Basic Real Analysis
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Lecture 20
Topology of Real Numbers: Connected sets, Limits and Continuity – Part II

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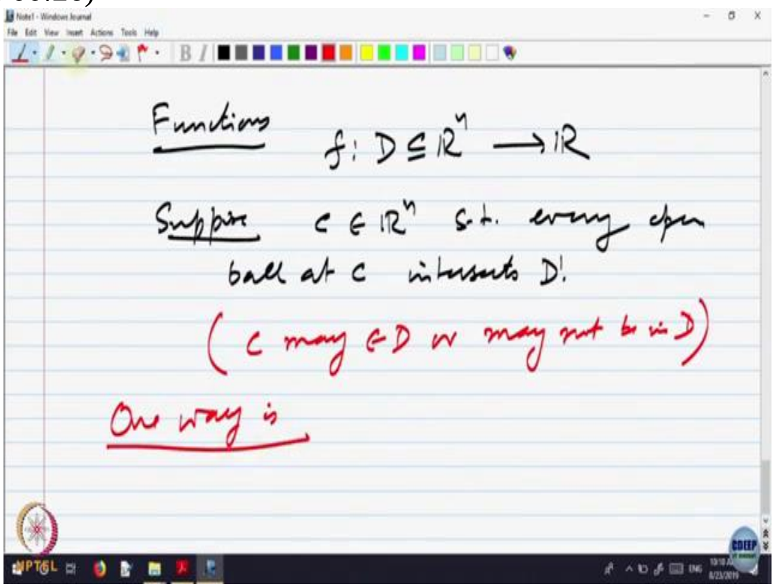

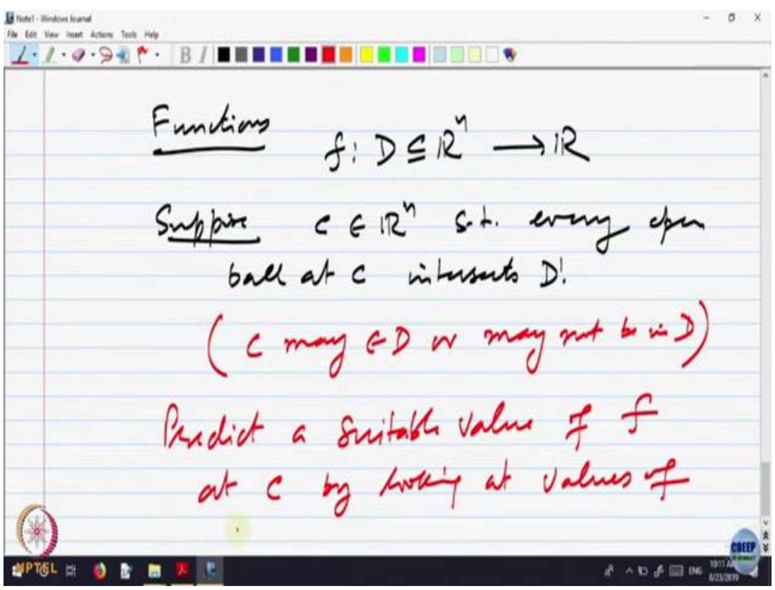

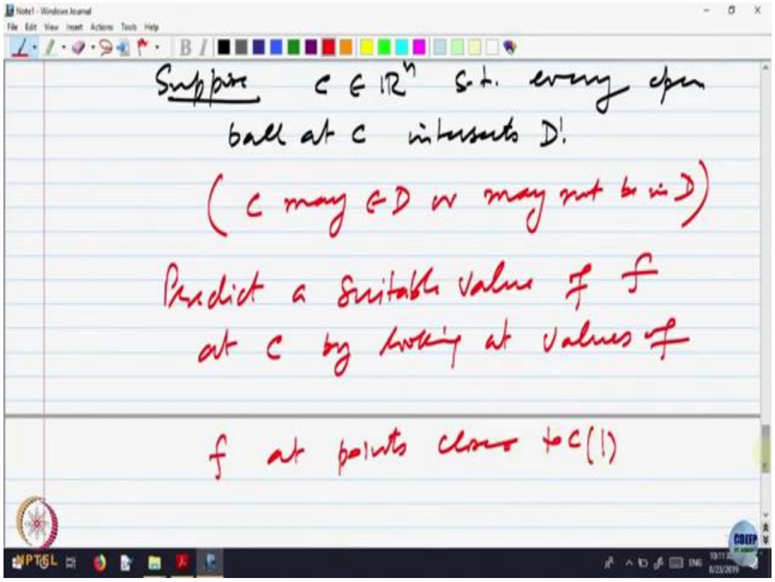

So, now let us start looking at functions defined on real line. So, we are going to look at now next, looking at functions, f will we defined in some domain, some set of say Rn to R. Let us look at what I want to do to motivate the at let us look at the following situation. Suppose C is a point belonging to Rn such that let me write every ball, every open ball at C intersects D.

So, this is I am assumption I am making so what does it mean, what I am saying is C is point so I think let me mention C may belong to D or may not be in D. f is function with the domain D, look at a point C, at point C may or may not be in the domain, function may or may not be defined we do not bother about it whether it is defined or not.

But the domain should have the property, there are points closed to C in the domain where the function is defined. So, one way of ensuring that is every neighborhood of the point C intersects D. for example at C I can take a ball of radius 1 over n small or any epsilon that should intersect D. So, points close to C are in the domain that is the assumption I want to make. The function is defined at points as close to C as you want.

But we do not know whether function is defined at the point C or not. So, that is the situation I am having. The question is by looking at the values of the function at points close to C where it is defined, can I predict some suitable value for the function at the point C? Is the question clear to everybody. So, let me repeat again, I want to look at the problem f is a function in some domain, I am given and C is a point which may or may not be in the domain it may or may not be define we forget whether it is defined or not.

But positively the function is defined at points closed to C how close, as close you want, function is defined. So, I know the behavior of the function at points close to C, that is idea I want to predict what could be the behavior of the function at the point C by looking at its behavior at points nearby.

So that is a prediction problem I being looking at, I am trying to predict a suitable value for the function at the point C by looking at its values at points close to it. So, what could be a way of doing that? One way could be that let us try to come closer and closer to C approach C. and close to C the function is defined, so look at the value of the function at points close to it, look at those images.

Those images if they are coming closer to something then that could be a suitable value for the function to be predicted. Is it a natural thing to do? Is it ok, initiatively very clear? So, how do I approach a point in D, I can take a sequence in D which is converging to C, then the points of the sequence will be coming close to C as close as you want, is it ok.

So, that is why I have put the condition that close to C every neighborhood must intersect. So, there are sequences which are coming to C. So, let us look at a sequence one way is, so I have not written what is the problem, so let me write the problem. So, the problem I am looking at is predict a suitable value of f at C by looking at values of f at points close to C. So, that is what we are trying to do.

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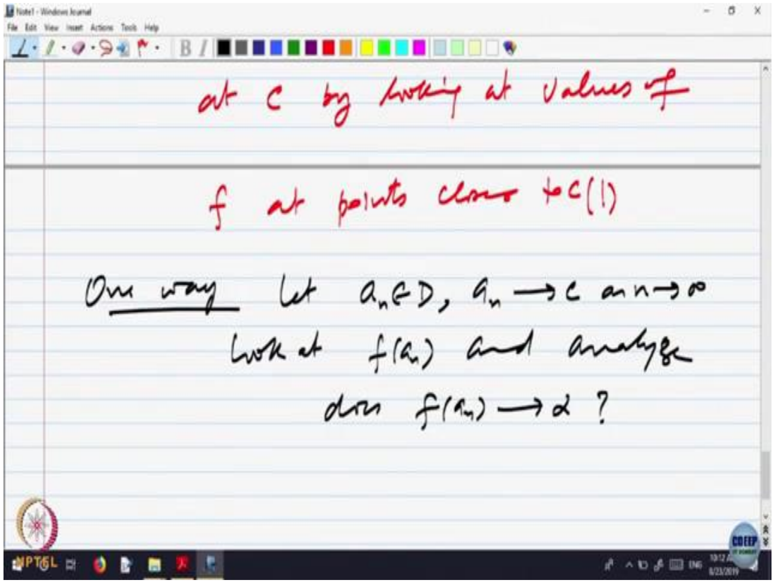

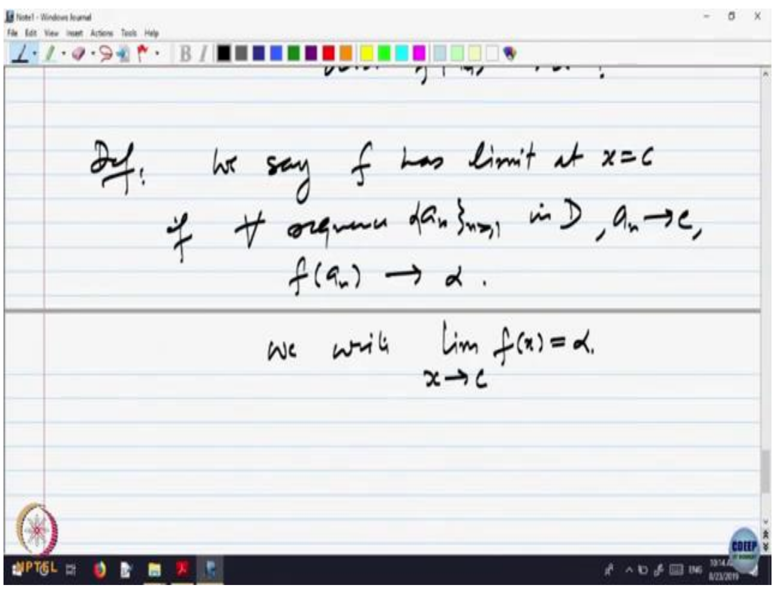

So, one way could be, so one ways let an belong to D and an come to C as. Look at, I am trying to predict a value or the function at the point C at a point close an, this is the value. So, look and analyze, does f of an come closer to a value, what is the meaning of that? Converges to some value alpha. So, what we are saying is, look at sequences in the domain which converts to C and look at the image sequence for those points f of an.

If they are converge to some value then that value could be a suitable value. As I am approaching the point C, I am approaching the value alpha. But it should not happen one sequence an is conversing to the C and Bn is also conversing but f of an actually converges to some value and f of Bn conversing to some other value.

Then will be in a problem which value is a suitable value. So, we should have for every sequence an converging to the point C, f of an should converge to the same value. Then that value will be a suitable value for the function at that point C. So, we are trying to predict by looking at the behavior at nearby points, we are not concerned what is a value at the point C, is it ok, clear the problem?

And if that happens one says the function has a limit at the point C. So, let us put a definition, we say f has limit at say x is equal C if for every sequence an in the domain an converging to C. f of an converges to alpha. We write limit x going to C f of x is equal to alpha. So, keep in mind limit is something to which the function is approaching as you approach the point in the domain.

So, naturally conditions are there should be a sequence at least one sequence converging to that point in the domain. Otherwise what are you going to predict, you are not going to predict anything. So, close to C there should be points in the domain as close as you want so that you can analyze the behavior of the function at points close to C.

And once the data is given I look at the behavior of the image sequences, if all the image sequences converges to the same value then we say that is a value the function should take and we say mathematically that function has a limit as x goes to C, clearly function need not to be defined at that point.

It is a suitable value for the function by looking at its behavior nearby. That is a way we should understand limit of a sequence and that is why we have understood for example for a sequence what was the limit, for a sequence an the limit, an need to be equal to C at all only they are coming closer and closer that was then sometime they may be equal to the value that does not matter.

But we do not bother about whether elements of the sequence are taking that value of where it is going to converge or not. Same for the function the limit is something which we want to predict a suitable value where the function is coming closer and closer to some value. As in the domain you come closer to the value C and that closeness mathematically is measured by a sequence in the domain converging to C that is an is converging to C, so an are coming closer to C, f of an are coming close to some value alpha whatever be.

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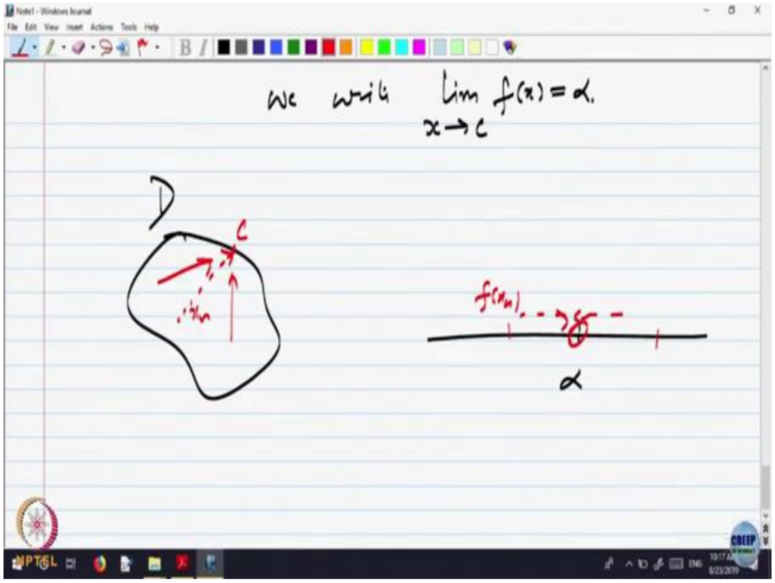

So, if you want to look at a picture in Rn say R, so this is the domain and here is the value alpha, so where any point. So, for example the point C could be here. So, if you take a sequence xn which is converging into C so look at f of xn, it will be here, it could be here, but they are all coming closer to the value alpha.

And you see in Rn there could be many ways of approaching C, there could be many paths. But whatever way whichever way xn is converging, we know how to xn converge, absolute value of or the magnitude of xn minus C goes to 0 we know that, the distance goes to 0. So, path is not important, it is the closeness which is important and incase the function is defined at the point C.

Supposing it so happens that the point C is in the domain that means the function is given some value. That may or may not be same as the value that we are trying to predict. So, if the value we are predicting for the function at the point C by looking at its behavior at nearby points happens to be same as the value or that function at that point C then it is natural to say there is a continuity in the behavior of the function.

Whatever we are trying to predict function behaves nicely there is a continuity in this behavior. So, that gives us a notion of what is called continuity of a function at a point if the value we are predicting, is the actual value taken by the function. So, continuity of the function at a point means the point is in the domain that means f of C is defined and for every sequence xn converging to C, the limit is equal to f of.

So, value predicted is the actual value taken that is how we should understand what is called the continuity of a function at a point. All of you have done this limits and continuity in your previous courses but just this perspective you should keep in mind.

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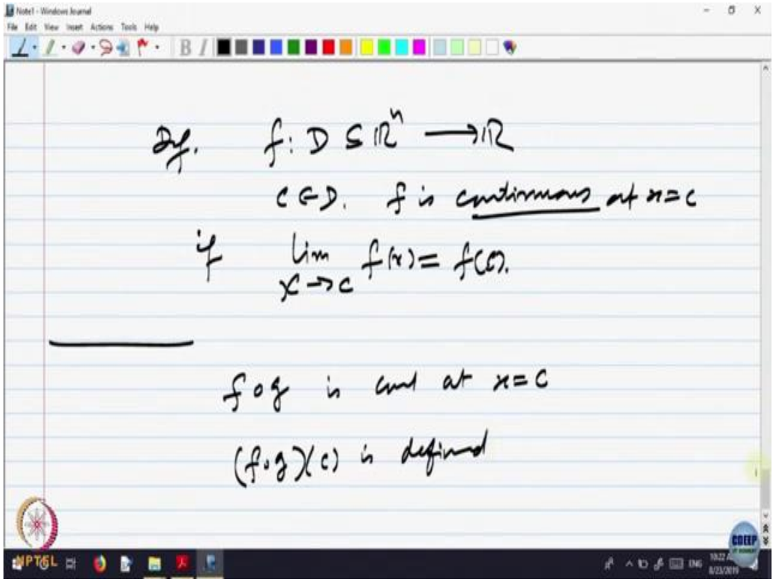

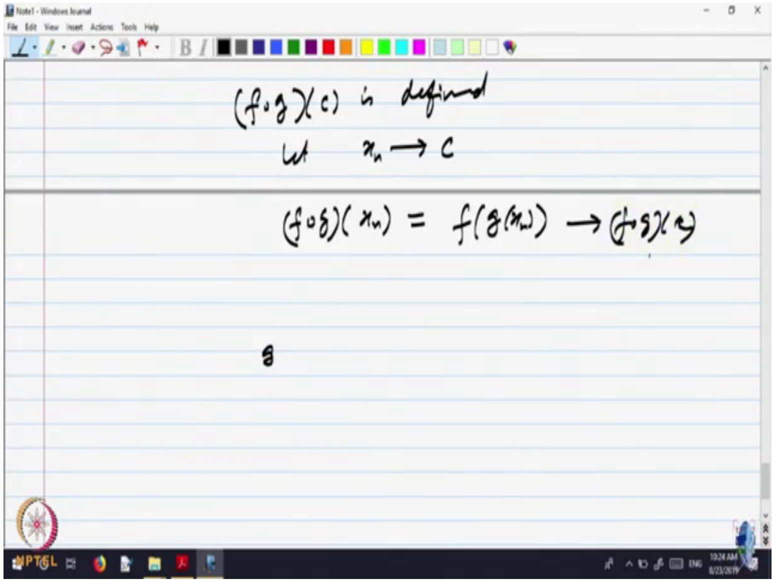

So, let me define, so limit we have defined. So, let us define what is called, so definition f defined in a domain in Rn to R, C belongs to D, f is continuous at x is equal to C if limit x going to point is C, f of x is equal to xs and is the value or the function at that point f of C. The value predicted is the actual value taken. So, that is continuity and now you all must have done this kind of theorems namely if the limit. You see now what we are looking at, we are looking at some property of functions now.

So, our aim… our domain of attraction or investigation is the class or functions defined on real lines or Rn. Now you can add functions. So, what are the operations possible on this class, you can add functions, you can multiply functions, you can divide functions, you can compose functions.

And the theorems are over the limits which we have already done. If f has a limit at a point C, g has a limit at a point C then f plus g has a limit at the point C and the limit is equal to sum of the limits. Similarly difference product, you can do also coefficient, you have to keep in mind limit of coefficient is not equal to 0 as in sequences we have done that.

So, those are called algebra of limits, limit of the sum is equal to sum of the limit, limit of the product is equal to product of limits and so on. I am just recalling you what you what you have might have already done. So, translate this for continues functions. If f and g are continues, f plus g is continues because the limit is equal to sum of the limits. Product of continues functions is continues.

Composition of composite functions in continues. Let me just indicate. So, supposing f is a function and such that and g is another function that f composite g is defined on some domain. I want to say that is continues at x is equal to C. So, f compose with g at C is defined. So, that is assumption, for composite of function is not always that I have composite g is defined whenever the range of g is in the domain of f it will be defined.

To show continuity let xn converge to C, we want to analyze f composite g of xn, what is that equal to by definition? It is f of g of xn, and where is xn converging to C. But I should have now g of xn converging somewhere otherwise I cannot say anything. So, condition is if f composite g is defined and g is continuous at C where domain and f is continuous at the g of xn then this is continuous and this will converge to of composite g of C.

So, appropriately you can out conditions, f composite g, so what should happen? I am looking at the point C limit at the point C. So, first of all… composite g should be defined, first g is operating and then f is operating. If xn is converging to C, look at the image that is g of xn that should converge to g of C.

So, g of xn is converging to g of C, f is defined there and if f is continuous at the point g of C then f composite gn will converge to f composite g at the point C, is that ok. So, those are the appropriate conditions that you should put by looking at what is to be done. So, conditions are not put just for the sake of being happy, it is because each one is required at some stage, is that clear to everybody what I am saying, all of you have done this but I am just revising and trying to make you understand, how you should understand something.

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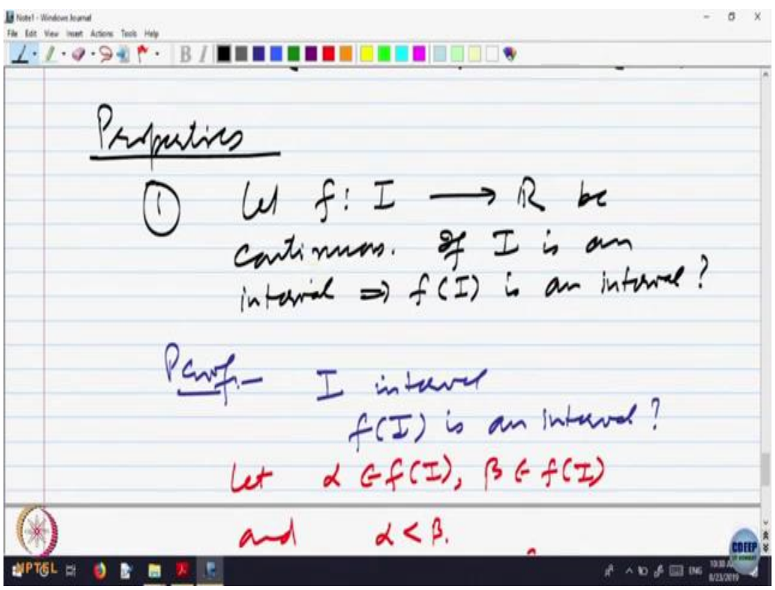

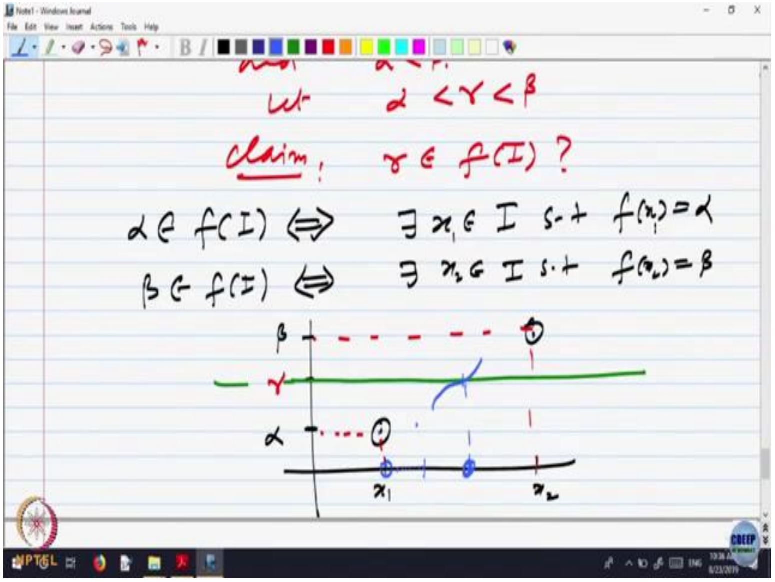

So, we will not spend time on composition of, on this properties of continuous function and so on. What we were going to look at is, some properties of continuous function. So, let me look at algebra of limits, algebra of continues functions as we are assuming, that you have all gone through and if you are not gone through or you would like to be more comfortable do it again, look at the proofs of limit of the sum is equal to sum of the limit and so on.

Be happy so that you understand, be I think is other way around. Understand and be happy, you will be always be happy if you understand. So, properties so what we were going to look at is, we are going look at properties of continuous functions, vis a vis special properties of subsets of real line.

For example if the function is defined on an interval, domain is a interval. What can I say about the range or the function, can I say it is a interval? If the domain is the interval can I say the range is interval or another property if the domain is a closed bounded interval can I say the range should be a close bounded interval. So, such kind of properties we want to look at of continuous functions.

So, continues functions, vis a vis special properties of subset of the domain. So, the first one is let f be a function defined on I to R, where f is continuous why write where be, so let f be continuous such that be continuous let me just write there is no such that meaning if I is an interval implies f of I is an interval.

I want to look at this property. So, let us see how does the proof work, I want to look at, so I interval, f of I is an interval that is a question we are looking at. How do you approve f of I is an interval? How can I prove? The only way we know is take two points in that set, take a point in between that is you were also be in that set. So, let us check alpha belong to f of I, beta belong to f of I and either alpha will be less than beta or, so let us write and alpha less than beta.

Let alpha less than gamma less than beta. Take a point in between, we are taken two points in f of I and we are looking at a point in between, claim we want to analyze gamma belongs to f of I, now what is the meaning of saying alpha belongs to f of I means what, trying to understand that means alpha is in the range of the function.

You see what we are trying to analyze the range and we know only something about the domain that is a interval. So, somehow I have to shift my attention to the domain and use the fact that domain is a interval. So, the proof has to be in such a way, I have to drive my proof in such a way that it goes to the domain somehow and then use that fact.

I can do something there only, so how do I go to the domain, it belong implies there is some x belonging to I such that f of x is equal to alpha, beta belongs to f of I that is same as saying there exists a point so let us call it x1 and call it f of x2 there is a point x2 where the value beta is equal to beta.

At a point x1 because it is alpha is in the range so some value, some point x1 should be mapped to alpha other point x2 should be mapped to beta. Now x1, x2 both are in I, so now the idea is, so let me draw a picture probably that we, I am going to give you a picture which is only sort of real line. So, here is the domain, here is the function, here is x1 here is x2, here is the value alpha and here is the value beta and here is the value and gamma is in between.

So, here is the point gamma. Now, I can directly work with alpha, beta, gamma and so on straightly proof will be understandable more, does not matter. So, let me I think probably do it straight away here itself, let me write, so draw this line. So, what I want to show, aim is to show that gamma belongs to f of I that means what, there is a point in the domain where the value gamma is taken, is it ok.

So, we want to show there is, this is the value gamma and what is geometrically if you look at what is the meaning of saying that the function will take the value gamma somewhere. I am institutive notion of a function, if you will have the picture of the function, what does it mean? What is the picture of a function?

You all done calculus, what is the picture of a function, that is the graph of the function, graph is the picture or the function and if you want to say that the value gamma is taken somewhere by the function that means the graph must intersect that line somewhere that is a geometric way of saying.

So, you want somewhere the graph should cut somewhere at a point, that is a point we are looking at, that is the point we are looking at. So, how do I capture that point that is the question? I want to capture this point in the domain. So, how do I capture it that is the question? So, now you see one way of capturing this point is, see look at this point this value alpha x1.

So, what I can do is I can start looking at points bigger than x1 where the value is this value alpha is less than gamma. So, look at all the points where the value is less than gamma, and try to look at the largest of these values in the domain. See I am going to look at the domain only, is it ok.

So, look at the largest so possibly that value, so at this point the value still remains below gamma, below the line, at the point next also it remains below gamma. So, look at the largest possible value in the domain bigger than x1 where the graph stays below that line. So, if I have the largest value that means after that it will attached to crossover otherwise it will not be the largest. So, that is one way of analyzing the proof. So, what should be we do, so the proof starts by looking at.

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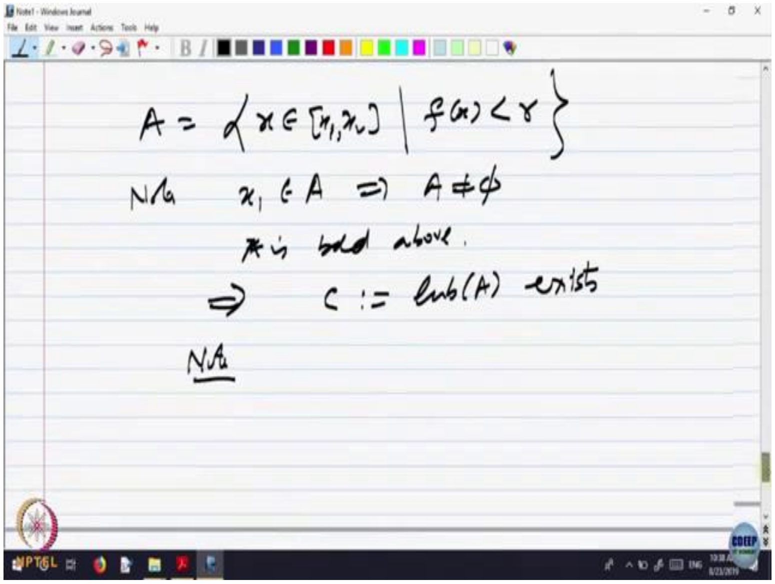

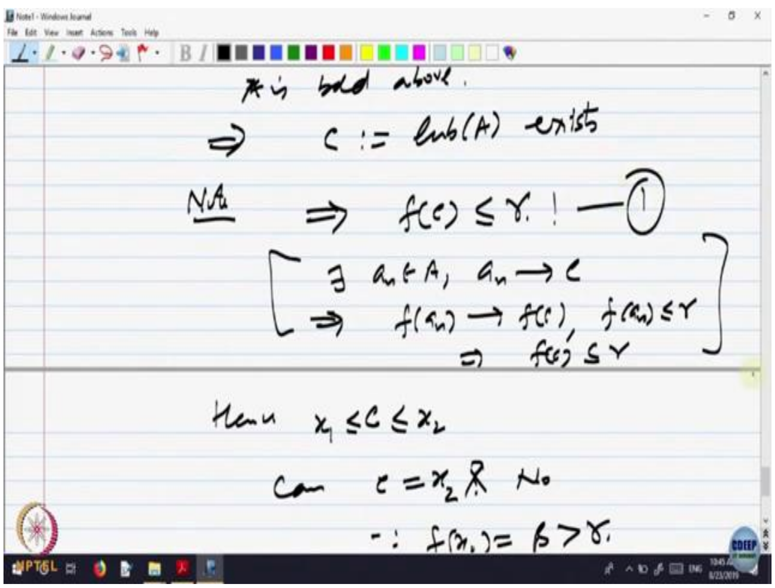

So, let us look at the set A, all x belonging to this x1, x2 such that f at x is less than gamma is it ok. So note, x1 belongs to A, because the value at x1 is alpha. So, A is non empty I am trying to look at the largest value, I have to ensure the largest value exist, so how do I ensure the largest value exist? The only way I know is lub property of real line.

So, x1 belongs to A implying A is non empty, A is bounded above, why it is bounded above, because it is inside x1, x2. We are choosing points between x1 and x2 only so it is bounded above. So, implies gamma equal to lub, not gamma something else I should write because gamma I already used so let us write C equal to lub of A exists. Now where you see note, what is the value of the so I have taken and I have gotten a point C which is lub of these points, you have the values less than gamma.

My aim is to show the value at the C is equal to gamma, value at the point C is equal to gamma. So, first of all look at note C is lub of something so I am saying this implies f of C is less than or equal to gamma. Why is that C is lub of a set A, in the lub of set A there must be a sequence in A converging to lub, there must be a sequence converging in lub.

So, I am not writing the proof I am just try to understand what I am saying, C is lub so that means what, that means there must be a sequence in A converging to lub, we have already seen that, C minus 1 over N cannot be there must be a point all that we have done. So, there is a sequence in A converging to C.

So, there is a sequence, so what is the value at that point or the sequence. So, what is the value at that points or the sequence they are in A. So, f of C is f of that element of the sequences is less than or equal to less than gamma, f is continues, so image must have the property so let me write here, their exist a sequence an belonging to A, an converging to C implies f of an converging to f of C because of continuity, f of an because it belongs A is less than or equal to gamma implies f of C is less than or equal to gamma that is a proof. So, I used continuity here.

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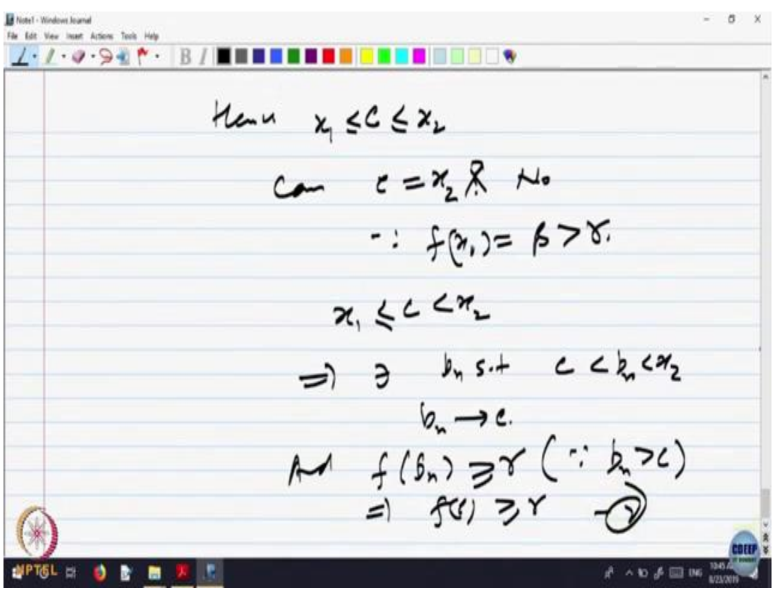

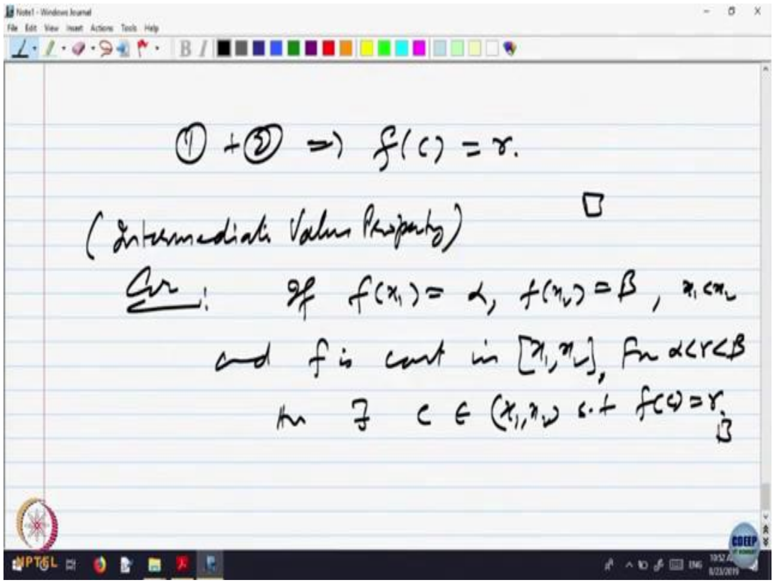

So, now what is the possibility. So, hence so what we were saying is that point C is bigger than or equal to x1, is less than or equal to x2. It is bigger than or less than because x2 is a upper bound and this is lub. So, it has to be less than or equal to x2. Now, the question is can C be equal to x2?

What is the value if C is equal to x2 what is the value at x2. That is beta which is bigger than gamma that cannot happen because just now I said f of C is less than or equal to gamma. So, this C cannot be equal to x2, so let us write no because f of x2 is equal to beta is bigger than gamma.

So, x1 less than C less than x2, it has to be but still we want to show that f of C is equal to gamma, it is less than or equal to gamma. If I can show it is also bigger than or equal to gamma we are through. But this implies, there is a sequence bn such that C is less than bn and less than x2 and bn converging to C, is that ok.

Because if C is less than x2 then there is a gap in between I can always have a sequence coming to C from that side. And where are these bn and what can say f of bn, bn is on the right side of alpha which is the supremum, where the value of the function is less than, so it has to be bigger than or equal to, that is only possibility. Alpha is the lub of all points in x1 x2 where the value is strictly less than gamma.

So, something bigger than lub at that point the value has to be bigger than or equal to gamma, is that ok. Because bn bigger than C, is that ok or not ok here look at the graph you will understand it better. So, here you see C and less than x2 so there is b1, and so on, bn converging to C. What can be the value at this point, it has to be bigger because what is C, C is the least upper bound of all those points where the value is less than gamma.

So, something bigger than that least upper bound the value has to be bigger than or equal to gamma. So, these are points here, in this file. So, that means what so implies f of C by continuity again is bigger than or equal to gamma because f is continues. Each element of the sequence b and f of bn is bigger than gamma, bn is converging to C so where does f of b and converge.

It converges to f of C by continuity, each bn is bigger than or equal to gamma so limit has to be also bigger than or equal to gamma. So, this is second and the first one was it is less than or equal to. So, this is one and you can call this as two if you like. One into imply f of gamma is equal to, f of C is equal to gamma sorry not 0, f of C is equal to gamma and that finishes. We are not doing anything surprising very naturally, what we are saying is here and if you look at the picture at this point the value is alpha.

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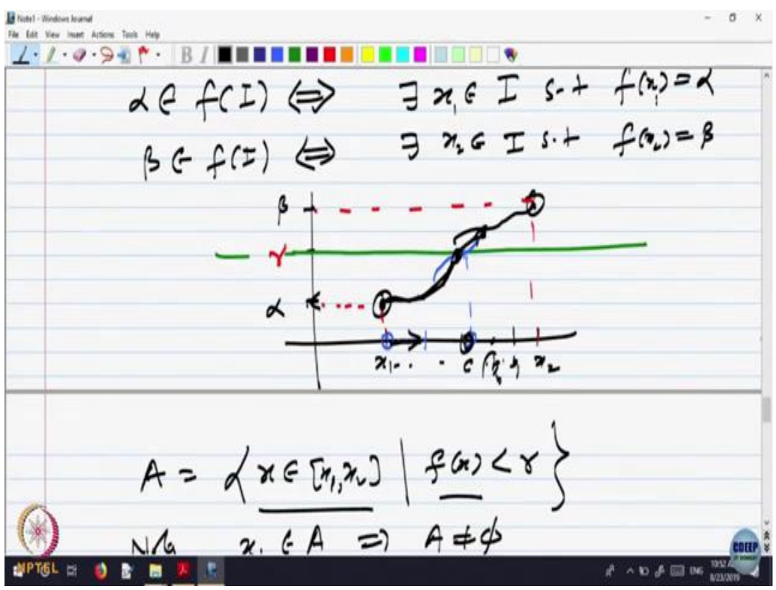

So, let me start travelling along this route, go on moving till you remain below the graph and look at the largest value of this where you stay below. That must be the point where you crossover, that must be the point that you crossover. So, written mathematically look at all the points in x1 x2 where the value f of x is less than gamma, that is a non empty set less than or equal to gamma, yes that is non empty set.

It is bounded above by x2 so it must have a least upper bound and that least upper bound is the required one. Because everything on the right hand side will be bigger than or equal to, here it is less than or equal to that all it is equal to gamma. So, that is one way of proving that if a function takes two values alpha and beta then it takes every value in between.

Another way of saying the same result what we have shown, if I is a interval f of I is a interval, but what is meaning of saying. If I is interval, alpha and beta are two values then every value in between also has taken that is what are showing. So, this theorem can also be interpreted as saying for a continues function if two values are taken alpha and beta then every value in between also must be taken at some point.

So, in that way it goes by the name of intermediate value property for continues functions. So, the proof that image of a interval is interval also gives you what is called intermediate value property, it is same. So, corollary if will… what is the doubt, where is bn? It is on the right side, bn is on the right side of C and what is C, bn is not belong to A, who said bn belong to A bn is the set of those points in a where f of x is less than gamma.

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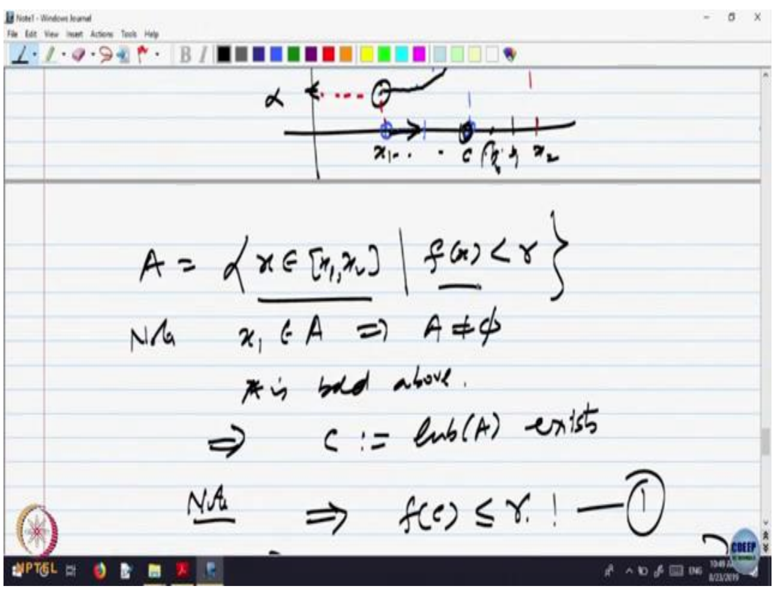

It is not all x1 x2. Look at the definition of A, what is A? x belonging to x such that f of x. So, for all points of A f of x is less than gamma and we are taken the supremum of it, supremum of A is alpha, so any point on the right side of alpha cannot be an element in A, because alpha is the least upper bound of A.

So, if it is not in A or any point f of x is less than gamma. So, it is not it is outside that means what, the value should be bigger than or equal to gamma. So, for all point on the right side of C the value is bigger than or equal to gamma only we have to ensure that there are points on the right side and for that C cannot be equal to x2.

Because f at C is less than or equal to gamma we have just now shown, f of C is less than or equal to gamma so C cannot be equal to x2 because the value at x2 is beta which is bigger than gamma. So, there has to be a gap in between that means C has to be strictly less than x2. So, it is interval in between. So, take any sequence converging and that does the job.

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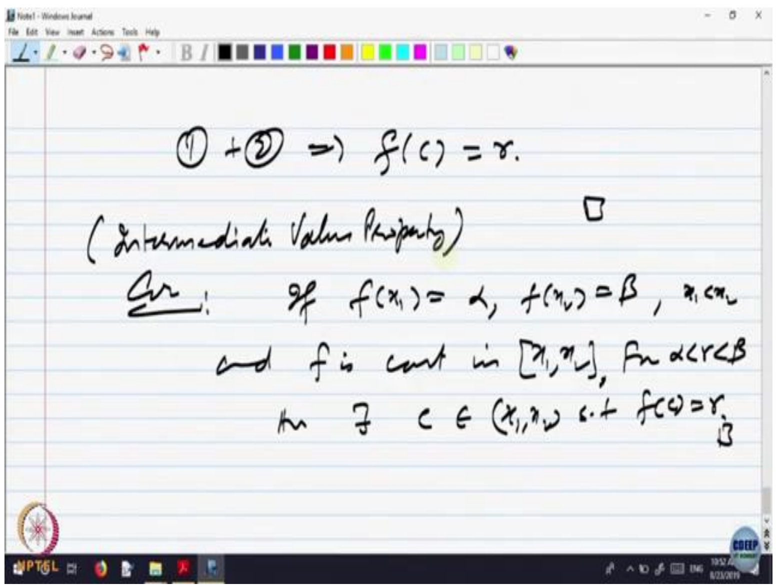

So, I was writing another way of this or a consequence of this whichever way you like. If f of x1 is equal to alpha, f of x2 is equal to beta and f is continues in x1. I am not saying x1 should be less than it could be other way around, x2 could be less than x1. So, let us say x1 less than x2 or it will be other way around then there is a point C belonging to x1, x2. Say that f of C and gamma is graph continuous for alpha less than gamma less than beta there is a point C such that f of C is equal to gamma.

So, if a continuous function takes two values alpha and beta it must take every value in between. So, that is what is called intermediate value property. This is very useful result, for example at some you want to locate where does the graph of a function, let us first interpret this geometrically, geometrically they says if you are here and you want to go here, there is a value at alpha, there is value at beta.

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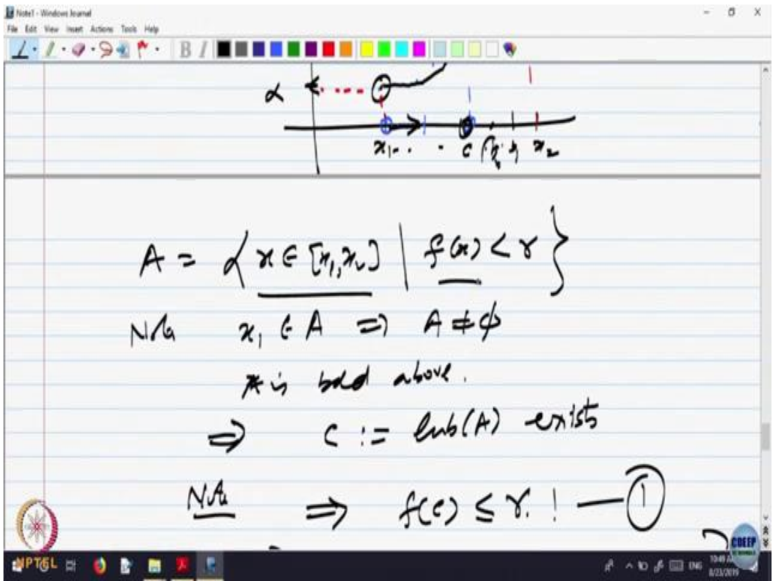

If you want to go and every value must be taken, this value will not be taken if you say that you go up to here and then you start your graph somewhere else. So, geometrically saying intermediate value property holds for continues function means the graph of the function has no breaks.

That is a geometric interpretive of this, so continuity of a function at a point means if you start somewhere and domain is a interval then and you end somewhere, then once you start drawing the graph you should not life your pen or pencil whatever you are doing to draw the graph, you should continuously go on doing.

There is no break in the graph of the function, this is why it is important in calculus when you want to get a picture of the function there is a first tool which gives you a picture of the function. Namely continuity implies there is no break in the graph of the function.

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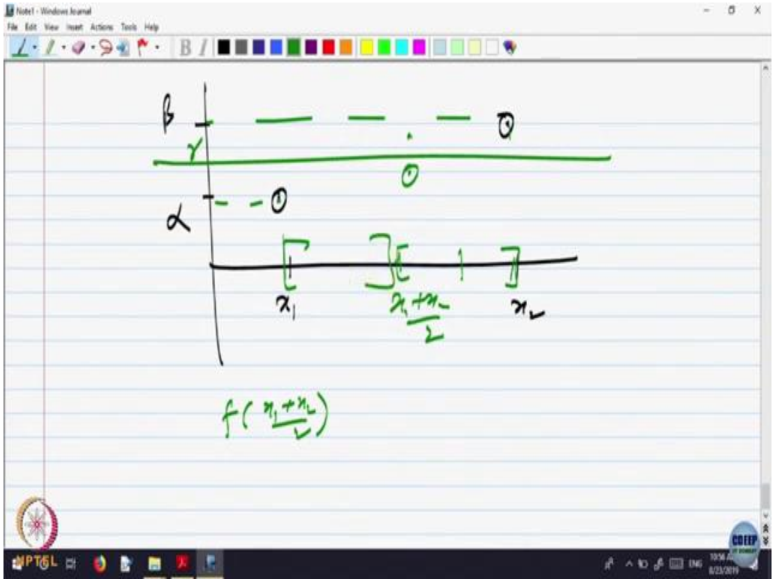

So, let us look at some, here is another way of proving this that is interesting. So, let us just look at x1, at x2 here the value is some alpha here the value is something beta, and here is gamma in between. See we are trying to locate a point in between x1 and x2 where things crossover. So, another very intuitive way of doing this is the following, at x1 the value is alpha and x2 the value is beta.

What I can try to do is look at the midpoint of this and look at the value at the midpoint. So, look at f of x1 plus x2 by 2, what are the possibilities. Either it is equal to gamma that is a place where actually the function crosses one possibility, you are very lucky or it is below or it is above. Only three possibilities if it is cutting at that point the line gamma then we are through, if not let us assume it is below.

So, let us assume the value here that is still below, now value at x2 is above. So, now let us only concentrate only in this part of the graph. The original one at x1 would has below, x2 it was above. Now, I have shrunk my vision, I know I do forget about everything else, look at only the midpoint and x2 at the midpoint the value is below, at x2 value is above.

So, you see now I am trying to capture that point kind of thing. Now again I will look at the midpoint of that again possibility same either I hit the jackpot, I get the value or it is below or it is above. So, if it is below I continue that process and go on shrinking. Now, if at this stage, the value is above, supposing at this stage I assume the value is below.

Supposing it was above then I will consider only on this part. So, what I am doing is at x1, x2. x1 the value is less x2 the value is bigger, cut it into half and keep only that half at one end point the value is less, the other end point the value is more again cut it into half and see which one is working.

So, go on shrinking this interval. At one end point the value is less, the value is more so what you get? You get an nested sequence of intervals. Such that at the left end point the value is less at the right end point the value is more and the intersection of all of them has to be a single point. What will be the value at the single point?

It has to be equal to gamma because if I look at the left hand points, it should be less than or equal to gamma, if you look at the right end points it should be bigger so they have to be same. So, this is called the bisection method of capturing a point, making it smaller and so you can write down the, I am not writing the proof it is second proof, you can try to write out the proof yourself.

The basically the idea is x1, x2 at x1 the value is less, x2 the value is bigger cut it into half either this half or this half will have the same property again at the left end point the value will be less, at the right hand point the value will be more, go on doing it if you do not hit the jackpot in between then eventually you should capture that point.