Basic Real Analysis
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Lecture 22
Continuity and Uniform continuity – Part 1

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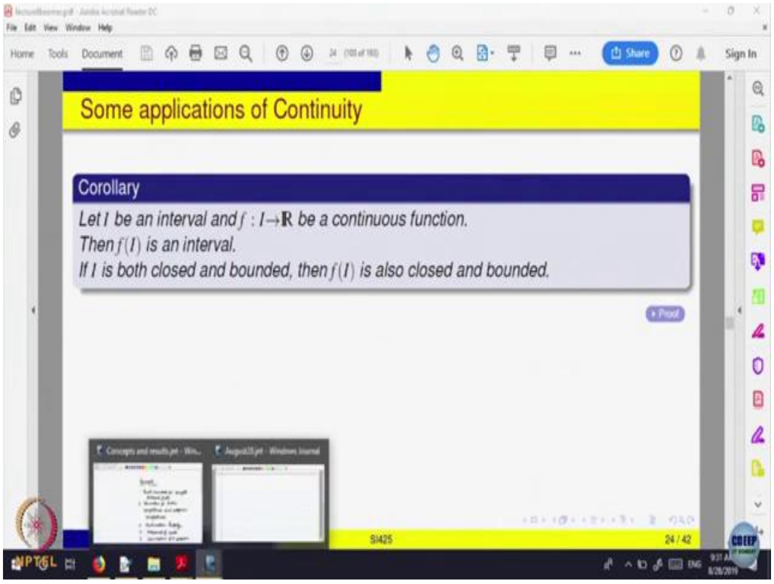

So, last time we were looking at a properties of continuous functions and we proved that as a consequence of intermediate value property that if, I is an interval the domain is an interval than the, arrange of the function is also an interval. When started looking at the special case, when I is a closed bounded interval what can we say about the range so let us look at that.

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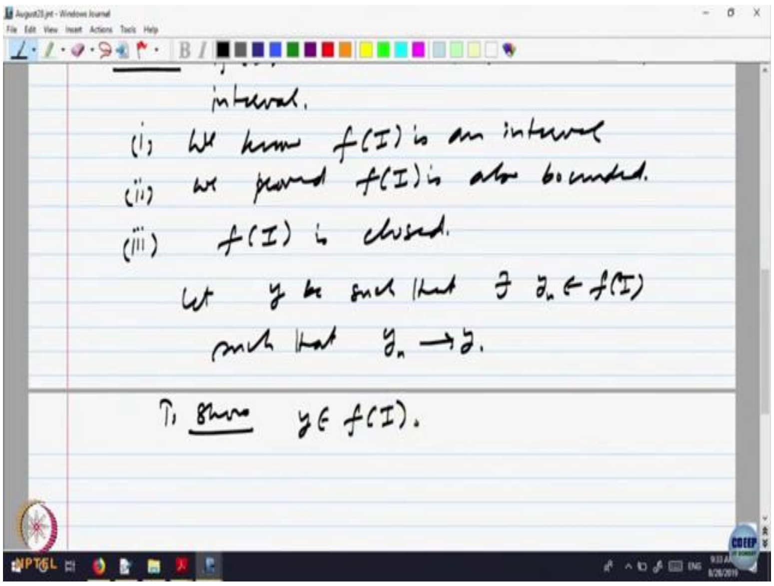

So, f is defined on I and taking values in R let us say, I is a, b closed bounded and f continuous. So, claim is that f of I is also a closed bounded interval. We already know, 1 we know f of I is an interval so let us and we had proved the last time that it is also bounded. So, we also proved, we proved f of I is also bounded and the basic idea was that if it is not bounded then there is a sequence in the range which is goes to absolute value of that goes to infinity. Look at the images pre images that being in I which is closed bounded should have convergent of sequence.

But that is not possible because if it has a convergent of sequence then the image must be convergent but that is unbounded so that cannot converge. So, that was idea we had proved that so let us prove the third that newly f of I is closed. So, let us look at y be such that, there exist a sequence yn belonging to the range there is a sequence yn belonging to f of I such that yn converges to y.

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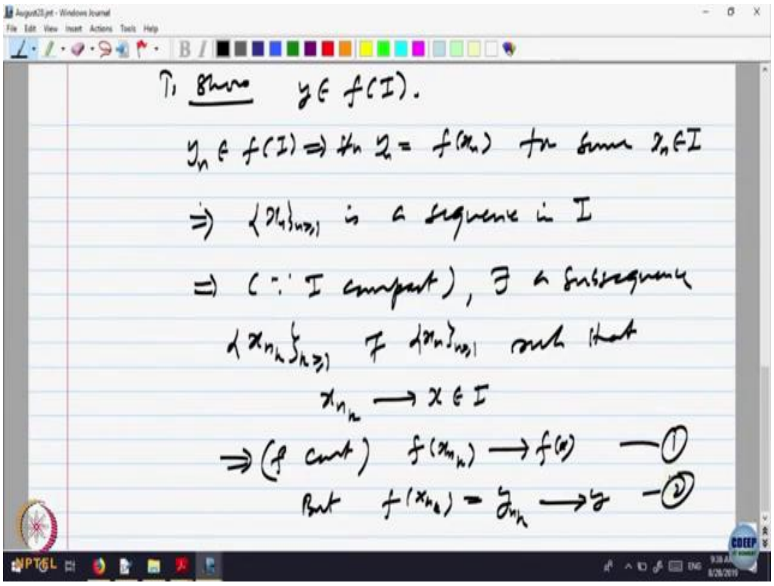

So, what is to be shown? To show that, f of I is closed we have taken sequence yn which is convergent to sub point y. So, to show that y belongs to f of I, that will prove that f of I is a closed set. So, let us start looking at because what was given is I is closed bounded so let us shift everything to the domain. So, yn belongs to f of I implies yn must be equal to f of xn for some xn belonging to I.

So, for every n so for every n this is happening so implies xn is a sequence in the interval I which is given to be closed and bounded so it must have a convergent subsequence. So, implies because I compact or closed bounded, there exist a subsequence say xnk of xn such that xnk converges to a point x belonging to the interval I.

So, that is compactness, every sequence has got a subsequence which is convergent in the set. But, then f continuous implies f of xnk converges to f of x but what is f of xnk? F of xnk is equal to ynk which converges, where does ynk converge because the sequence yn is convergent to y, so that is what is given to us yn converges to y. Every subsequence must also converge to y.

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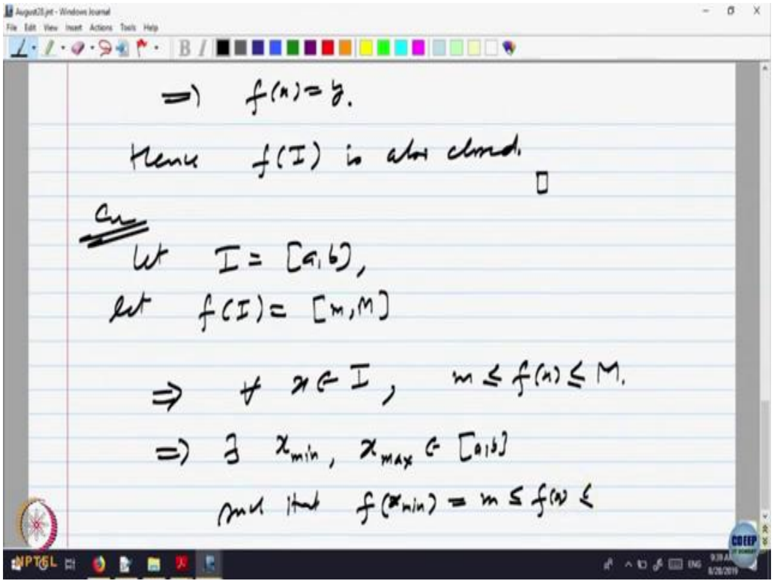

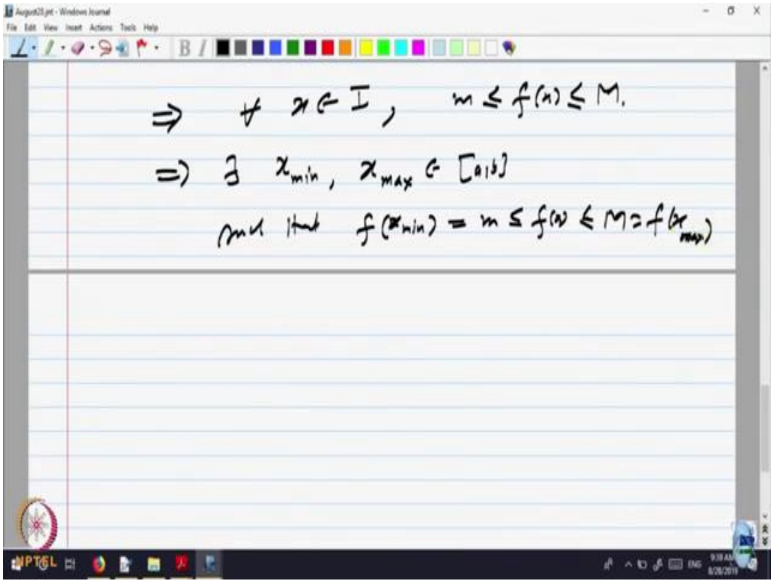

So, now look at this statement 1, look at this statement 2 so 1 and 2 limit being unique implies f of x is equal to y. So, idea is from the range shift everything to domain analyse and comeback so that, proves hence f of I is also closed. So, what we have proved is if I is a closed bounded interval then f of I also is a closed bounded interval.

So, let us write this as it is slightly more elaborate way solve this. So, we had I is a, b f of I is a closed bounded interval so it has some end points, it is closed, it is bounded. So, let us say this is small m and capital m so let us write corollary of let. Then what does it imply? What does it mean? That the range f of I is the closed bounded interval small m and to capital m that means small m is in the range of the function.

So implies for every x belonging to I, m is less than or equal to f of x is less than or equal to capital m. Range is equal to small m to capital m that means what? Small m is the smallest value of the function and capital m is the largest value of the function on the interval a, b. Not only that small m is in the range so small m must be equal to image of some point.

So, implies there exist some let us call x min and some x max belonging to a, b such that, the value at point x min is equal to small m less than or equal to value at every point is less than or equal to capital M which is the value at the point x max. Is that clear I am just re interpreting that result for the if the domain is a closed bounded interval a, b we just now proved range must be also a closed bounded interval. So, if domain is a, b range is small m to capital M.

What does it mean, that range is equal to this? That means every value of the function is between small m and capital M that means function is bounded. Small m is the minimum value, capital M is the maximum value and small m and capital M being in the range, they are the values taken at some point.

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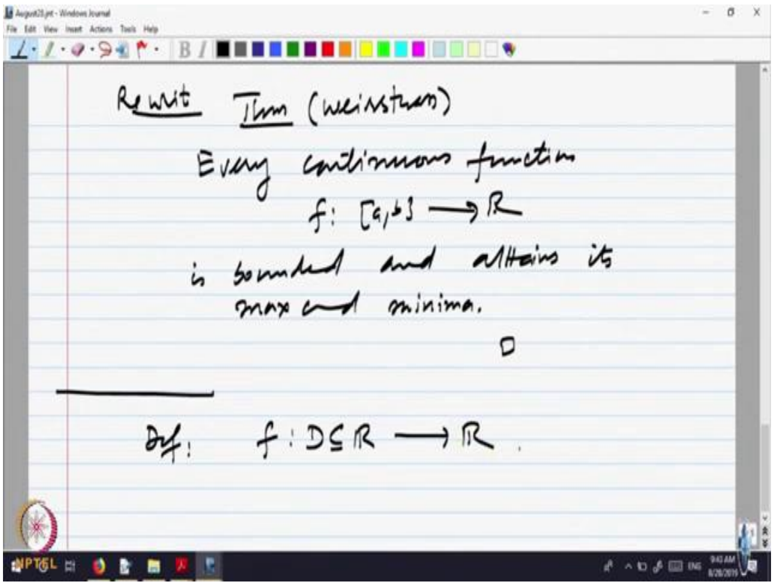

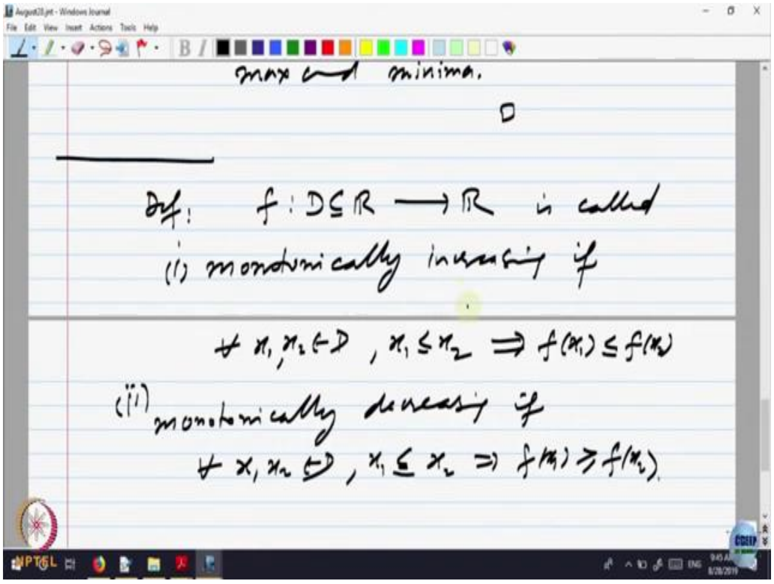

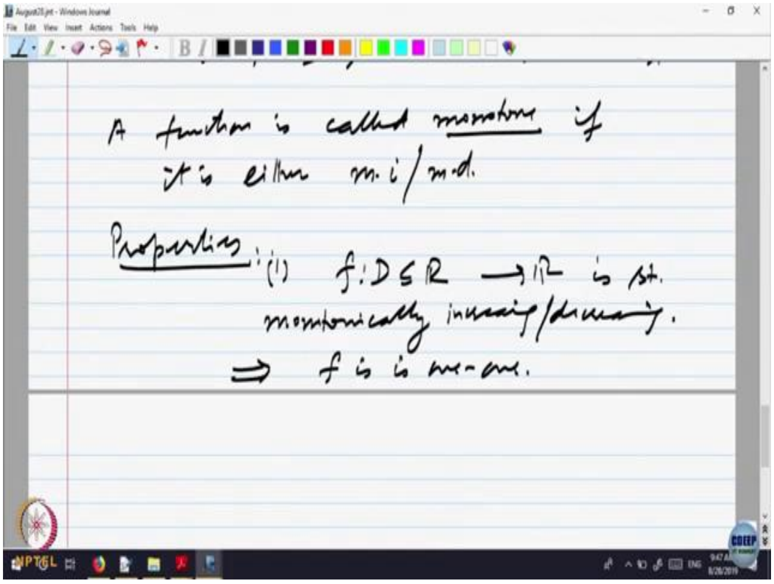

So, what we are saying is so you can this corollary can be rewritten as so rewrite if you like you can write it as a theorem. It is weir stress, so the theorem says every continuous function f from a, b to R. I am just rewriting nothing doing nothing more than that is bounded and attains its maximum and minimum.

There are points where the maximum value with the function the range is bounded so (())(10:33) bound property it must have greater over bound and least upper bound and what we are saying is this theorem says that not only they exist by the completeness property, they are actually attained that some points in the domain.

So, this is very useful in proving when we analyse maxima, minima’s of functions and optimisation problems. So, that is… so we showed in fact if we look at the proof, the proof does not use the fact that other than the fact that you are in an interval. If you do not want to claim then the range is interval you can just assume domain is a compact set then the range is a also compact set, it is closed bounded that is all.

Only because of intermediate value property we get intervals get mapped into intervals otherwise the same proof works by saying that, if you have domain to be a compact set then the range is also a compact, range is also closed and bounded. So, we did not use the fact that the domain was an interval in proving that range is a closed bounded set only when interval we use intermediate value property and those things.

So, for a connected set continuity preserves connectedness, image of a connected set is connected, image of a compact set is compact. Let us look at one or two some more properties of continuous functions before we go over to something else. So, let us look at… there are some, you must have already come across what it will may defined it, what is called a monotone function?

A function f in a domain D contained in R to R is only for real valued functions of on reals is called monotonically increasing if for every x1, x2 belonging to the domain. The name is indicative of what we are looking at? x1 less than or equal to x2 should imply the image f of x1 is less than or equal to f of x2 increasing.

As we move from left to right your function is going up and up geometrically. So, this is monotonically increasing and similarly you can define what is monotonically? So 1 to motono monotonically decreasing if x1, x2 belonging to the domain x1 less than x2 less than or equal to the point x2 implies f of x1 is bigger than or equal to f of x2.

The graph is going down and down as you move from left to right so that is decreasing. Keep in mind we are saying less than or equal to monotonically increasing, I am not saying redistrict in equality. If you want to say, the function … we want to indicate that f of x1 is strictly less than, there will add the words strictly monotonically increasing and strictly monotonically decreasing will qualify what is increasing and decreasing?

If you want to say strict. So, let us just a function is called monotone if it is either monotonically increasing or monotonically decreasing. If the function is either increasing or decreasing you are not vary particular whether it is increasing or decreasing, we are only interested in knowing that it is increasing or it is decreasing everywhere.

See increasing and decreasing are not properties of the function at a point, these are properties of the function over the domain. While, limit, continuity or properties of the function at a point so that is the difference. This is the property or the function over a domain over a subset of the domain if at least.

So, monotone if it is either increasing or decreasing so let us observe or observations or properties. Let us look at 1 suppose f in a domain d to R is monotone is strictly let us says strictly monotonically, strictly monotone so let us says strictly monotonically increasing or let us say decreasing one of them.

Either it is strictly increasing or strictly decreasing it is given to us then this implies obvious property that f is one-one, f is to be one-one is that okay? A function is either strictly increasing or strictly decreasing it said to be one-one is it clear everybody why is so? Because if not then what will happen at 2 different points x1, x2.

F of x1 is equal to f of x2 but either x1 will be less than x2 or x2 is bigger than x1 either of it, so either of it will contradict the fact that is monotonically increasing or decreasing. Let us look at, what about the converse of this statement?

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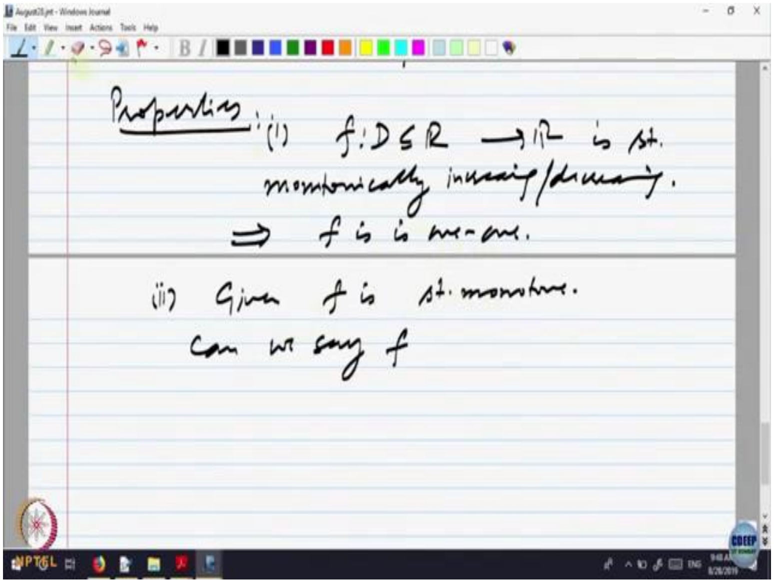

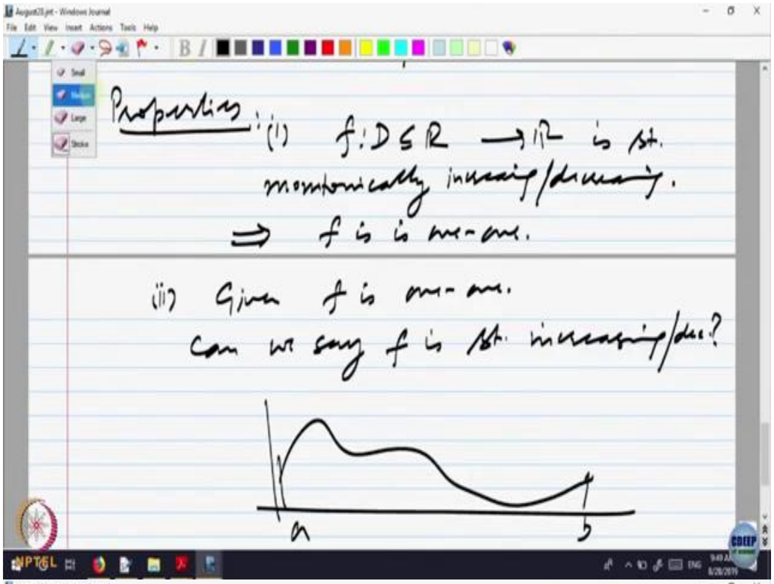

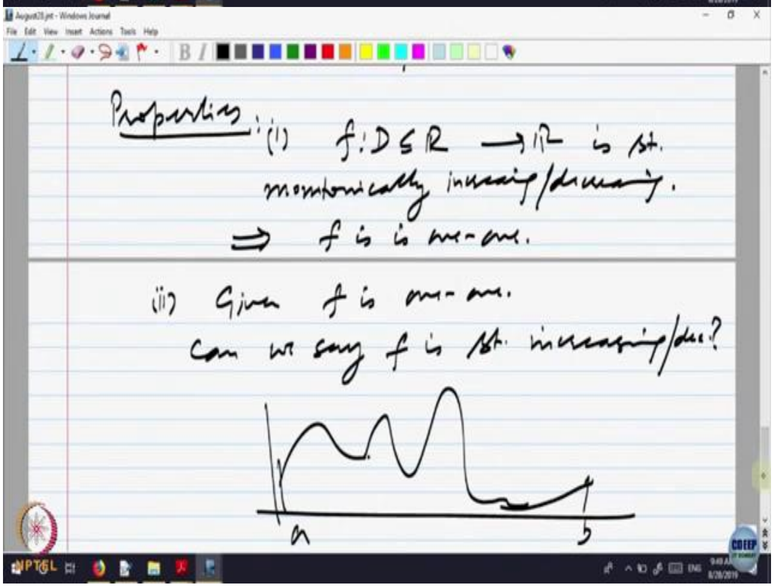

Suppose we are given f is strictly monotone, can we say given f function is one one sorry other way round so converse will be if f is given f is strictly we are looking at the converse of it. Suppose f is one-one, can we say f is strictly increasing or decreasing? Obviously not. Many examples, let y equal to x square for example if you can look at, okay or any function could be anything one-one, graph could be a weird function so this is the graph of the function say on the interval a to b.

It is one-one every horizontal line cuts the graph only once so it is one-one. We just want to look at the geometrically but it is not increasing or decreasing is clear or if we want you can make it more complicated you can just change the graph to something like you can do that anything is just possible one-one but still it is not monotonically increasing or decreasing.

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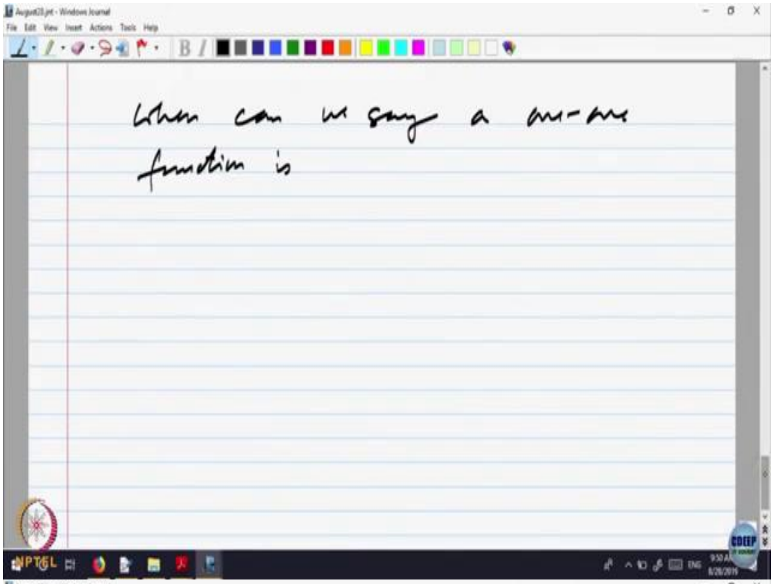

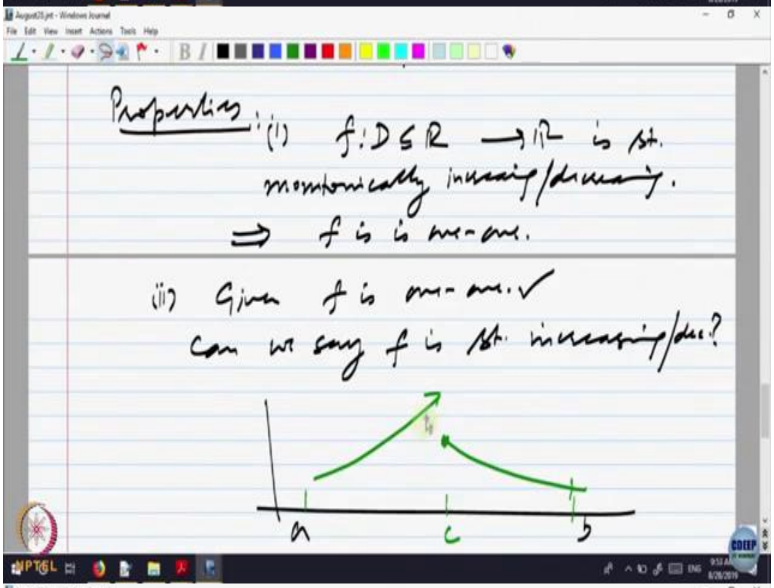

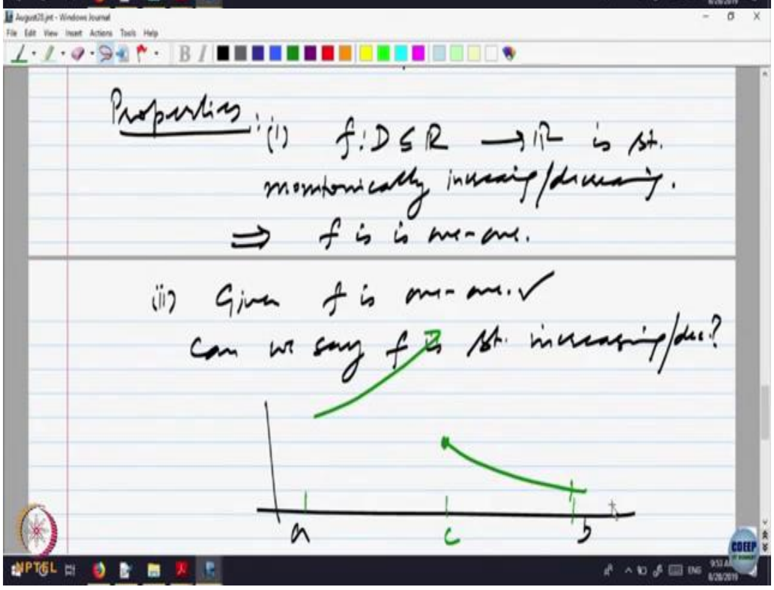

So, what is it? So, when can we say it is given to be one-one? But still okay. When can we say a one-one function is… this is not one-one sorry I gave very wrong graph and everything. I want example of a function which is one-one but it is not strictly increasing or decreasing, so can we think of a graph of such a function.

A function which is one-one but is not increasing or decreasing quite simple again. Let me guess change this to what we want? You want a graph of a function which is one-one can think of a drawing a picture of such a thing. I want a function which is one-one but it is not monotonically increasing or decreasing just a picture I do not want a formula.

So, it is seems it is not possible to draw it looks like you may not but let us try to draw, if we want to draw it what should happen? So, this is a, point a and this is a point b. I start here and it keep on increasing, it is one-one. But, I want to break the property it is increasing but I still want to keep their property it is monotone. So, I cannot go from here, but I can go like this.

Is this okay function? It goes up to here and then it starts here at this point, this is the point c, it goes up to here. On the left side a, to c it is monotonically increasing, on the right side it is monotonically decreasing and it is one-one. This function is not one-one, shall we make it one-one?

So, let us make it one-one still it is not very good so let me make it one-one so how do I make it one-one? So, let me take this, now it is one-one. So, that is how we experiment in mathematics so I have got a function and which is one-one but it is not monotone in the interval in the domain a to b.

Why it is not happening? Because there is a break in the graph of the function so probably it is, true. If the function is continuous and one-one then it should be monotone. So, let us write that is in theorem so in the function is one-one, what can we say if a function is one-one and continuous.

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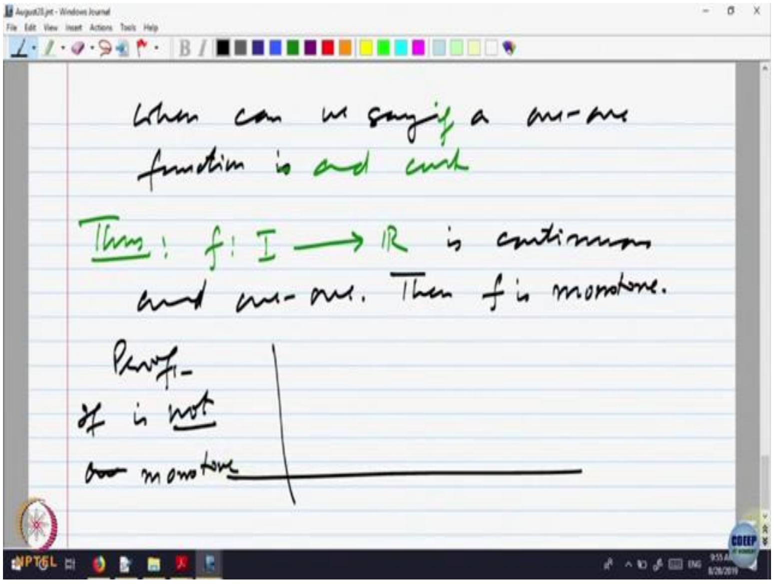

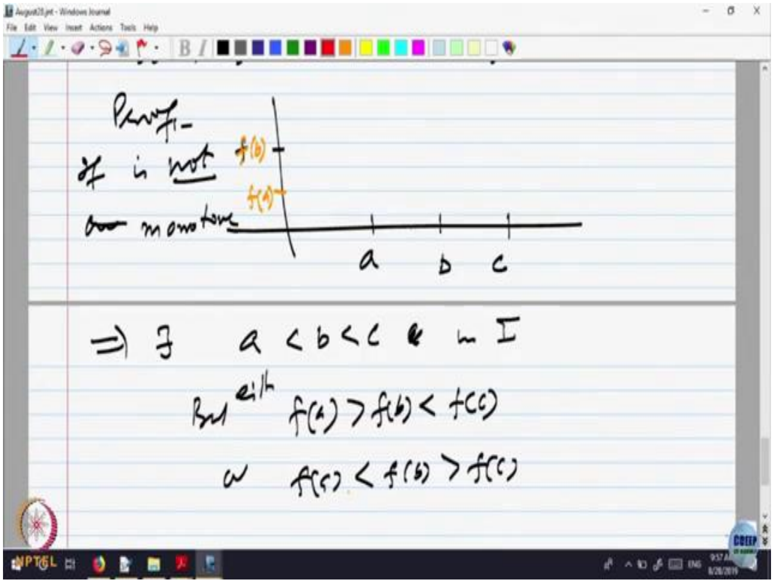

So, let us try to analyse theorem f from on interval I let us keep it on interval I to R is continuous and one-one then f is monotone. If it is continuous and one-one that should be one-one. The essentially what we are saying is, we start save at the point a, and you decide whether you want to go up or you want to go down.

So, on the start drawing, we start going up you cannot come down because one-one will be contradicted so the function should be monotonically increasing. So, let us write a proof of that so proof, so let me let us discuss the proof first. So, let us say so if let us assume f is not so this is also a good idea of not one one not monotone.

If the function is not monotone then what has should happened? Mathematically what I should I say, say for example then it should happen I should have 3 points a, b and c so then implies there exist a less than b less than c below in I. But what should happen? F of a, so either f of a, what should happen?

If it is monotone f of a will be less than f of a is bigger than f of b and f of b is less than f of c but either this or other way right something other inequality should happen. So, what is happening? Here is a, f of or what should happen f of a or let me write that also, if is less than f of b is bigger than f of c, one of these should happen, one of the thing should be contradicted. So, now see let us to visualize f of b is somewhere here so let me write f of b and f of it is bigger than f of a so probably here is f of a it is bigger than f of a.

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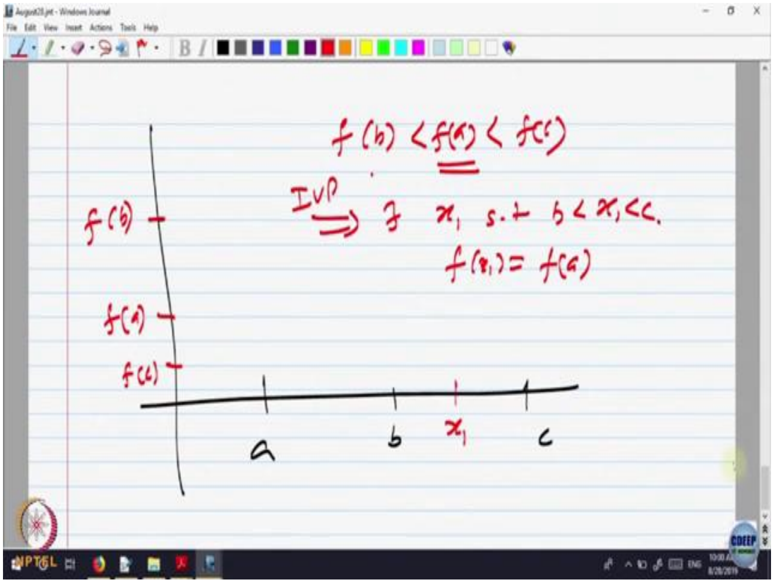

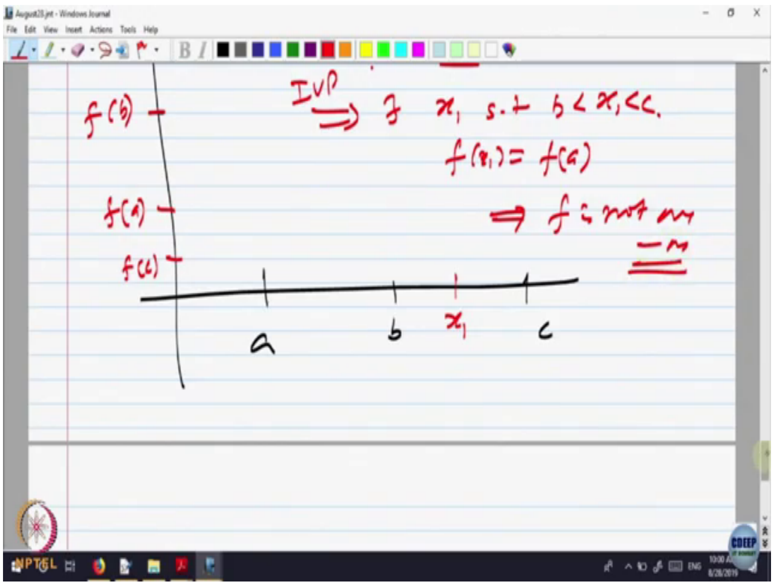

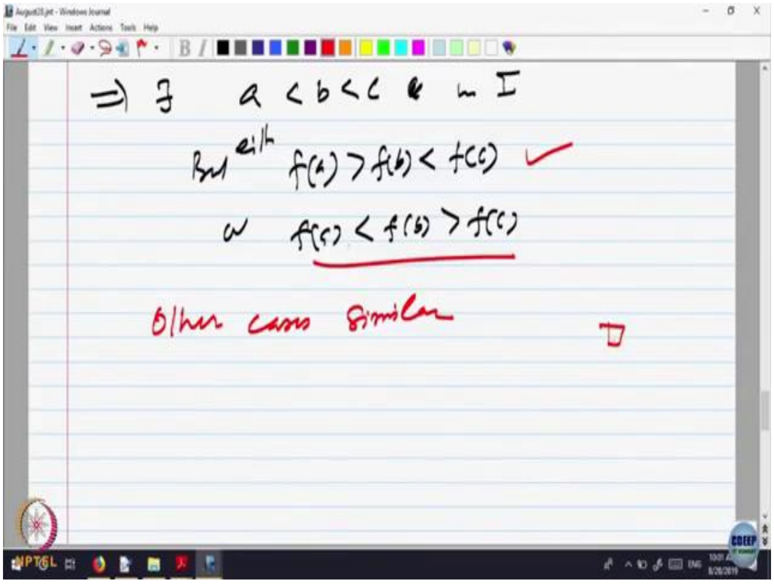

Are you able to see or should I draw a bigger picture? May be I think a bigger picture will be able to try here so let me draw somewhere here. So, here is a, here is b and here is my sorry is b and here is my c, 3 point we have got. So, here is f of b this is f of b and probably here is f of a, because we are looking at the case f of b is bigger than f of a as well as bigger than f of c.

So, f of b and f of c either is here or it is in between somewhere here so two possibilities either it is f of c is less than f of b but it can be bigger than f of a we do not know what is or this could be somewhere in between here. So, let us say this is one of the case is everything will be similar.

Now, what is happening so look at this, this value f of a, is between c and b. This value f of a, is between f of c and f of so f at b is less than f of a, is less than f of c. Function is given to be continuous so f of a must be attained at some value in between b and c by intermediate value property. This value is between these two values so implies by intermediate value property there exist a point x1 such that f of x1 is equal to a and where is x1, x1 is between b and c is that okay?

Now, f of x1 is a, f of x1 is not f of a sorry not a, f of a in between values f of a. So, there is a point where the value f of a, is taken and where is x1 here is x1, x1 is between b and c so f of a, is taken at a as well as at x1 but x1 cannot be equal to a so this contradicts one-one ness of the function f. Is that okay for everybody clear.

If f of c was up and f of a, was down then we would have same c is taken somewhere else also between the point a and again there will be a contradiction. So, this is implies f is not one-one. So, we analyse this case (())(30:24) property that it is f of b is bigger than f of a, f of a so we specialised it say one of the case is this.

If is other way around a and c then again there will be a contradiction similarly if you can analyse this one also. So, basic idea is the one-one property and continuity implies it has to be monotone. If not it will contradict intermediate value property somewhere so write all other cases other cases similar so write down the other possible cases.

We looked at when this is in between other case is, f of c is in between f of b less than f of c less than f of a so that gives you two and similarly two other cases. So, a function which is continuous and one-one has to be monotone. So, this is the theorem that we just now proved. A one-one continuous it has to be monotone. Let us look at monotone function slightly more.