**Basic Real Analysis,**
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**Lecture 17**
**Topology of Real Numbers: Compact Sets and Connected Sets Part 2**

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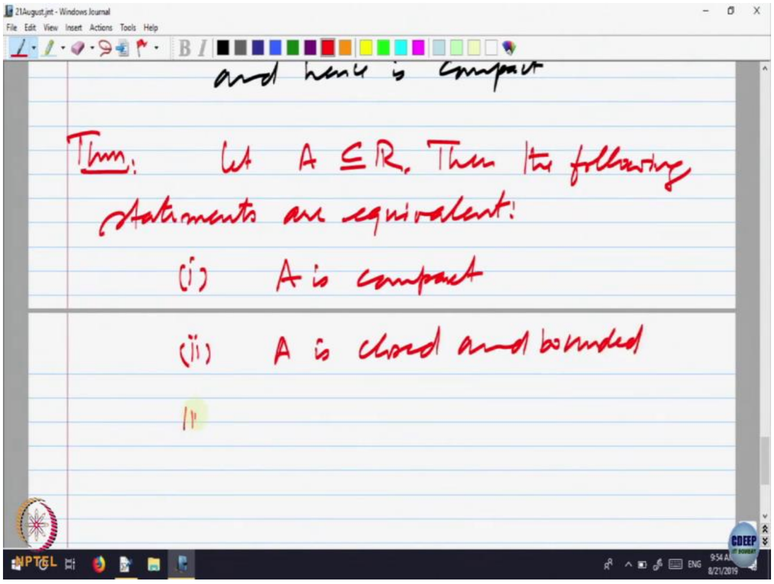


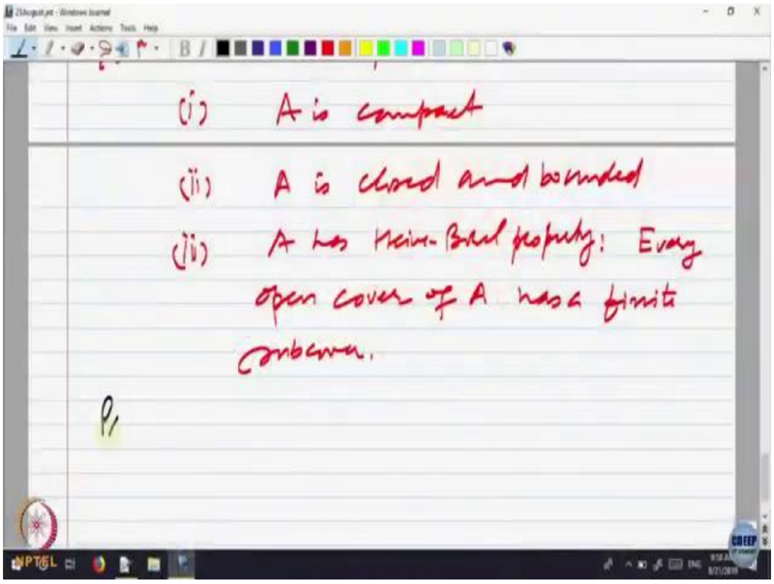


Let A contained in R, so the following statements are equivalent then the following statements are equivalent. So, what are the statements? One A is compact, two A is closed and bounded and third when says A has Heine-Borel property and what is that property? (())(01:19) every open cover of A has a finite sub cover. So, recall what was the our definition of compactness that was in terms of sequences (())(01:44) is called compact if every sequence has got a sub sequence which is convergent in the set.

Second property closed and bounded, we define closeness of a set again in terms of sequences (())(02:03) is closed whenever as you can see an elements of A converges somewhere then the limit must be inside the set A that was closeness. Bounded that is interval of intervals. There is a, some interval big enough which contains the set A that is boundedness. So, closeness and boundedness are mixture of some property of sequences and a property of intervals.

Now, saying A has Heine-Borel property it is just purely in terms of open sets. But recall, what was an open set? A set is open if every point is interior point that is same saying every point given any point in the set there is a open ball around it which is inside the set there was a open set. Again in terms of the sets open balls, every open cover has got a finite sub cover.

So, the important thing of this theorem is that later when there is no notion of sequences in higher courses in mathematics there is, there are courses where sequences are not possible that is called if you happened to study what is called (())(03:29) it is a course and topological spaces.

They one defines only what are open sets one does not go to sequences or anything everything is dealt in terms of declaring in some sets to be open and then defining a set to be closed if it compliment is open and then doing whatever we are doing kind of a thing. So, there the notion of compactness is defined in terms of Heine-Borel property. Because saying that a set is close or bounded, closeness you can define in terms of openness but boundedness requires notion of distance, boundedness.

So, spaces where this is not possible to define notion of distance, but we want to have a notion of compactness. So, there the definition is the third one that every open cover of that set has got a finite. So, that is a definition that is taken as a definition in topological spaces of compactness. So, I am just giving you a snap shot of something which you may come across.

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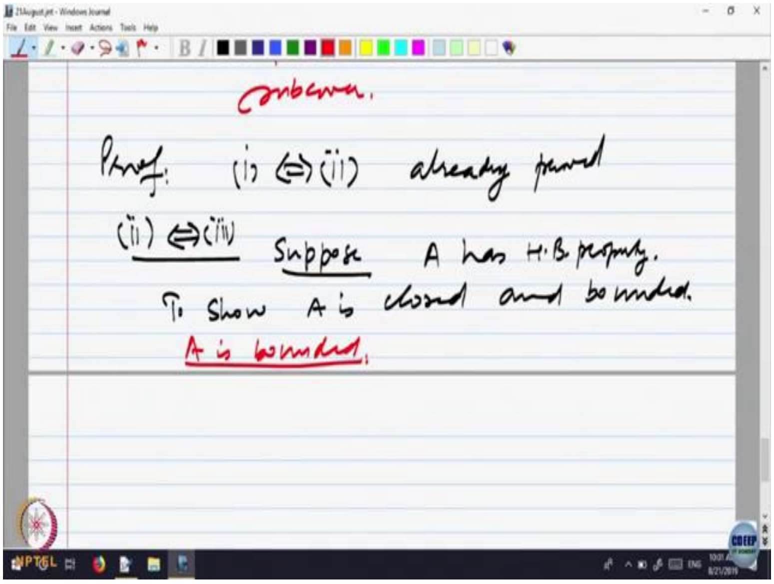


So, let us write a proof of this 1 is equivalent to 2 that we have already proved and A is compact if and only if it is closed and bounded already proved. Let us prove 2 if and only if 3. So, that we want to proof. So, let us look at suppose A has, I will just short a Heine-Borel property as H.B property, given any covering has got a finite sub-cover we want to show A is closed and bounded we want to show it as, it is closed and bounded.

So, let us first proof A is closed or it is bounded either way it is, easier to show bounded so let me just show that first and that does not matter actually because the both have to be shown, so that A is bounded. Now to show that A is bounded A is a subset of everything is in real line by the way. A is a subset of real line we are in the real line. A is bounded that means it lies between bounds.

Now given any set A and I have to use somehow my tool that every covering has got a finite sub cover. So, keep these two things always in mind what we want to show and what is given. So, to show bounded I have to somehow manufacture some covering of the set A and use the fact that it has got a finite sub cover and conclude as a consequence that it is bounded. Now to show it is bounded if we can show A between minus n to plus n every A is a subset of the interval minus n to plus n that will be bounded.

Now once I want to show it is between minus n to n can I sort of take that interval and make it a covering of the set A. How do I get the covering of, so that one of this elements is minus n to plus n. So, first of all to apply Heine-Borel property I should have open interval. So, let us take the open interval minus n to plus n, then if A is contained in minus n to plus n then A is bounded obviously.

And how can you manufacture a covering so that minus n to plus n is one the elements in that covering. I can let vary n minus 1 to plus 1 minus 2 to plus 2 and go on expanding it. So, that will be a covering of the whole real line actually. And in particular it will also be a covering of the set A because A is a sub set of.

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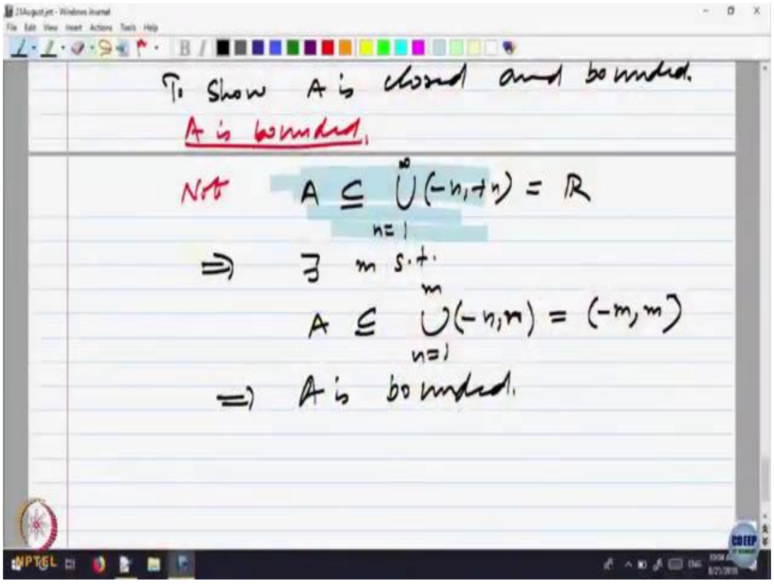


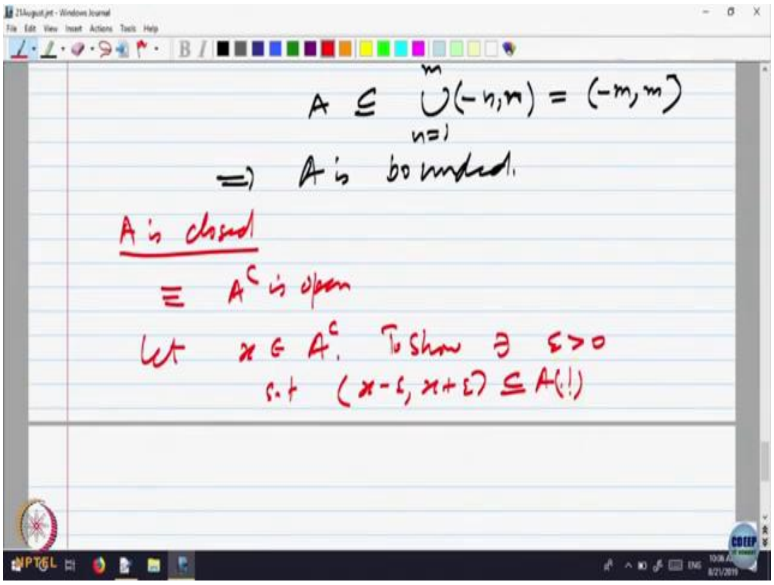


So, let me write, so note so this is how one thinks of a proof A is contained in union minus n to plus n, n equal to 1 to infinity which is contained in R or which is actually A is a subset of R and this is actually equal to y sub set this is equal to, this is equal to R. So, what you have done is we are given a covering of R and hence it is a covering of A also. But now this being a covering of this being a covering of this is a covering of A and all elements are open interval.

So, open sets, so it is open covering of A. So, by Heine-Borel property it must have a finite sub-cover. So, let us apply that, so implies, implies there exist some m such that A is subset of n equal to 1 to m minus n to 1m and plus m and that is nothing but minus m to plus m implies A is bounded.

How do we think of the proof, we want to show A is bounded it lies between some limits. So, and that should something like interval, and we should go to covering. So, we manufacture a covering, so that this particularly element and use Heine-Borel property. So, A is bounded let us show it is also closed, so A is closed.

Now here lies a very nice idea as far as the notion of distance is concerned, see I want to show A is closed I can just give the proof directly and be happy about it and but I just want to give feelings for how does one think of a proof if A is closed, how do I show something is closed? That 2 has 1 take a sequence and say it is converging somewhere and try to show that limit is inside.

But how is that our convergence of a sequence will relate it to open coverings then we got my tool available is every open cover is got a finite sub-cover some how I have to bring in those things. To show closeness I should bring in coverings somehow. Now, instead of, showing in A is closed it is equivalent to showing A compliment is open.

So, here is idea why A compliment is open is useful because openness is in terms of neighbor hoods every point, every point in A compliment if I want to show is open then I have to show given any point in A compliment there is open interval around it, plus somehow I am coming to some kind of a covering kind of a thing.

So, let us use that definition to show A to show, A is closed I will try to show A compliment is open. So, for that let us take a point x belonging to A compliment. To show there I guess some epsilon will be greater than 0, such that x minus epsilon to 2x plus epsilon is contained inside A. So that is what is to be shown. A compliment. Now how do I manufacture coverings from here every point I want to show this

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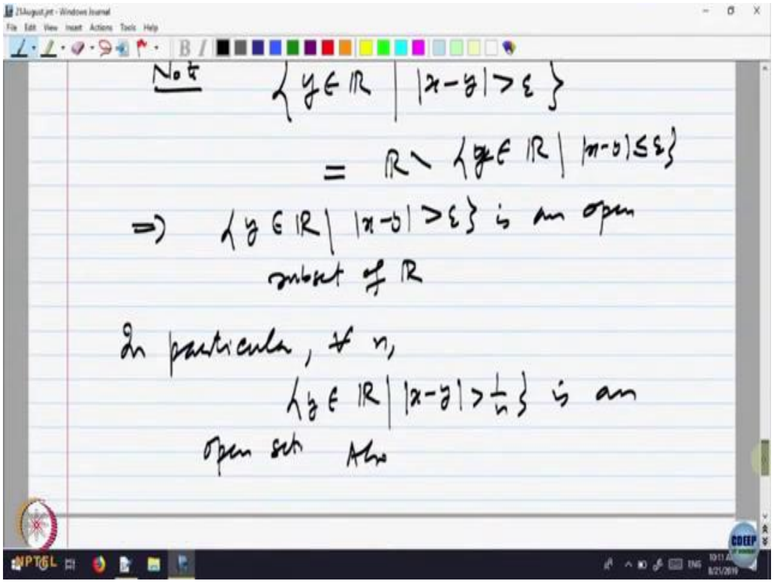

Now here is something which is very useful. So, here is a note how to get open sets? So, look at x is a point look at all y belonging to real line such that the distance between x and y, x is a point, if x is less than or equal to then it is a close set, it is a close ball. If I make it bigger than epsilon what kind of a set is this in the real line x is fixed I am looking at all y such that the distance of x and y is bigger.

Obviously it is a compliment of the close ball. So, it is an open set you can look at that way. So, this is equal to R minus y belonging to R such that x minus y is less than or equal to epsilon and that is a close set. So, that means, so implies that this y belonging to R such that x minus y is bigger than epsilon is an open subset, a general fact it is an open subset. You can do it also in R n, in R n also it is a look at all the distance between the x and y bigger than epsilon in R n that also is an open sub set of R n.

Because only the notion of distance is required. Now I have already got an open set and now I want to manufacture a covering, I want to manufacture a covering, so how do I go to the covering this is happening for every epsilon, this is happening for every epsilon. So, how do I go to the covering kind of a thing so that is the question? For every epsilon I want a family and how can you generate a family by varying epsilon? 1 over n.

So, in particular for every n y belonging to R such that x minus y bigger than 1 over n is an open set, x is fixed. Is it okay it is an open set, open set also what is a relation of this open set with A where was the point x? X was in A compliment.

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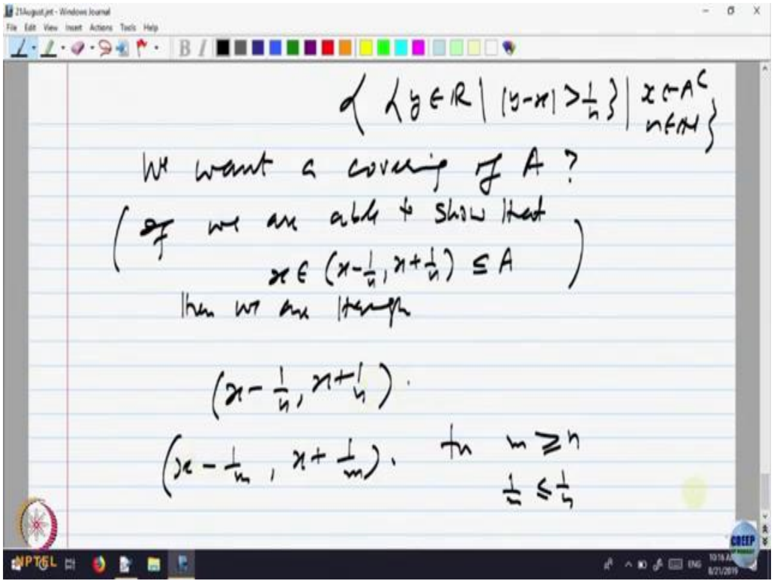


So, what is the relation of this with A compliment? Here is a A compliment can you say A compliment here is the, are the sets y belonging to R such that mod of x, y minus x is bigger than 1 over n. So, what is my aim? I want to get a covering of, see what is given to me? Given to me is A is I want to show A is closed and given to me is A is, has Heine-Borel property. So, that means every A has Heine-Borel property.

So, every covering of A has got a finite sub cover. So, what is relation between, so let me also let us look at this family these are sets where x belongs to A compliment n belongs to n. So, am I getting some kind of covering of air from this things or not? Or of A compliments something are you able to see something or not, I have got a set A, I am looking at A compliment.

So, if point is, so if x is in A compliment then all things which are bigger they are open sets and they and x is a an A compliment. So, from here I want to covering, so we want a covering of A from these things. So, what could be a family which will give me a cover of A getting something some idea anybody has? If I can show that the ball see this is what is this? This is a ball this is open set open interval around x of distance 1 over n.

So, now if I am able to show, so if we are able to show that x belongs So, x is an A compliment that is given to me if x belongs to x minus 1 over n x plus x 1over n if x belongs to this which is inside A than I am through, then we are through.  Because I want to show x is a interior point, so I want to there is a I am trying to show that this kind of things comes inside. So, I am trying to work backward now to indicate how is the proof is going on go n.

Now, look at this x minus 1 over n to x plus 1 over n this interval if m is bigger than n what is happening? If, what is the relation between this for m bigger than or equal to n what is the relation between these two intervals, which one is contained?

Student: x minus 1 by n to x plus 1 by n.

Professor: Because if m is bigger than n than 1 over m is less than or equal to 1 over n, both are centered at the point x. So, this is smaller than this one.

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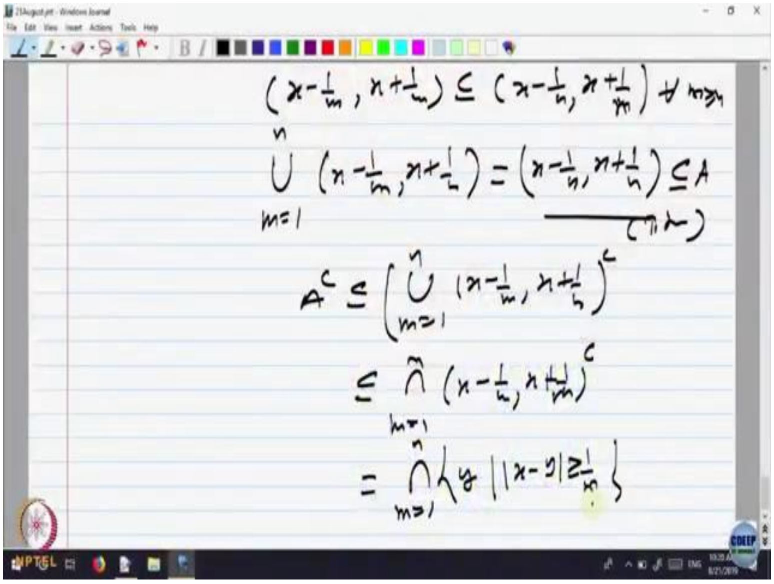

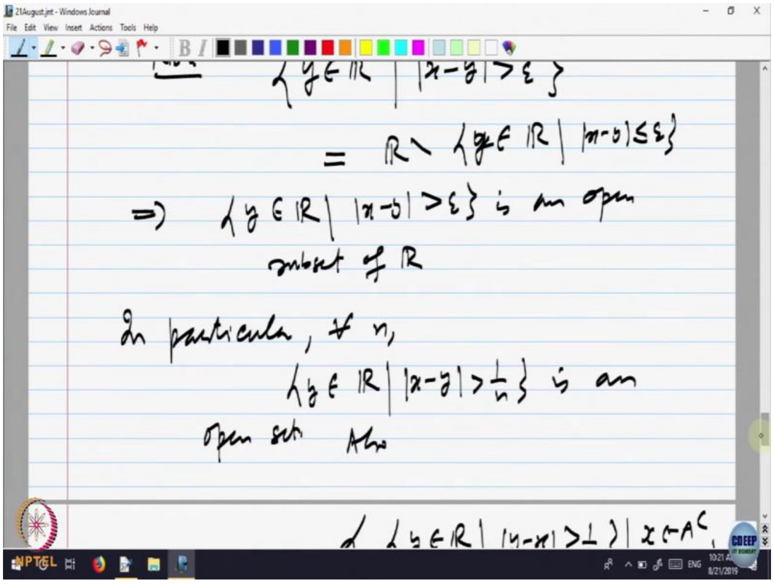

So, let us write than this, so for x bigger than 1 let us say x minus 1 over m to x plus 1 over m is inside x minus 1 over n to x plus 1 over m, n for every n bigger than or equal to n. That means the union of we take this union m equal to 1 to n x minus 1 over m x plus 1 over m. Actually that is equal to, I am just playing with intervals, and what I want to show, I want to show this kind of a thing I am just working out a argument which will work. So, I want to show this is contained in A. So, to show this is what I want to show. If I revert it that means what? A compliment should be inside.

So, where should be A compliment if this unions are inside A where should be A compliment in the union m equal to A this is inside this. So, x minus 1 over m x plus 1 over m compliment. If something is inside A compliment is inside the compliment of that, so inside intersection of m equal to 1 to n, x minus 1 over m to x plus 1 over n compliment of that set.

So, let me write compliment of this interval and what is this thing, so this is equal to intersection m equal 1 to n of what is this thing the same as y such that x minus y is bigger or than equal to 1over m, 1 over this is 1 over. So, you see those sets come here again here yes? So, let us now go back I want finite number of them should contained it, A compliment should be inside finite number of them.

So let us go back, so these were all open sets and here is, here is that, so y this is an open set. And x belongs to it singleton x belongs to it. So, A should be finite union of, so we want to show it is open. So, every point as a neighborhood is in, if I put bigger than or equal to than it will be a closed set.

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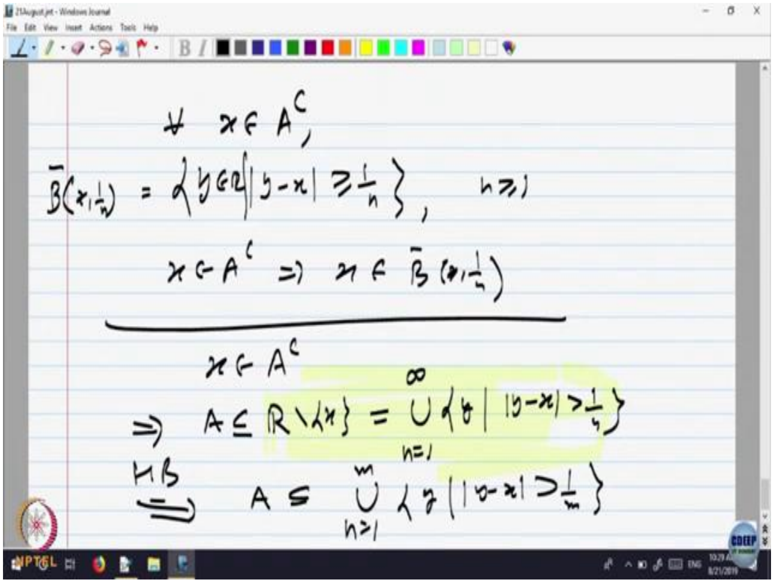

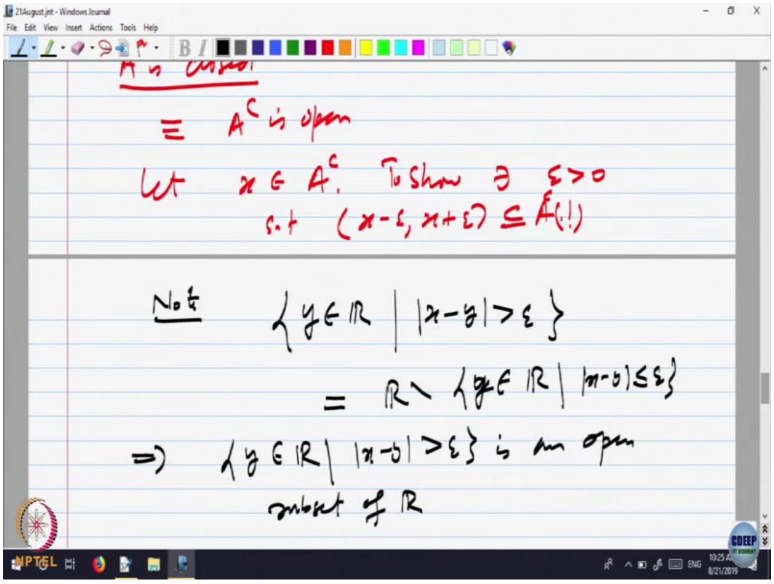

So, for every x belonging to A compliment we are looking at the sets y belonging to R such that y is bigger than or equal to, now mod of y minus x is bigger than or equal to 1 over n y is bigger than or equal to n and n bigger than or equal to 1. So, if x belongs to A compliment and if we call this sets, so this is the ball centered at x of radius 1 over n and the close thing, so implies x belongs to ball.

I think I will have to postponed the proof because see the basic idea is I want to show A is close that means it should be can I use that is open it is open. I have to go to a covering of A. So, how do go to a covering of A? A minus why do I vary? x belongs to A compliment, I think x belongs to A compliment that implies R minus singleton x A is a sub set of, if x belongs to A compliment A must be a subset of the compliment of the single ton.

Because x does not belongs to A. So, even if I remove it, it is still a subset, and what is this? This is equal to union, I think this seems to be, y minus x is bigger than or y bigger than or equal to let us just write bigger than  yeah I think this is what I was trying to reach 1 over n, n equal to infinity. Just look at the statement x belongs to A compliment, so A is a subset of R minus x if I will take any point in R minus x than why A is not equal to x if I take any point in y, in this than that point is not equal to x.

So, distance between x and y absolute value of x and y should be bigger than something. So, sufficiently large and it will be bigger than 1 over n. So, I am saying this two are equal, is that this is equality, is this equality ok for everybody? Because right hand side is union of subsets of R so that, so right hand side is a sub, and x does not belong to it, so that is inside this and conversely if I take a point y in R minus x there is a distance something and that distance will be bigger than or equal to, or bigger than 1 over n for some n.

So, the left hand side is a subset of right hand side, right hand side is a subset of left hand side. So, A now that is what I wanted to say that for A I have got a covering. Now A is compact and I have got these are all open sets we have already shown these are open sets they are the compliments of the closed intervals. So, there must be a finite sub cover there must be a finite sub cover. So, implies by lub property I was just trying to motivate that this should be, so this means A is a subset of R minus x but R minus x now it is gone we are directly using A is the subset of this union.

So, that implies A is n equal to 1 to m y mod y minus x bigger than 1 over m. Because not lub Heine-Borel property what is we are given is, Heine-Borel property. A is inside R minus x and R minus x is union of this open sets we got compliments of close intervals.

So, A must be covered by finitely many of them by Heine-Borel property compactness of, so if you this is what I was trying to motivate you that A should be inside finite number A should be finite number of the anyway. So this is think sometimes trying to explain to much complicates the things. So, it is simpler to say, this is very simple. So this implies what?

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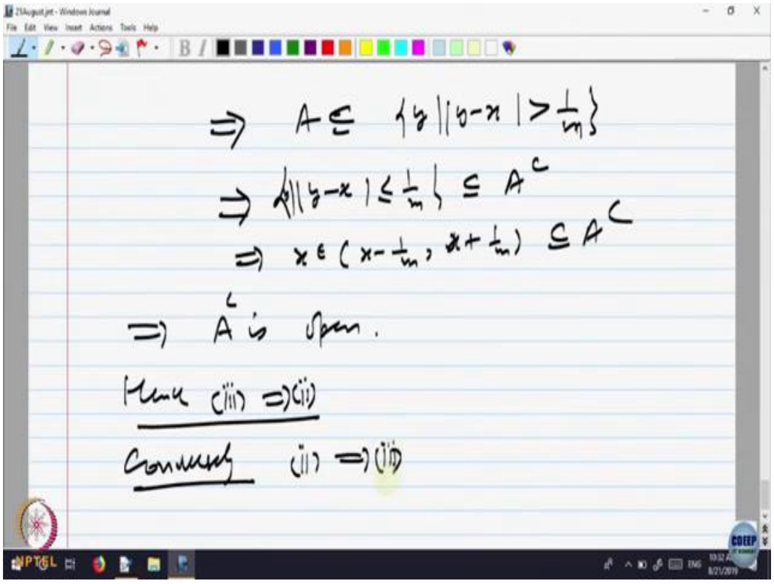

Now this bigger than 1 over m bigger than 1 over n, n equal to 1 to m what is this right hand side. So that means why A is a sub set of what is this union we just now observe for m bigger than n 1, this is smaller 1, So what is the largest 1. So, this is all y such that y minus x is bigger than 1 over n equal to 1 to m.

So this is m, 1 over m is the largest length possible, and that implies now go to compliment what is the compliment of this, that means, so y minus x less than or equal to 1 over m that set. All y such that in sub set of A compliment that implies the interval x minus 1 over m to x plus 1 over m, x plus 1 over m is inside A compliment and x belongs to it. So, the crucial thing is only through observe that these are all mod y minus x bigger than epsilon, for x fix epsilon fixed is an open set for every.

So, that proves that A is also, so implies A is, not an A compliment less than or equal, so imply A compliment is open, so that proves, so what we have proved is that if A has Heine-Borel property than it is closed and bounded. So 3 implies 2 hence three implies two.

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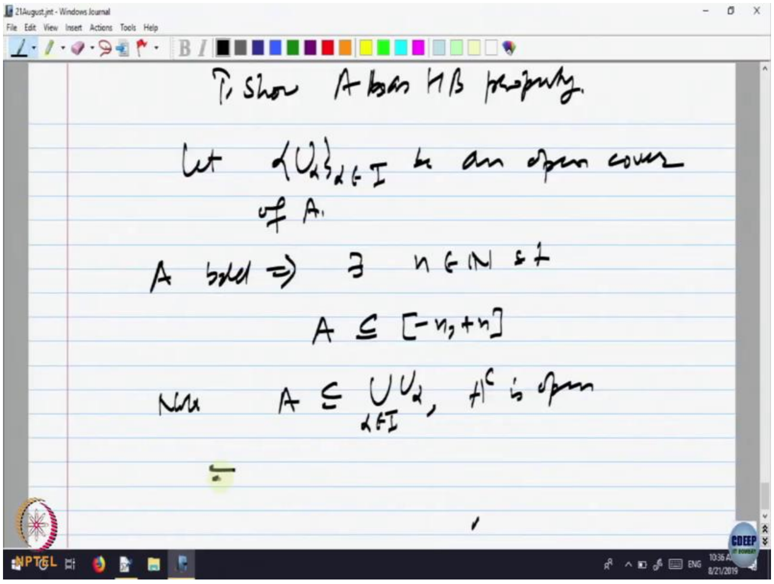


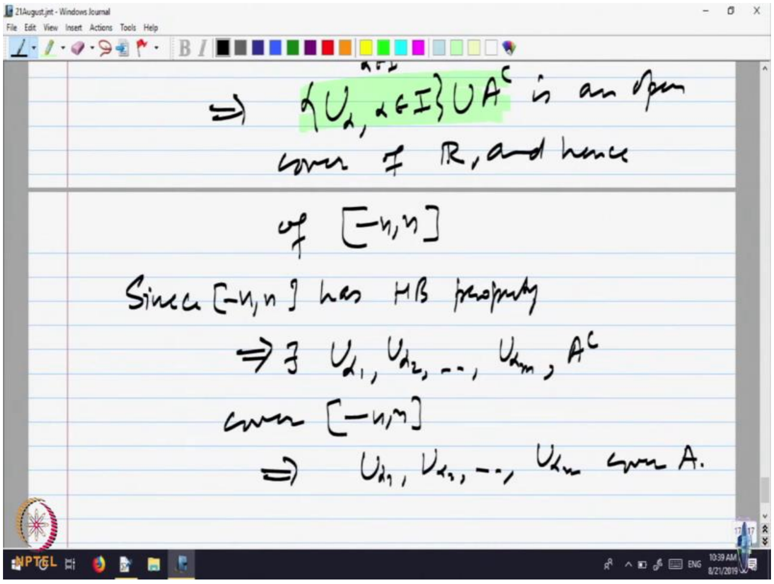


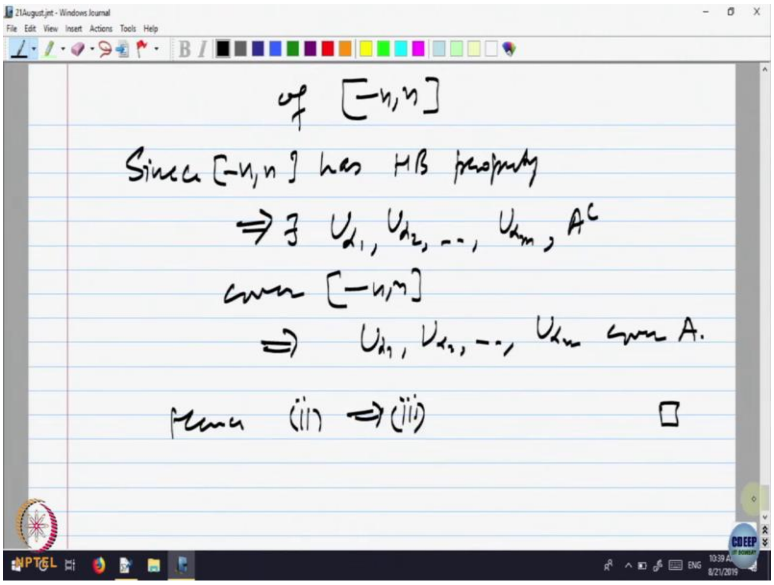

So, let us look at converse what I want to prove 2 implies 3. So, that means given is always good idea try to what is given what is two here given A is closed and bounded A is closed and bounded, to show A has Heine-Borel property. So, where do we start to show it as Heine-Borel property I should start with an open covering of A and get a finite sub cover.

So let say U alpha, alpha belonging to I be an open cover of A. So, I have to construct a finite sub cover. We only know A is closed and bounded if A were interval that we have already proved if A were an interval which has not proved in the beginning that if A is a closed bounded interval than it has Heine-Borel property but A may not be a close bounded interval, but it is a close bounded set.

So, that means what if it is a bounded set at least it will be inside a close bounded interval. So, A bounded implies there exist n long natural number such that A is inside minus n to plus n. Now, the aim should be from a covering of A, I should try to go the covering of minus n to n and then use the theorem therefore minus n to n I got a finite sub cover. So, from a covering of A I want to go to a covering of minus n to n for A I already have a covering, for A I already have a covering.

So, note A is inside union U alpha, alpha belonging to I. So, what I want? A is inside this A is already covered and I also know A, is so what is outside A that is not covered. What is not outside that is A compliment what can you say about A compliment, what can you say about A compliment? It is open. So, what is not covered by A it covered by A compliment which is open. So, covering of A and A compliment will give me a covering of everything in particular of minus n to plus n, so that is the idea.

A compliment is open, implies U alpha, alpha belonging to I this sets union with A compliment is an open cover of R and hence of minus n to n. Because minus n to n is a subset. How we use that fact but close bounded interval have Heine-Borel property.

Since minus n to n has HB property this implies I got a covering. So, that means, there is a finite sub cover, that finite sub cover of minus n to n may include A compliment may not include A compliment but positively it will have some finite sub collection of U alpha. So, implies U alpha 1, U alpha 2, so there exist U alpha m may be A compliment cover minus n to n. From this collection how do I get a finite sub cover keep A compliment and pick up probably some finite number of elements from the given cover.

So, that is what we have written. Now if I delete from this if I delete A compliment, what will be that cover? This cover minus n to n A is a subset of it, so that implies what the remaining one should cover A because this is only covering A compliment. So, implies U alpha 1 U alpha 2, U alpha m cover A. So, that proves that given a covering of A have got a finite.

So, the idea is boundedness puts it inside A interval and closeness gives me covering of compliment put together I get a covering, select a finite sub cover of that interval minus n to n remove A compliment from that because we do not need we need a covering of only A. So, remaining things should cover A. So, that proves so hence we approve that close bounded implies Heine-Borel property. So, we have proved a important theorem namely for a sub set of real line.

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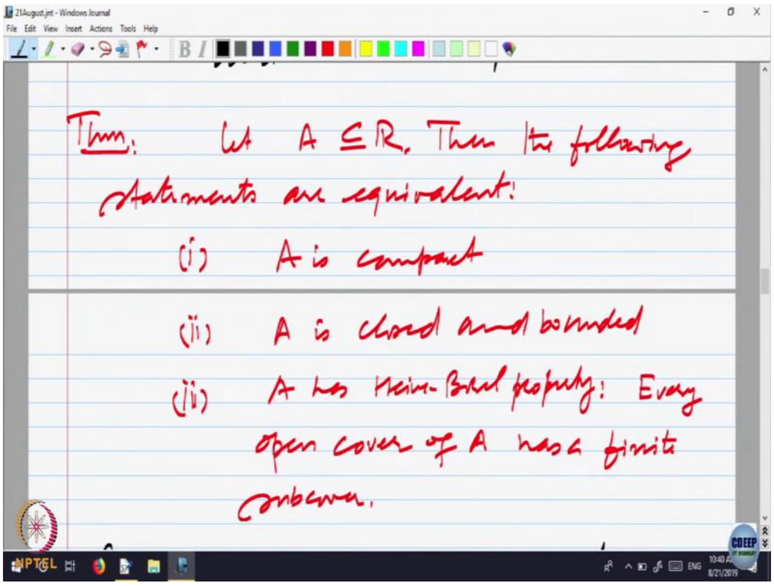


Say in this proof one, implies 2 and 2 implies 1 and we did not use the fact that we are really in the real line that is anywhere it is to R n is also true only proving two implies three and three implies two we heavily used the notion of distance that we are on the real line and the distance given by the absolute value, So, this proof actually works for Rn also, because in R n you can define a notion of the sequence closeness openness covering open covering but the proof become slightly more involved.

So we will not be proving those theorems for R n but with a remark that these theorems also, this theorem also remains true for R n, in fact we will define later on what is called the matrix base just we will define and is a generalization of R n basically notion of distance and this theorem remains true even for matrix bases, but the proves are slightly involved because there is no order there is no way of saying bigger than or less than.