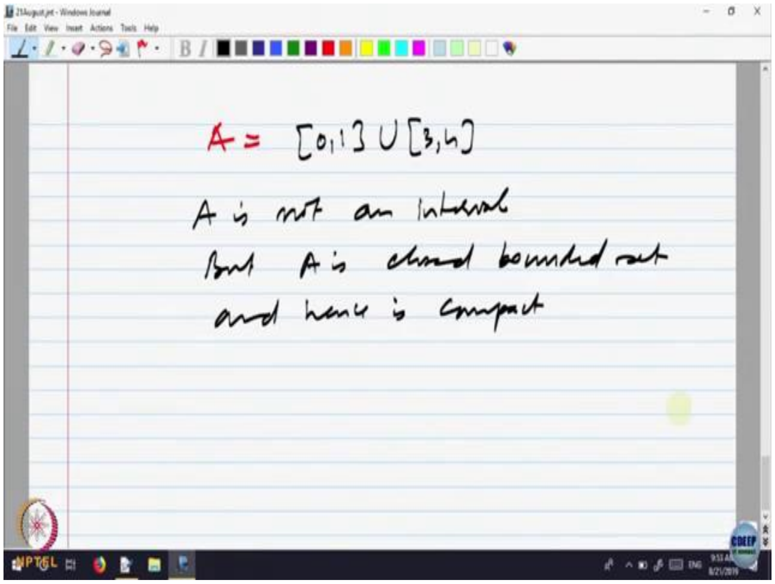

So look at the set A which is equal to say 0 to 1 union 3 to 4. A is not an interval, it is not an interval, but A is, is it a closed set? Yes, it is a closed set because we shown that finite union of closed set is a closed set. So, this is a closed set and it is bounded, it is bounded between 0 and 4, so it is a closed bounded set, is a closed bounded set and hence is compact. It is not an interval but it is a compact sets, so compact subsets even of real line need not be intervals, closed bounded interval. But what we want to show is it has those properties that Heine-Borel you can call it, so let us write this as a theorem.